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p-Modes of Low Order in a Bi-Polytropic Model of the Sun

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Abstract. Based on the Solar Standard Model SSM we developed a solar model in hydrostatic equilibrium using two polytropes, each one associated to the *radiative* and *convective* zones of the solar interior. Then we apply small periodic and adiabatic perturbations on this bi-polytropic model in order to obtain eigenfrequencies and eigenfunctions which are in the **p-modes** range of low order $0 < l < 5$; for $l = 2, 3$ these values agrees with the observational GOLF data within a few percent.

1. Introduction

Polytropic models have largely been used in the study of NRO of a gaseous sphere (Kopal 1949). We have computed the first modes of a bipolytropic model whose indices $n_2 = 3.85$ and $n_1 = 1.5$ describes both the *radiative* and *convective* zones respectively of the solar interior. We used Cowling's approximation (Cowling 1941) which reduces the order of the system of differential equations to 2 (instead of 4). The radial part of the perturbation obeys equations (1) and (2) (Ledoux & Walraven 1958):

$$\frac{dv}{dr} = \left[\frac{L_l^2}{\sigma^2} - 1\right] \frac{P^{\frac{2}{\Gamma_1}}}{\rho} w \qquad (1)$$

$$\frac{dw}{dr} = \frac{1}{r^2}\left[\sigma^2 - N^2\right] \frac{\rho}{P^{\frac{2}{\Gamma_1}}} v \qquad (2)$$

where

$$v = r^2 \delta r P^{\frac{1}{\Gamma_1}} \qquad (3)$$

$$w = \frac{P'}{P^{\frac{1}{\Gamma_1}}} \qquad (4)$$

are proper functions, l is the degree of the spherical harmonic, Γ_1 is the adiabatic exponentent equal to $\frac{5}{3}$, σ is the angular frequency, N is the *Brunt-Väisälä* frequency and L_l is the *Lamb* frequency.

The equations (1) and (2) and the boundary conditions lead to a eigenvalue problem with eigenvalue σ^2, which is the problem to be solved.

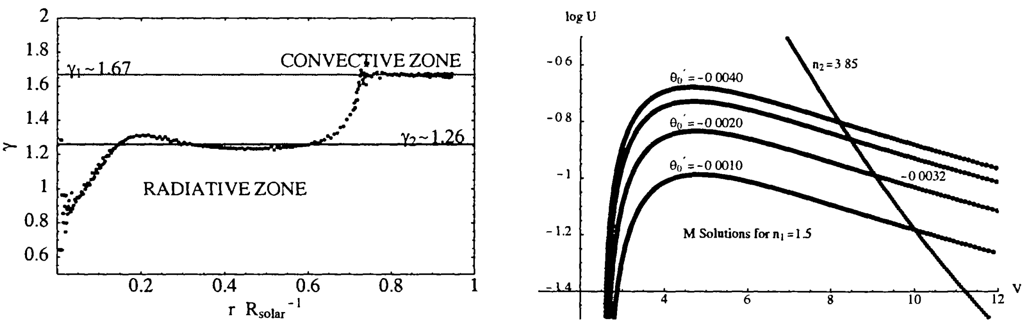

Figure 1. **Left.** γ at the solar interior due to SSM (Bahcall & Pinsonneault 2000). Regions with γ constant are described by polytropic process. The bipolytropic model is obtained if one assume two values for γ; 5/3 in the *convective* zone, and 1.26 in the *radiative* zone. **Right.** Solutions of the **Lane-Emden** equation ath the UV plane; the center of the Sun as at $U = 3.0$, $V = 0$ and its surface at $U = 0$, $V = \infty$. We choose $\theta_0' = -0.0032$ as a **M**-Solution for the *convective* zone.

2. Polytropes

Our unperturbed model consists of a gas of particles with spherical symmetry, selfgravitating, in hydrostatic equilibrium and with its state equation given by:

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{5}$$

K and γ are parameters that depend only on the polytropic index n, and the mass and radius of the configuration. The polytrope theory developed by the ends of the XIX century, can be used to know the dynamical structure of a star, within which local quasistatic thermodynamic changes follows a *polytropic process*, i.e. one in which the specific heat remains constant. This approach can be used in some regions of Sun's interior. We have used the pressure and the density data from the sophisticated SSM of Bahcall and Pinsonneault to plot $\gamma = \frac{dLnP}{dLn\rho}$ vs x with $x = r/R_{\odot}$. Various regions clearly emerge. Of these regions, the outermost one ($\gamma_1 = 1.677$, i.e. $n_1 = 1.50$) represents the *convective* zone where heat transport is achieved by adiabatic convection. It's SSM output in Fig. 1 is approximated rather well by a constant straight line indicating a polytropic behavior. The second intermediate zone labeled *radiative zone* in Fig. 1 can be approach by a polytrope $\gamma_2 = 1.26$, i.e. $n_2 = 3.85$. This two polytropes have been used by Hendry (1993). However, there are analytic approaches to the nuclear zone developed by Bludman and Kennedy (1999), which can be explored in multipolytropic models of Sun's structure.

2.1. Two Polytropes Within the Sun

Following Hendry (Hendry, 1993) we used ξ, θ as the variables in the **Lane-Emden** equation for the *convective* zone with index n_1, and η, ϕ as the variables for the *radiative* zone with index n_2. The parametric polytropes for the *convective* and *radiative* zones are respectively

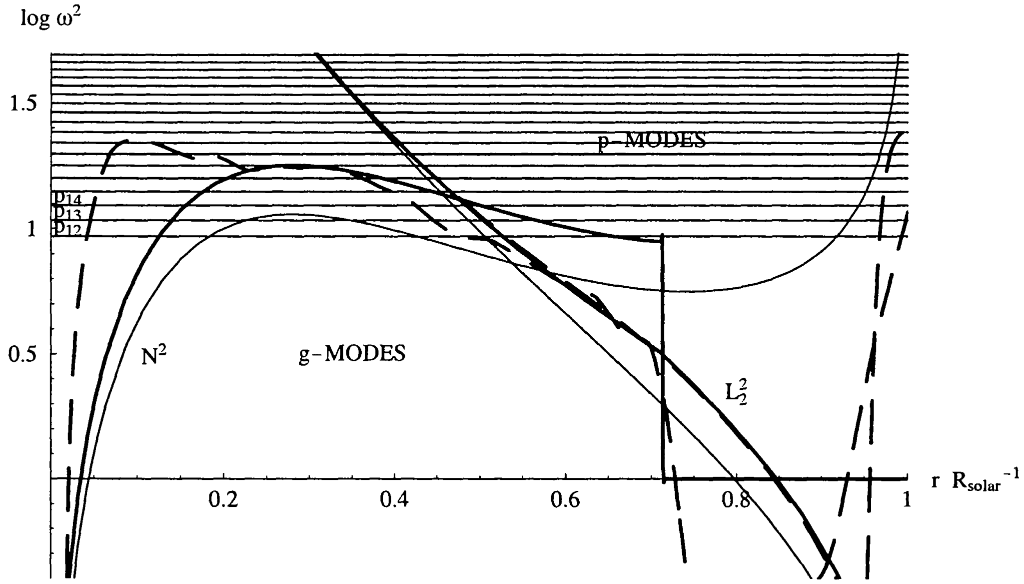

Figure 2. Propagation Diagram for the **bp** model (solid line), for the polytropic index 3 (thin line) model and for the MSE (dashed lines).

$$P = K_1 \rho^{\frac{5}{3}}, \text{ with } \rho = \lambda_1 \theta(\xi)^{1.5} \tag{6}$$

$$P = K_2 \rho^{1.26}, \text{ with } \rho = \lambda_2 \varphi(\eta)^{3.85} \tag{7}$$

The main challenge is to learn how to fit these two polytropes together. Since the physical quantities P, ρ and M are continuous across the interface (not θ or ϕ), the variables U and V given by

$$U = \frac{\xi \theta^n}{-\theta'} \tag{8}$$

$$V = \frac{(n+1)\xi(-\theta')}{\theta} \tag{9}$$

are be very useful (Chandrasekhar,1939).

Let us start by considering the *convective* zone. We take $n_1 = 1.5$ thus $\xi_1 = 3.6538$ and $\theta_0' = -0.2033$. Though, this polytrope is not being used in the vecinity of $\xi = 0$, it is possible to consider all of the solutions of the **Lane-Emden** equation for $n_1 = 1.5$. These may be generated beginning at ξ_1 with an arbitrary starting slope and integrating inwards. Solutions with starting slopes less negative than θ_0' are of particular interest (in the literature, they are referred as **M**-solutions) since these are the ones which intersect the polytrope that represents the *radiative* zone. Four such solutions, translated into the U, V variables, are shown in Fig. 1 (Right). Solutions with starting slopes more negative than θ_0' (**F**-solutions) do not intersect the *radiative* polytrope and so do not need to be considered here.

Knowing $\theta(\xi)$ and $\phi(\eta)$ we can deduce the density and pressure curves $\rho(r)$ and $P(r)$. The above model yields a central density of $\rho_c = 1.299 \times 10^5\ Kgm^{-3}$ and a central pressure of $P_c = 2.237 \times 10^{16}\ Nm^{-2}$.

3. NRO in a Bi-Polytropic Model (BPM)

Space oscillation properties of the solutions of equations (1) and (2) are related to the signs of the coefficients given in the second members of these equations. Space oscillations are allowed only in the regions where these coefficients have opposite signs. The limits of these regions are defined by

$$\sigma^2 = \frac{l(l+1)\Gamma_1 P}{\rho r^2} = L_l^2 \tag{10}$$

$$\sigma^2 = N^2 \tag{11}$$

In the (x, ω^2) plane, these equations define two curves. In Fig. 2 we have plotted them as a continuous line for the bipolytropic model. Have been denoted *p-modes* and *g-modes* the regions of this plane corresponding to the conditions of position in the star and frequency, allowing spatial oscillations. These regions are characterized by the possibility of existence of progressive acoustic waves and progressive gravity waves respectively (Scuflaire 1974). Thus we shall refer to these regions as the acoustic and the gravity regions. We have also plotted in the same figure the frequencies of the first *p-modes* for bipolytropic model.

The equations (1) and (2) are very convenient for the analytical discussion, but for the numerical computations we use the more appropriate form:

$$\frac{dy}{dz} = \frac{l+1}{x}\left[-y + \frac{l}{\omega^2}z\right] + \frac{x}{\Gamma_1}\frac{GM_\odot \rho}{R_\odot P}\left(\frac{q}{x^3}y - z\right) \tag{12}$$

$$\frac{dz}{dx} = \frac{1}{x}\left[\omega^2 y - lz\right] + R_\odot A\left(\frac{q}{x^3}y - z\right) \tag{13}$$

where we have put:

$$\frac{r}{R_\odot} = x \tag{14}$$

$$\frac{m}{M_\odot} = q \tag{15}$$

$$\frac{\xi_r}{R_\odot} = x^{l-1}y \tag{16}$$

$$\frac{R_\odot P\prime}{GM_\odot \rho} = x^l z \tag{17}$$

$$\frac{R_\odot^3 \sigma^2}{GM_\odot} = \omega^2 \tag{18}$$

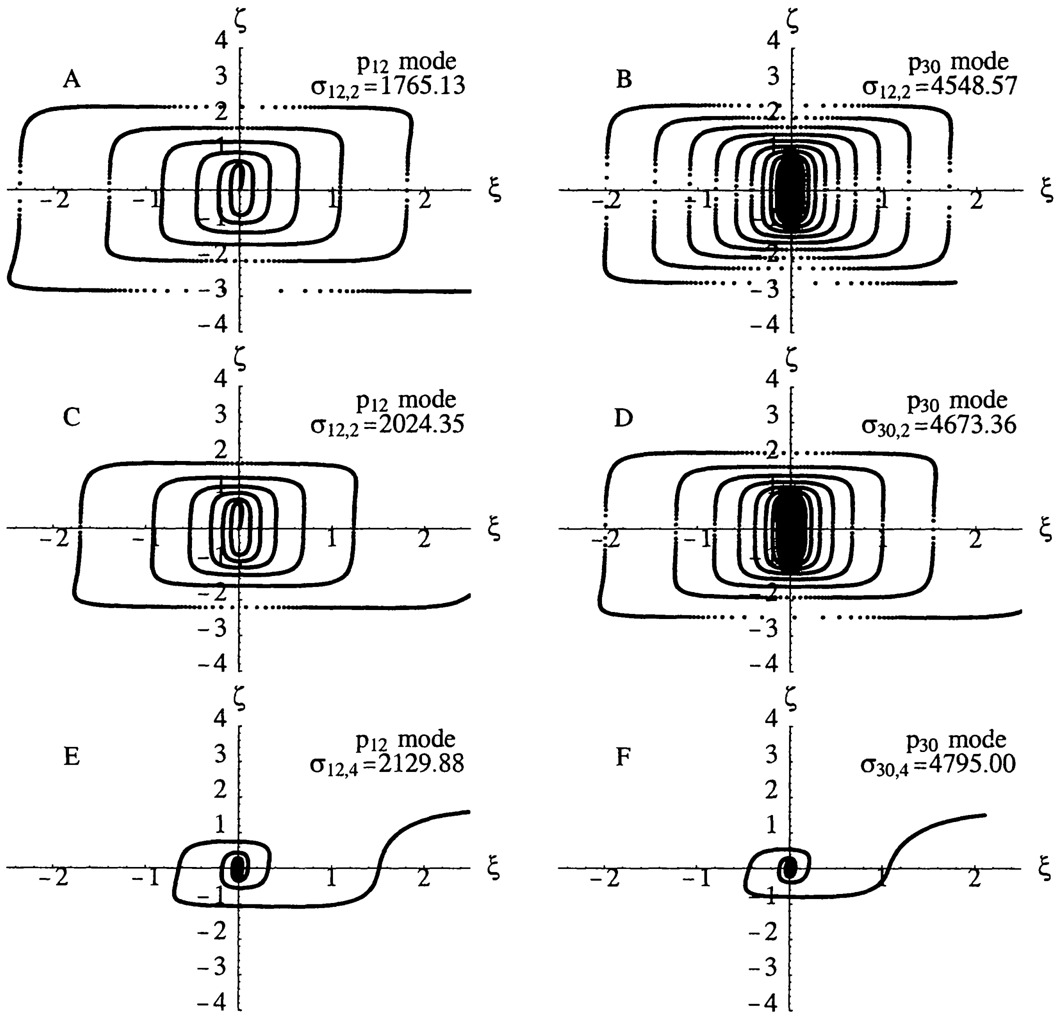

Figure 3. **Some *p-modes* for a polytropic models.** Figures **A** and **B** corresponds to the $n = 3$ model. **C**, **D**, **E**, and **F** are the same modes but in the bipolytropic model. The units of σ_{nl} are μHz.

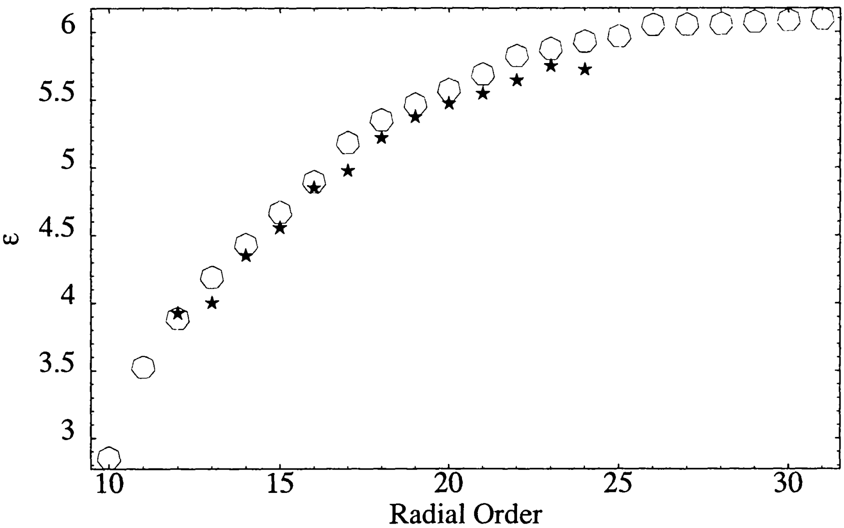

Figure 4. Comparison with GOLF data for $l = 2$ (circle) and $l = 4$ (star). The ε value is given by $\varepsilon = 100\% \times (\sigma_{\mathrm{BPM}} - \sigma_{\mathrm{GOLF}}/\sigma_{\mathrm{BPM}})$

The regularity condition at the centre, first requires that

$$\omega^2 y - lz = 0 \tag{19}$$

and second, at the surface, the cancellation of the lagrangian perturbation of the pressure is written by

$$\frac{q}{x^3} y - z = 0 \tag{20}$$

In order to determine the solution uniquely, we impose the normalizing condition

$$y = 1 \tag{21}$$

at the centre. With a trial value for ω^2 we integrate equations (12) and (13), with initial conditions (19) and (21) using Runge-Kutta method, with a step size taken from paper of Christensen-Dalsgaard [Christensen-Dalsgaard et al.,1994]. Usually this solution does not satisfy equation (20) and a new integration is performed with another value of ω^2. This procedure is repeated until equation (20) is satisfied, using a Newton-Raphson method to improve the value of ω^2.

The radial displacement $\delta(r)$ and the pressure perturbation P' are periodic space functions; the variables $v(r)$ and $w(r)$ vary strongly from the center to the surface, then it is impossible to plot them directly along the axes. However, the most appropiate functions $\xi = \pm \log_{10}(1 + \left|\frac{\delta(r)}{R_\odot}\right|)$ and $\zeta = \pm \log_{10}(1 + \left|\frac{R_\odot P'}{GM_\odot \rho}\right|)$, have been plotted in Fig. 3, their signs are chosen according to the signs of the variables $\delta(r)$ and P'.

4. Conclusions

Although we do not use an atmosphere model and the input physics is described by polytrope structure, the modes obtained from the bipolytropic model are close to the observacional data. We show in Fig. 4 a comparison with GOLF data[1], showing an agreement within a few percent (roughly between 4.5% and 7%) up to a radial order of 30.

References

Bahcall, J. N., & Pinsonneault, M. 1995 in Rev.Mod.Phys., 67, 781

Chandrasekhar, S. 1939 in An Introduction to the Study of Stellar Structure, Dover Publications, Chicago

Christensen-Dalsgaard, J. & Mullan, D.J. 1994 in MNRAS, 270, 921

Cowling, T.G. 1941 in MNRAS, 101, 367

Hendry, A. 1993 in Am.J.Phys, 61, 10

Kopal, Z. 1949 in ApJ, 109,509

Ledoux, P. & Walraven, T. 1958, in Handbuch der Physik Vol LI, Springer Verlag, Berlin

Mullan, D. & Ulrich, R. 1988 in ApJ, 331, 1013

Scuflaire, R. 1974 in A&A, 36, 107

Tassoul, M. 1980 in ApJS, 43, 469

Unno, W., Osaki, Y., Ando, H., Saio, H., Shibahashi, H. 1989 in Nonradial Oscillations of Stars, University of Tokio Press, Tokio

[1] taken from `www.medoc.ias.u-psud.fr/golf.html`