

POLYNOMIAL APPROXIMATION IN WEIGHTED SOBOLEV SPACE

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1. Introduction. The so-called BRAMBLE-HILBERT lemma [1,2] plays an important role in estimating the error of the finite element method (FEM). This lemma is based on the inequality of the form

$$\inf_{g \in P} \|f - g\|_{W^{m,p}(\Omega)} \leq c(\Omega) \sum_{\alpha \in A} \|\partial^\alpha f\|_{L_p(\Omega)},$$

where P is a class of polynomials, A is an associated class of multi-indices, $\|\cdot\|_{W^{m,p}(\Omega)}$ denotes the norm in the classical Sobolev space $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $p > 1$, $W^{0,p}(\Omega) = L_p(\Omega)$, and Ω is an open and bounded domain of $\mathbb{R}^n$.

Using the Sobolev's integral representation, DUPONT and SCOTT [3] have obtained the above inequality for a more general collection P and A

$$(1.1) \quad \inf_{g \in P} \|f - g\|_{W^{m,p}(\Omega)} \leq c(\Omega) \sum_j \|P_j(\partial)f\|_{L_{p_j}(\Omega)},$$

where $\{P_j\}_{j=1}^k$ is a collection of homogeneous polynomials of degree l_j and P is the intersection of the kernels of the operators $P_j(\partial)$, $j = 1, 2, \dots, k$.

Recently EL HATRI [4] has found (1.1) in a weighted Sobolev space

$$(1.2) \quad \inf_{p \in P} \|f - p\|_{W^{m,p}(\Omega, \sigma)} \leq c(\Omega) \sum_j \|P_j(\partial)f\|_{L_{p_j}(\Omega, \sigma)},$$

where $\|\cdot\|_{W^{m,p}(\Omega, \sigma)}$ denotes the norm in the weighted Sobolev space $W^{m,p}(\Omega, \sigma)$ which is defined with respect to the weight function $w = (d_M(x))^\sigma$, $\sigma \in \mathbb{R}$, $x \in \Omega$, $\Omega \subset \mathbb{R}^2$, see [5].

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In this paper we establish (1.2) for a more general domain Ω of $\mathbb{R}^n$, $n \geq 3$. Our proof is analogous to that one of DUPONT and SCOTT [3]. It is based on Sobolev's integral representation, combining the method of local coordinates, and using the continuity of the Riesz potential in weighted space. This result is justified by the estimate of the finite element error of some degenerated and singular boundary value problems. We mention here that in the case of nonsmooth domains the use of some weight a priori estimates is natural because of the existence of some singular solutions even for the elliptic equations.

2. Notations and preliminaries. Let Ω denote a bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$, and let $x, y, \dots$ denote points in $\mathbb{R}^n$. Given a point x in $\mathbb{R}^n$, $x = (x_1, \dots, x_{n-1}, x_n)$, we shall write: $x = (x', x_n)$, where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Henceforth, we suppose that $\partial\Omega$ satisfies the Lipschitz condition and Ω is of the class $C^{0,1}$, i.e. for each point $x \in \partial\Omega$ there exists a neighbourhood U_x , such that in an appropriate cartesian system of coordinates (y', y_n) , the set $U \cap \Omega$ is represented by the inequality $y_n < a(y')$, where $a = a(y')$ is a Lipschitz-continuous function, see ([6], p.22). Thus, Ω verifies the inner-cone and the outer-cone property see ([5], p.25) and Ω can be considered as a finite union of domains that are star-shaped with respect to some corresponding balls, see ([6], p.23).

Seeing that Ω belongs to the class $C^{0,1}$, then for each point $x^0 \in \partial\Omega$ we have the following properties:

(A) The point x^0 is the origin of the local coordinate system (y', y_n) and the y_n -axis is the axis of an unbounded outer-cone K , $K \cap \bar{\Omega} = \emptyset$, with an opening angle ω , $0 < \omega < \pi$.

(B) x^0 belongs to one and only one set $\Gamma = \{(y', y_n) : y' \in \Delta_\delta : y_n = a(y')\}$ of $\partial\Omega$; where $a = a(y')$ is a Lipschitz-continuous function on the closed cube:

$$\Delta_\delta = \{y' = (y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-1} : |y_i| < \delta, i = 1 \div n-1\}, \delta = \text{const.}$$

The hyperplanes $\{y_i = 0\}$, $i = 1, \dots, n-1$ divide Δ_δ to some cubes $D_i \subset \Delta_\delta$, $i = 1, \dots, 2^{n-1}$.

Let denote by $V(x^0)$ a neighbourhood of x^0 , and let define

$$(2.1) \quad \Omega_i = \{x = (y', y_n) \in \Omega : y' \in D_i\}, \quad i = 1, \dots, N$$

such that:

(C) The number $\delta > 0$ is selected such that $V(x^0)$ is entirely covered by the union of $\bar{\Omega}_i$, $i = 1, \dots, 2^{n-1}$, and there is an open ball $B_i, \bar{B}_i \subset \Omega_i$, for which the domain $\bar{\Omega}_i$ is "star-shaped" with respect to B_i (the last condition follows using some arguments from ([6], p.23).

From now on, let x^0 be a fixed point of Γ for which the conditions (A), (B) and (C) are fulfilled, then we have

Lemma 2.1. Let Ω_i be a subdomain of Ω defined in (2.1) for a given i and let $x \in \bar{\Omega}_i$. Let denote by $B(x)$ the segment joining the point x to any point of a fixed ball B_i corresponding to Ω_i according to the condition (C). Then, there is a constant $c_i > 0$, not dependent on x , such that

$$(2.2) \quad |x - x^0| \leq c_i |z - x^0|, \quad \forall z \in B(x).$$

3. Integral representation. Next we need a basic result, its proof can be found in [7,3]. Let $\bar{D}$ be a bounded domain which is star-shaped with respect to a ball $B \subset D$. Let $f \in C^\infty(\bar{D})$, and let $\varphi \in C_0^\infty(B)$ is such that $\int_{B_i} \varphi(y) dy = 1$, and let $x \in D$, then

$$(3.1) \quad f(x) = Q^l f(x) + R^l f(x), \quad l \in \mathbb{N},$$

where $Q^l f(x) = \sum_{|\alpha| \leq l-1} \int_B \varphi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy$, $R^l f(x) = \sum_{|\alpha|=l} \int_D K_{\alpha,D}(x,y) f^{(\alpha)}(y) dy$,
 $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_k \in \mathbb{N} \cup \{0\}$, $f^{(\alpha)} = \partial_x^\alpha f = \frac{\partial^{\alpha_1+\dots+\alpha_n} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

The kernel $K_{\alpha,D}$ is given by $K_\alpha(x,y) = (l/\alpha!)(x-y)^\alpha K_D(x,y)$, $K_D(x,y) = \int_0^1 \varphi(x+s^{-1}(y-x)) s^{-n-1} ds$.

As a function of y , $K_{\alpha,D}(x, \cdot)$, and, therefore, $K_D(x, \cdot)$ is supported in the convex hull of $\{x\} \cup B$ denoted by $D(x)$.

The domain D will be considered as an element of the set $\{\Omega_i, i = 1, \dots, N\}$, where the union of subdomains Ω_j for $j > 2^{n-1}$ covers $\Omega \setminus \overline{V(x^0)}$. Each one of the subdomains Ω_j for $j > 2^{n-1}$ is connected to some boundary point (like the domains Ω_j for $j \leq 2^{n-1}$ in (2.1), connected to the point x^0) or is a ball inside the domain Ω . It follows from the above construction that each one of the subdomains Ω_j is star-shaped with respect to a corresponding ball B_j .

Let $M \subset \Gamma$ be a manifold of dimension m , $0 \leq m \leq n-1$, and let us denote for $x \in \Omega$: $d(x) = d_M(x) = \text{dist}(x, M) = \inf\{|x-y| : y \in M\}$. In this paper, we shall deal in more details in the case $M = \{x^0\}$ ($m = 0$).

Let us give some technical results that will be used to prove the main theorems in this paper. Denote $K_i \equiv K_{\Omega_i}$.

Lemma 3.1. Under the assumptions (A), (B) and (C), $\forall x \in \Omega_i$, for all $z \in \Omega_i(x)$, there is a constant $c > 0$ such that

$$(3.2) \quad K_i(x, z) \leq c(d_M(z))^\nu |x-z|^{-n-\nu}, \quad \forall \nu \geq 0, \quad i \in \{1, \dots, N\}.$$

Proof. For all $x \in \Omega_i$ and $z \in \Omega_i(x)$, using the property of function φ and Lemma 2.1, we have

$$|K_i(x, z)| \leq c_i \int_{\frac{|z-x|}{r_i}}^1 s^{-n-1} ds \leq c_{1,i} (d_M(z))^\nu |x-z|^{-n-\nu}, \quad i = 1, \dots, 2^{n-1},$$

where $r_i = \text{diam}(\Omega_i)$. For $i > 2^{n-1}$, the above inequality is trivial.

Remark 3.1. In view of definition of $K_{\alpha,i}(x,y)$, it follows as (3.2) that $\forall x \in \Omega_i$, $\forall y \in \Omega_i(x)$, $i = 1, \dots, N$ and $\nu \geq 0$

$$(3.3) \quad |\partial_x^\gamma \partial_y^\lambda K_{\alpha,i}(x,y)| \leq c(d_M(y))^\nu |x-y|^{|\alpha|-n-\nu-|\gamma|-|\lambda|}.$$

4. Embedding theorem in weighted Sobolev space. Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$. Let's consider the weighted Sobolev space $W^{m,p}(\Omega, \sigma)$, defined in [5], by means

of the weighted function $(d_M(x))^\sigma$, and which is a Banach space with respect to the norm

$$(4.1) \quad \|u\|_{W^{m,p}(\Omega,\sigma)} = \begin{cases} \left(\sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x)|^p (d_M(x))^{\sigma p} dx \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{|\alpha| \leq m} \text{vraisup}_{x \in \Omega} (|D^\alpha u(x)| (d_M(x))^\sigma), & p = \infty \end{cases}.$$

For $m = 0$ we use the notation: $W^{0,p}(\Omega, \sigma) = L_p(\Omega, \sigma)$ and by $|\cdot|_{W^{m,p}(\Omega,\sigma)}$, we denote the semi-norm in $W^{m,p}(\Omega, \sigma)$:

$$|u|_{W^{m,p}(\Omega,\sigma)} = \left(\sum_{|\alpha|=m} \int_{\Omega} |D^\alpha u(x)|^p (d_M(x))^{\sigma p} dx \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$|u|_{W^{m,p}(\Omega,\sigma)} = \max_{|\alpha|=m} \text{vraisup}_{x \in \Omega} (|D^\alpha u(x)| (d_M(x))^\sigma) < \infty, \quad p = \infty.$$

Notice that the space $C^\infty(\bar{\Omega})$ is dense in $W^{m,p}(\Omega, \sigma)$ with respect to the norm (4.1), see ([5], p.79).

The evoked polynomial approximation in the introduction of this paper is based on the estimate of the following error

$$R^l f(x) = f(x) - Q^l f(x),$$

with respect to the norm (4.1). This estimate follows from the continuity of the Riesz potential

$$(4.2) \quad T(f)(x) = \int_{\Omega} (d_M(y))^\theta \frac{f(y)}{|x-y|^{n-\sigma}} dy, \quad \sigma \in \mathbb{R}, \quad \theta \in \mathbb{R}$$

in $L_p(\Omega, \theta)$. The proof of the following technical lemma does not present essential difficulties, see also ([8], Lemma 2, p.26).

Lemma 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $x^0 \in \partial\Omega$. Let $\alpha, \beta \in \mathbb{R}$ verified $\alpha > -n, \beta > -n$. Then, there is a constant $c > 0$ such that for all $x \in \Omega, x \neq x^0$, we have

$$(4.3) \quad \int_{\Omega} d_M(y)^\beta |x-y|^\alpha dy \leq \begin{cases} c |x-x^0|^{\alpha+\beta+n}, & \alpha+\beta+n < 0, \\ c, & \alpha+\beta+n > 0. \end{cases}$$

Proposition 4.1. Let $1 \leq p \leq q < \infty, \frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$, and let $\mu \in \mathbb{R}, \sigma \in \mathbb{R}$; such that

$$(4.4) \quad r(\sigma - n) + n > 0, \quad \mu q + n > 0 \quad \text{and} \quad r(\sigma - n) + \mu q + n > 0.$$

Then, there is a constant $c > 0$ such that, for the Riesz potential (4.2) we have

$$(4.5) \quad \|Tf\|_{L_q(\Omega,\mu)} \leq c \|f\|_{L_p(\Omega,\theta)}, \quad \forall f \in L_p(\Omega, \theta).$$

Remark 4.2. For $1 < p < \infty, q = \infty$, we get the following result

$$\text{vraisup}_{x \in \Omega} (d_M(x))^\mu (Tf)(x) \leq c \|f\|_{L_p(\Omega,\theta)}, \quad \forall f \in L_p(\Omega, \theta)$$

if $\mu \geq 0$ and $p'(\sigma - n) + n > 0$, $\frac{1}{p} + \frac{1}{p'} = 1$.

We say that a normed space X is continuously injected in a normed space Y if $X \subset Y$ and there is a constant $c > 0$ such that $\|u\|_Y \leq c\|u\|_X, \forall u \in X$, and we write $X \hookrightarrow Y$.

Theorem 4.1. Let $1 \leq p \leq q < \infty$, $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$, $\mu \in \mathbb{R}$, and $\theta \in \mathbb{R}$ such that $r(k+1-m-\theta-n) + n > 0$ and $\mu q + n > 0$.

Then, if $\mu q + r(k+1-m-\theta-n) + n > 0$, we have

$$(4.9) \quad W^{k+1,p}(\Omega, \theta) \hookrightarrow W^{m,q}(\Omega, \mu).$$

Further, if $p > 1$ and $p > \frac{n}{m-\theta}$, $m \in \mathbb{N}$, then

$$(4.10) \quad W^{m,p}(\Omega, \theta) \hookrightarrow C^0(\overline{\Omega}).$$

Proof. The proof of Theorem 4.1 uses the integral representation (3.1) in Ω_i , $i = 1, \dots, N$, with $l = k+1$ and taking into account the inequality (3.3) with $\nu = \theta \geq 0$ and the Proposition 3.1. For $\theta < 0$, we use the trivial injection $W^{k+1,p}(\Omega, \theta) \hookrightarrow W^{k+1,p}(\Omega)$, $\forall \theta < 0$.

5. Polynomial approximation. The proof of the next theorem follows from the integral representation (3.1), using the Proposition 4.1 and taking into account the same approach applied in the proof of Theorem 5 in [3].

Theorem 5.1. Let $\{P_j\}_{j=1}^k$ be a set of nontrivial homogeneous polynomials (in n variables) of degree l_j , $j = 1, 2, \dots, k$, respectively; having no common (nonzero) complex zero (this forces $k \geq n$). Define

$$\mathbb{K} = \{v \in C_0^\infty(\mathbb{R}^n) : P_j(\partial)v = 0, \quad j = 1, 2, \dots, n\}$$

Then $\mathbb{K} \subset P_r$, for some integer r , where P_r is a class of polynomials of degree $\leq r$. Let $l = \min_{1 \leq j \leq k} l_j$, m be a nonnegative integer less than l , and let $1 \leq q \leq \infty$, $1 \leq p_j \leq \infty$, $\frac{1}{r_j} = 1 - \frac{1}{p_j} + \frac{1}{q}$, ($j = 1, 2, \dots, k$), such that

$$\begin{cases} -n < r_j(l - \theta - m - n), -n < \mu q, & j = 1, 2, \dots, k, \text{ for } 1 \leq q < \infty \\ -n < \mu, -n < l - \theta - n, & \text{for } q = \infty \end{cases}$$

Then, if $\mu q + r_j(l - \theta - m - n) + n > 0$, $j = 1, 2, \dots, k$, there is a constant $c = c(\Omega)$ such that

$$\inf_{Q \in \mathbb{K}} \|f - Q\|_{W^{m,q}(\Omega, \mu)} \leq c(\Omega) \sum_{j=1}^k \|P_j(\partial)f\|_{L_{p_j}(\Omega, \theta)}.$$

Corollary 5.1. Let $L : W^{k+1,p}(\Omega, \theta) \rightarrow \mathbb{R}$, $\theta \in \mathbb{R}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$ be a linear functional satisfying the following conditions:

- (i) L is continuous in the space $W^{k+1,p}(\Omega, \theta)$: there exists a constant $c > 0$ such that

$$|L(f)| \leq c\|f\|_{W^{k+1,p}(\Omega, \theta)}, \quad \forall f \in W^{k+1,p}(\Omega, \theta);$$

- (ii) $L(p) = 0, \forall p \in P_k(\Omega)$, where $P_k(\Omega)$ is the class of complete polynomials of degree k defined in Ω . Then

$$|L(f)| \leq c |f|_{W^{k+1,p}(\Omega,\theta)}, \quad \forall f \in W^{k+1,p}(\Omega,\theta);$$

- (iii) if $L(q) = 0, \forall q \in Q_k(\Omega)$, then there is a constant $c = c(\Omega)$ such that

$$|L(f)| \leq c [f]_{W^{k+1,p}(\Omega,\theta)}, \quad \forall f \in W^{k+1,p}(\Omega,\theta),$$

where $Q_k(\Omega)$ is the class of complete polynomials of degree k in each variable, $[.]$ is a semi-norm in the space $W^{k+1,p}(\Omega,\theta)$ without the mixed derivatives of orders $k+1$.

Remark 5.2. Corollary 5.1 may be used to estimate the error of the finite element method applied to Boundary Value Problem with the differential operator of order $2m$

$$(\ell u)(x) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))$$

and whose ellipticity is violated

$$\sum_{|\alpha|, |\beta| = m} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq c (d_M(x))^\sigma |\xi|^{2m}, \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

REFERENCES

- [¹] BRAMBLE J. H., S. R. HILBERT. SIAM J. Numer. Anal., **7**, 1970, 112–124. [²] Id. Numer. Math., **16**, 1971, 362–369. [³] DUPONT J., R. SCOTT. Math. Comp., **34**, 1980, No 150, 441–463. [⁴] HATRI M. EL. RAIRO M²AN, **21**, 1987, No 1, 27–62. [⁵] KUFNER A. Weighted Sobolev spaces. New York, John Wiley and Sons, 1985. [⁶] MAZ'YA V. Sobolev spaces. Berlin, Springer-Verlag, 1985. [⁷] BESSOV O. V., V. P. IL'IN, S. M. NICOL'SKI. Integral representation of function and embedding theorems. Moscow, Nauka, 1975 (Russian). [⁸] FRIEDMAN A. Partial Differential Equations. New York, Holt, Rinehart and Winston, 1969.

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