ADeLA2002 - Astrometry in Latin America
ADeLA Publications Series, No. 1, 2003
R.Teixeira, N.V.Leister, V.A.F.Martin, and P.Benevides-Soares, eds.

Laplacian Orbit Determination

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Abstract. Laplace's method is a standard for the calculation of a preliminary orbit. Certain modifications enhance its efficacy: reduce the observations, if necessary, by use of the L_1 criterion; use a polynomial, whose order is determined by impersonal criteria, to calculate the first and second derivatives of observational quantities; combine the separate equations, one to determine the heliocentric distance of the object and the other its geocentric distance, into one polynomial equation for the heliocentric distance, whose roots are found by a standard algorithm; use recursion to calculate the f and g series. At least one differential correction is recommended to increase the accuracy of the computed orbital elements. Difficult problems, lack of convergence of the differential corrections, for example, can be handled by total least squares or ridge regression. The method is first applied to calculate a preliminary orbit of Comet P/ 1846 D1 (de Vico) from 59 observations made during five days in 1995 and then to a more difficult object, the Amor type minor planet 1982 DV (3288 Seleucus).

1. Introduction

Many have recognized the merits of Laplace's approach to orbit determination because it offers the substantial advantages that there is no restriction as to the number of observations to be used, three is the minimum, and the only number that can be used by Gauss's method, but if more are available so much the better, and the method works for orbits of any eccentricity, including hyperbolic orbits. Gauss's method as usually presented in the literature is restricted to low eccentricity orbits because the radius of convergence of the f and g series becomes smaller and smaller as the eccentricity approaches unity, although this restriction can be removed if we use the f and g functions rather than series.

But the restriction to use of only three observations remains. This can be a conundrum for the neophyte when more than the minimum three observations is available. The first example presented later uses 59 observations. There are over 32 thousand ways to select the three observations. How do we decide? Experience plays a role, but books extolling the virtues of Gaussian orbit determination give no hints on how to do it and acquire that experience. Laplace's method avoids this difficulty and, moreover, by making use of all of the observations offers a more satisfactory solution from the point of view of information theory. If we use Jaynes's definition of "amount of uncertainty" (1957), Laplace's method has zero uncertainty whereas the 32,509 ways of selecting three observations with Gauss's method result in considerable uncertainty.

2. Treatment of the Observations

Rather than work directly with right ascension and declination, I prefer to use the unit vector of position

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \tag{1}$$

and its first and second derivatives. These are needed to calculate for a given moment of observation a heliocentric distance r and a geocentric distance ρ. Not only will observational error be present in the spherical coordinates, but they also will be uncorrected initially for geocentric parallax. Let $\Delta\alpha$ and $\Delta\delta$ be the error in the spherical coordinates. Then the error $\Delta\mathbf{u}$ in $\mathbf{u}$ becomes, after differentiating Eq. (1) and taking L_2 norms of both sides,

$$\| \Delta\mathbf{u} \| \leq \| \begin{pmatrix} \cos\delta\Delta\alpha \\ \Delta\delta \end{pmatrix} \| . \tag{2}$$

Thus, $\Delta\mathbf{u}$ will in general incorporate less error than do the spherical coordinates.

The derivatives $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ should not be calculated from finite difference approximations. Differencing is an intrinsically noisy operation. With observational noise thrown in, the calculated first and second difference approximations likely bear little relation to the real first and second derivatives. But as Taff (1985) and others have recommended one can pass a polynomial through the components of $\mathbf{u}$ and calculate analytic first and second derivatives. Polynomial fitting, unlike finite differencing, is a smoothing operation.

This gives rise to two questions: what order should we use for the polynomial? and, when we use more than the minimum number of observations, the coefficients of the polynomial will be calculated from an overdetermined system; should we use a least squares fit?

To address the second concern first, if it seems that discordant observation are present, then one should use the robust L_1 criterion (Branham 1990), little influenced by outliers, to calculate the coefficients of the polynomial.

Regarding the second concern because we need the first and second derivatives of $\mathbf{u}$, a parabola is the minimum order polynomial that will do. An extremely high order polynomial would be unjudicious because of possible numerical instability. A cubic or quartic would seem to be the highest order called for consistent with a stable determination of the coefficients. How do we decide which order? One could employ Eichhorn's notion of efficiency (Eichhorn, 1990). Eichhorn defines the efficiency as the ratio of the product of the eigenvalues of the covariance matrix to the product of the diagonal elements of the same matrix raised to the power $1/n$, where n is the number of variables present or, for the problem at hand, the order of the polynomial. The efficiency, invariant to scaling, varies from 0 to 1. An efficiency close to 0 indicates that the overdetermined system incorporates too many variables, some of which must be otiose; an efficiency close to 1 indicates that all of the variables are required.

Laplace's method, by imposing dynamical and geometrical conditions on the heliocentric and geocentric distances, results in two equations for these quantities. If ρ is eliminated, we obtain a single polynomial equation for r whose roots can be found by any number of standard algorithms, Jenkins-Traub, for example.

Once good values for r and ρ have been found for a given time of observation, rectangular coordinates for the other times of observation can be calculated by use of the f and g series. Geocentric parallax can be eliminated by use, for each time of observation, of topocentric coordinates. Calculation of the f and g series is facilitated by use of recursion, offered by modern programming languages. Bond (1966) gives equations for the recursive calculation of the coefficients needed in the f and g series. These expressions are far simpler than the lengthy series expansions found in the literature.

3. Differential Corrections

Having a preliminary orbit one can correct the observations for planetary aberration. The preliminary orbit can then be corrected by use of differential corrections. As always with differential corrections uncertainty exists whether the iterates will converge. Sufficient conditions for convergence are known (Zadunaisky and Pereyra 1965). Without going into the mathematical details, one can state heuristically that what may destroy convergence is: an initial approximation far from the final solution; an ill-conditioned data matrix; the presence of large residuals. The initial approximation for a Laplacian differential correction is given by the observations, and little can be done to refine it. But the conditioning of the data matrix can be controlled by various techniques, such as scaling the matrix. Likewise, use of the L_1 criterion guards against large residuals.

Divergence can, moreover, often be avoided by use of techniques such as total least squares (TLS) regression and ridge regression. I discuss TLS in a review paper (Branham 2001) and ridge regression in my book (Branham 1990).

4. Two Examples

To illustrate the use of Laplace's method I will make use of fifty-nine observations made during five days during September 1995 of Comet P/ 1846 D1 (de Vico). I choose this example out of many possible ones because I have calculated an orbit for this comet and can compare the preliminary orbit with the definitive orbit. Because the orbit is highly eccentric (e =.96297) and we have only five days of observations, this should provide a good test of the efficacy of Laplace's method to calculate a preliminary orbit. Table 1 shows the comparison of the preliminary with the definitive orbit.

The second example is more demanding, the Amor-type minor planet 1982 DV (catalogued as 3288 Seleucus in the Minor Planet Center on-line catalog), because the object passes close to the earth, minimum distance 0.18 AU, resulting in substantial perturbations. I used twenty-seven observations for this object covering a time span from 28 Feb. to 14 Sept. 1982.

The initial orbit, calculated from least squares, is so bad that no realistic geocentric distance for the object can be calculated. But if instead we use the

Table 1. Preliminary and Definitive Orbits for Comet 1846 D1

Orbital Element	*Preliminary Orbit*	*Definitive Orbit*
M_0	$15.^\circ 14328$	$319.^\circ 36736$
$a(AU)$	19.7667450	18.0186023
e	1.0336102	$.9629730$
$q(AU)$	$.6643650$	$.6671749$
Ω	$79.^\circ 74035$	$78.^\circ 37858$
i	$89.^\circ 77687$	$89.^\circ 22955$
ω	$322.^\circ 68884$	$230.^\circ 10644$

robust L_1 criterion we find a highly hyperbolic, $e = 722$ (!), orbit. But at least it exists and is sufficient for ridge regression to be applied and results in the elliptic orbit shown in Table 2. Also shown is the orbit given in the catalog of the Minor Planet center. The sum of the squares of the residuals decreases from an atrocious $0.38784 \cdot 10^0$ for the initial hyperbolic orbit to $0.30712 \cdot 10^{-3}$ for the final orbit. Even so, the preliminary orbit can hardly be considered satisfactory because of failing to include perturbations. But if the preliminary orbit is further corrected and perturbations included, the agreement becomes acceptable.

Table 2. Orbits for 3288 Seleucus

Orbital Element	*MPC Orbit*	*Preliminary Orbit*
$Epoch(JD)$	2452600.5	2445049.540
M_0	$40.^\circ 74300$	$359.^\circ 88668$
$a(AU)$	2.0327881	32.7894630
e	0.4568761	0.9634881
$q(AU)$	1.1040558	1.1972059
Ω	$218.^\circ 76585$	$220.^\circ 69983$
i	$5.^\circ 92929$	$8.^\circ 00404$
ω	$349.^\circ 20616$	$347.^\circ 81594$

5. Conclusions

Laplace's method, when modified to use polynomial fitting rather than finite differencing to calculate time rates of the observational quantities and incorporating ridge regression for the differential correction of recalcitrant objects, represents a highly satisfactory method for initial orbit determination. It offers certain advantages over Gauss's method.

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Discussion

Assafin: How does your Laplacian routines compare with the Gaussian Method procedures for orbit computations?

Branham: They are comparable in precision, but I see some advantages on the Laplacian Method. For instance, on some cases the Gaussian Method won't converge, it gives negative roots, unlike the Laplacian Method. And I can use all observations, too.

Holvorcem: How does your implementation of the method of Laplace compare to Vaisala orbits in the generation of ephemeris for nearly discovered asteroids? In this case one normally has 2 nights/observation with 3-4 observations per night spaced between 0.5 to 1.0 hour apart.

Branham: It should work well. The example I gave, 59 observations of comet De Vico , were made during only 5 nights in 1995, with the observations grouped and not spread evenly over the nights. The preliminary orbit calculated from Laplace's method agrees well, nevertheless, with a definitive orbit calculated using all of the observations made in 1846 and 1995-1996.