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RESONANCE HOPPING IN COMETS

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ABSTRACT

The motions of ten short-period comets strongly perturbed by Jupiter are discussed in terms of the "hop" phenomenon, in which objects move dramatically from one resonance situation to another. It is shown that the "fuzzy boundary" concept is useful in trying to develop a theoretical understanding of the situation, and a sketch of a proof is presented. It is suggested that objects associated with the 3:2 and 2:3 resonances are the ones that can become involved in weak or quasi-weak capture as temporary satellites of Jupiter. Otherwise, resonance effects of quite high order, as well as double-resonance effects, play significant dynamical roles that also warrant a mathematical explanation. © *1997 American Astronomical Society.* [S0004-6256(97)02504-1]

1. INTRODUCTION

The appreciation that close encounters with Jupiter can dramatically change the character of the motion of a comet dates back more than two centuries, Lexell (1778) noting that when the comet that now bears his name passed by the earth in 1770 it had a revolution period close to half that of Jupiter, whereas the perihelion distance was evidently much larger and the revolution period much longer than that of Jupiter before 1767 and after 1779. A more thorough study by Leverrier (1857) confirmed these effects of the 2:1 mean-motion resonance and showed that the comet's path was completely unknowable after 1779—a classic case of what we understand nowadays as chaotic motion.

Although other cases of short-period comets that were substantially perturbed by Jupiter became apparent during the nineteenth century, knowledge of an increased number of dramatic cases had to await the twentieth century, when the use of photography permitted the discovery of fainter—and thus more distant—comets, including some traveling in low-eccentricity orbits more nearly similar to that of Jupiter itself. On one of only two occasions when a diagram has been published on an IAU Circular, Rasmusen (1943) mentioned the approach of 39P/Oterma to Jupiter in 1939 and that this comet was near the 3:2 resonance, a subsequent detailed study by the discoverer herself (Oterma 1958) showing that before the 1937 approach and again after the inevitable repetition in 1961 the perihelion distance was significantly larger and the comet then very roughly in 2:3 resonance with Jupiter instead.

In a popular account of his own work on 39P/Oterma, the second author (Marsden 1962) remarked that while closest to Jupiter the comet was for a time traveling in an elliptical orbit about the planet, and that all of the known asteroids in 3:2 resonance were configured so that they could not be near aphelion when in conjunction with Jupiter—the long-term stabilizing effect of these asteroidal librations later being discussed in detail by Schubart (1968). Improved computing capabilities during the 1960s made it feasible to study systematically the motions of all the available short-period comets over time intervals of at least several centuries (Marsden 1967; Kazimirchak-Polonskaya 1967a), and such computations led to the realization that comets could also exhibit librations, although these are only of a temporary nature, the orbital eccentricities are generally large, and there can be moderately close approaches to Jupiter. The first known example of a comet in temporary 3:2 libration with Jupiter was 50P/Arend, the orbital eccentricity being near 0.5, and the libration apparently persists for $4\frac{1}{2}$ of its 200-year cycles despite a passage only 0.49 AU from Jupiter well inside the interval during which the libration occurs (Marsden 1970).

A study by Rickman (1979) of the six short-period comets discovered in 1975 produced two more cases of low-eccentricity orbits near the 3:2 resonance. One of them, 82P/Gehrels 3, persisted as a temporary satellite of Jupiter for a full revolution period, while 74P/Smirnova-Chernykh exhibited a more complex situation that kept this comet within 1.5 AU of Jupiter for 12 years. Carusi *et al.* (1985) noted a number of other cases of both temporary librations and temporary satellites. In recent years, long-term numerical integrations of cometary orbits have become progressively more popular, and several investigators, including the second author of this paper, independently noticed that the 3:2 case of 111P/Helin-Roman-Crockett persisted as a temporary jovian satellite for several revolutions, Tancredi *et al.* (1990) remarking on an apparent tendency for the transition into and out of the vicinity of the 3:2 resonance to occur when Jupiter is itself near perihelion. However, in a statistical examination of the prevalence of the even longer satellite status

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(Carusi *et al.* 1994) of comet D/1993 F2 (Shoemaker-Levy 9), Kary & Dones (1996) found only a small preference for the transition to occur near Jupiter's perihelion, and they down-played any actual influence of the 3:2 resonance.

The rapid transition of comets into and out of the 3:2 resonance, with at other times perhaps an approximate association with the 2:3 resonance, is akin to the process discovered by the first author (Belbruno 1990) in discussing the motion of a particle in the earth-moon system. Just as the character of the motion of comets 39P/Oterma, 82P/Gehrels 3, and 111P/Helin-Roman-Crockett completely changes over a timespan on the order of Jupiter's revolution period, so does the resonance shifting in the earth-moon system occur on the order of a month. This is extremely fast in comparison with the general dynamical process of Arnold diffusion (Arnold 1989; Nekoroshev 1971), which describes a way for the solution to move between resonances in Hamiltonian systems. In a general case of Arnold diffusion it can take thousands, even millions, of years to move from the influence of one resonance to another. Because of the great speed of transition, the process of quickly jumping resonances is referred to as "the hop."

In the earth-moon-particle case, techniques for handling nonlinear dynamical systems allow one numerically to estimate a region around the moon in which the motion of the particle is chaotic (Belbruno 1990, 1994, 1995). A particle that initially orbits the earth in a particular resonance may move into this region. It then ceases to have a preferred central body, and there is the possibility of a temporary capture by the moon. The orbit will be elliptical with respect to the moon, yet unstable. When it leaves the sensitive region it can reorbit the earth with a different resonance. One of the main results of this paper is to give a dynamical explanation of why the hop occurs, and a sketch of a proof is made. This is presented in Sec. 4. This proof sketch is not completely mathematically rigorous, and the deeper mathematical reasons behind this process are suggested by Mather's (1982a, 1982b) basic work on the nature of unstable dynamics for area-preserving transformations for two degrees of freedom. As opposed to the Kolmogorov-Arnold-Moser (KAM) theorem (Arnold 1989; Moser 1973; Siegel & Moser 1971), which proves the existence of invariant tori supporting quasi-periodic motion, Mather's work demonstrates the absence of such invariant tori under certain circumstances. However, the real problems of earth-moon-particle and sun-Jupiter-comet involve more degrees of freedom, and Mather's work does not address the rapidity of the resonance shifting. All of the work described in this paper models the motions of the planets with DE403. In particular, the planetary orbits are modeled to first order as ellipses. The hop also occurs in more idealized versions of the three-body problem. It is found that this process occurs in both the three-dimensional and planar elliptic restricted three-body problems. Moreover, it occurs as well in the three-dimensional circular restricted problem. Each of these problems has more than two degrees of freedom, a probable requirement for the hop to occur with the given mass ratio of Jupiter to the sun. This is supported by the fact that the hop does not seem to occur in the planar circular restricted problem of two degrees of freedom, a point that is discussed in Sec. 5.

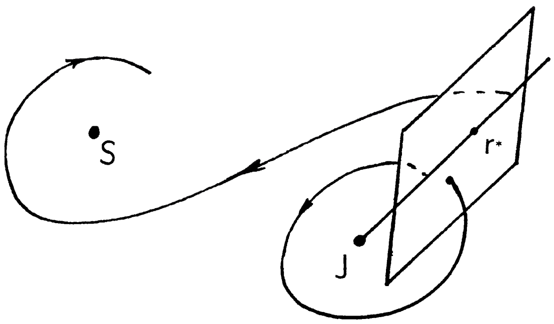

FIG. 1. Stable and unstable motion wrt Jupiter.

Results for a number of comets that perform hop dynamics are presented in Sec. 3. There is a pattern to the different resonance values depending on the special conditions.

2. THE HOP AND COMETS

The hop is a dynamic process that occurs near a region about Jupiter where sensitive motion occurs. This region is called the fuzzy boundary, fb, first discovered in 1986 (Belbruno 1994, 1995). Before more accurately defining this motion and presenting comets that perform it, we briefly mention some basic properties of the fb. The details can be found in the literature.

The fb is a region about Jupiter where a moving particle of negligible mass, say, a comet C, feels the gravitational forces of the sun and Jupiter in a nearly equal fashion. It can be viewed as a higher-dimensional analogue of the collinear Lagrange points L_1 and L_2 of Jupiter. These are points where a particle at rest relative to the Jupiter-sun line remains at rest. A slight perturbation of the velocity or position will cause C to move away from these locations near surfaces in the seven-dimensional phase space of position, velocity and time called invariant hyperbolic manifolds (Arnold 1989). L_1 and L_2 are contained within the fb of Jupiter, which is a five-dimensional set in the phase space. In the sense that L_1 and L_2 can be viewed as two places where the gravitational forces tend to balance for a fixed particle, the fb generalizes this notion for a moving particle. It is a region where the stability of motion of C is in transition.

The stability transition, ST, of C is defined by determining the transition between stable and unstable motion. We define the motion of C about Jupiter to be stable if, when starting with elliptical initial conditions with respect to Jupiter, C does a complete cycle around Jupiter and returns to a reference plane through its initial condition, in phase space, without first having moved around the sun. If it moves around the sun before returning to the reference plane, then the motion is called unstable. It is assumed that the reference plane is normal to the initial velocity direction (Fig. 1). Given a position x with respect to Jupiter and a fixed velocity direction, the ST location is found as a distance on a radial line through x from the center of Jupiter, lying in the reference plane. This distance, r^*, is determined iteratively on the computer by propagating trajectories from the line. At each initial point, the velocity magnitude is adjusted so that the eccentricity of C with respect to Jupiter is a fixed number e, where $0 \leq e < 1$. For distances $r < r^*$ the motion is stable, and for $r > r^*$ it is unstable. In order to keep track of all possible initial conditions, five parameters, and hence five dimensions, are needed: two to keep track of x, and the other three are the initial velocity direction, the epoch, and e. In position space, the fb can roughly be viewed as a solid region extending from the center of Jupiter out to a maximal distance that is, on the average, approximately 0.40 AU or 60 million km. Within this region, C is in the fb at a given position and velocity direction if it has the correct velocity magnitude. If C moves too fast, its motion will be unstable, and if it moves too slowly, the motion will be stable. Beyond the outer boundary of the fb, C will always have unstable motion. In this sense, C notices the gravitational effect of Jupiter in a negligible way. It is remarked that the fb was defined as an ST region after only one cycle of C about Jupiter. Two or more cycles could have been used in the definition, but one cycle was adopted to yield a minimal stability condition. As is discussed in Sec. 4, the fb may be related to a so-called hyperbolic tangle (Arnold 1989) about Jupiter consisting of a very complicated network of intersections of hyperbolic manifolds associated with L_1 and L_2. Manifolds from this network may extend far from Jupiter in many directions, and in this sense the fb is not associated with the classical Hill's regions. Also discussed in Sec. 4 is that some of these manifolds may be associated with special periodic orbits. These are so-called Birkhoff periodic orbits. In fact, the resonant cometary orbits themselves may be approximations to these periodic orbits.

To be in the fb a comet needs two conditions. It must have negative Kepler energy with respect to Jupiter, and it must be in an ST. A comet on or near the fb is captured by Jupiter, although this capture is in general temporary. This temporary capture while on or near the fb is called weak capture.

Eventually, the comet will in general escape Jupiter after being in weak capture, and the Kepler energy with respect to Jupiter will become positive, i.e., the jovicentric orbit of C becomes hyperbolic. The same occurs whether time is considered to move forward or backward. It is noted that a comet can have negative Kepler energy with respect to Jupiter, i.e., it can be temporarily captured by Jupiter, but this need not be weak capture. For weak capture to occur, the additional requirement that the comet should be in an ST is necessary. The verification of being in an ST is carried out in Sec. 3.

The Kepler energy with respect to Jupiter is defined by

$$E_J = \frac{1}{2}|\mathbf{v}|^2 - \frac{Gm_J}{|\mathbf{x}|},$$

where G is the gravitational constant, m_J the mass of Jupiter, and $\mathbf{x},\mathbf{v}$ are the three-dimensional position and velocity vectors of C with respect to Jupiter. Typically, during weak capture, E_J varies in a sensitive way over values that are negative but very near zero. Also, it is typical that E_S, the Kepler energy with respect to the sun, has large nonlinear fluctuations while the comet is near an ST and is nearly constant otherwise.

The hop can be defined as a process when the motion of C enters and exits weak capture in resonance states with respect to Jupiter. More exactly, let $m:n$ denote a resonance in which C performs m cycles about the sun in the time taken by Jupiter to make n cycles, m and n being small positive integers, with $m \neq n$. Thus, for a hop to occur, C is initially hyperbolic with respect to Jupiter and in an $m:n$ resonance. It becomes weakly captured by Jupiter and then hyperbolically escapes into another resonance, $i:j$, which may or may not be equal to $m:n$. As is discussed in Sec. 5, the speed of this resonance shifting is very fast relative to known mechanisms for such a shift, which therefore seems to need another explanation.

The hop is defined when the energy is negative during the ST (i.e., the comet is temporarily a satellite of Jupiter). This requirement of negative energy is used to ensure that the motion is near the fb. This condition can be slightly modified when the energy is slightly positive during an ST. In such a case quasi-weak capture is said to occur. If, during quasi-weak capture, there is a resonance shift, this is called a quasi-hop. Several comets perform the quasi-hop.

For all comets considered here, numerical work shows that if fast resonance shifting occurs, the comet was near an ST. Thus, weak or quasi-weak capture seems to be a necessary condition.

Several comets are shown to be performing the hop and quasi-hop. Moreover, the resonances separate into two categories. That is, hops and quasi-hops are associated with different types of resonances based on the sign of E_J. This is described in Sec. 3. In Sec. 4, a proof sketch is given that explains why the hop occurs, and this indicates that it is a robust process.

It is noted that because we are dealing with the real solar system, the existence of high-order nonlinear perturbations in the motions of the planets makes it unlikely that there would be precise resonances. An exact resonance occurs when C starts at a point $\mathbf{x} \in \mathscr{R}^6$ in the six-dimensional phase space of position and velocity, and returns to this same point after a rational multiple of Jupiter's period. When we speak of resonance, we mean near resonance, where C returns to a small neighborhood of $\mathbf{x}$.

In the final section, Sec. 5, a possible theoretical explanation of the hop is discussed, and its occurrence in versions of the more simplified restricted three-body problem is studied.

3. COMET EXAMPLES

The principles described in Sec. 2 are illustrated with ten comets, the initial conditions for the orbits of which are from observational data. Since these comets are well documented in the literature they are described here only briefly. One of the main results presented in this paper is that, when a comet has a resonance shift during an interaction with Jupiter, this occurs only during an ST, i.e., the criteria for a hop or quasi-hop are satisfied. Many different resonance values are observed, and another important result we present here is that they occur in a pattern. This pattern is described by noting

TABLE 1. Summary.

Comet	E_J at Periapsis ($\mathrm{km}^2/\mathrm{s}^2$)	Resonance Shift	Type
111P/Helin-Roman-Crockett	− 2.24	3:2 → 3:2	H
52P/Harrington-Abell	− 0.18	5:3 → 8:5	H
39P/Oterma	− 1.34	2:3 → 3:2	H
	− 0.08	3:2 → 2:3	H
82P/Gehrels 3	− 2.23	2:3 → 3:2	H
14P/Wolf	+18.74	7:4 → 3:2	QH
	+ 8.20	3:2 → 4:3	
	+10.40	4:3 → 3:2	
	+ 2.20	3:2 → 4:3	
	+ 2.20	4:3 → 3:2	
	+ 5.30	3:2 → 4:3	
	+27.50	4:3 → 3:2	
	+ 3.80	3:2 → 4:3	
D/1770 L1 (Lexell)	+24.40	4:3 → 2:1	QH
74P/Smirnova-Chernykh	+ 1.63	6:13 → 7:5	QH
59P/Kearns-Kwee	+18.58	3:13 → 4:3	QH
7P/Pons-Winnecke	+31.64	2:1 → 2:1	QH
50P/Arend	+18.08	None	No ST

Notes to Table 1.

ST = Stability Transition; E_J = Kepler Energy wrt Jupiter.

TABLE 2. Orbit events (resonant transitions).

Comet	Start	ST Start	ST End	Pre-Period	Post-Period
111P	1943 July 30	1968 Dec. 5	1984 Aug. 3	7.86	8.12
52P	1900 Jan. 1	1974 Jan. 21	1984 Feb. 7	7.22	7.67
39P	1918 Dec. 13	1936 Nov. 23	1938 Dec. 26	18.12	7.89
		1962 May 21	1964 Feb. 12	7.89	18.40
82P	1940 Jan. 10	1969 Nov. 6	1973 Dec. 5	18.24	8.11
14P	1900 Jan. 1	1922 July 7	1922 Dec. 18	6.84	8.30
D/1770 L1	1722 Aug. 19	1767 Jan. 12	1767 June 9	9.20	5.57
74P	1900 Jan. 1	1955 June 1	1956 Apr. 13	25.88	8.50
7P	1800 June 22	1800 Oct. 31	1800 Aug. 27	6.04	5.73
59P	1900 Jan. 1	1961 Aug. 21	1962 Feb. 4	51.61	8.95

Notes to Table 2.

All integrations carried out to 2000 Jan. 1.

Exact Resonances (years) — 1:2, 23.72; 2:1, 5.93; 3:2, 7.91; 2:3, 17.79; 5:4, 9.49; 5:3, 7.12; 4:3, 8.90; 7:4, 6.78; 7:5, 8.47; 8:5, 7.41; 3:13, 51.40; 6:13, 25.70.

the sign of E_J during the resonance transition, i.e., whether the comet is performing a hop, if $E_J<0$, or a quasi-hop, if $E_J>0$.

It is numerically observed that the resonances 2:3, 3:2 mainly occur for hops. These two resonances almost disappear for quasi-hops, where the principal resonances are 2:1 and of higher order. The results are summarized in Table 1 and are now described in more detail.

The differential equations used to integrate the motions of the comets are given by

$$\ddot{\mathbf{x}}=\sum_{i=1}^{10} Gm_i(\mathbf{x}-\mathbf{x}_i)|\mathbf{x}-\mathbf{x}_i|^{-3}, \tag{1}$$

where $\cdot\equiv d/dt, \mathbf{x}=(x_1,x_2,x_3)\in\mathscr{R}^3$ is the position of the comet, $\mathbf{x}=\mathbf{x}(t)$, and m_i and $\mathbf{x}_i\in\mathscr{R}^3$ $(i=1,2,3,\ldots,10)$ are the masses and positions of the planets and sun given by the ephemeris DE403 (Standish *et al.* 1995). For the comets under consideration, the gravitational effect of Jupiter is dominant. An eleventh-order numerical integrator is used throughout. The integrated positions of the comets are checked with observed positions to ensure accuracy.

The initial states of the comets can be found in Marsden & Williams (1996). All the orbits are numerically integrated over a sufficiently long time span in order to determine whether or not resonance transitions occur. As explained in Sec. 5, it is unlikely that an exact resonance will occur. Instead, what is observed numerically is a variation of the period of the comet about the resonance value.

The time of start of a hop is determined when the comet has both negative energy and is in an ST. The comet exits a hop when one of those conditions is not satisfied. For a quasi-hop, the start and end are determined by the time the comet is near an ST and has positive energy. The durations of these transitions vary from 17 years in the case of comet 111P/Helin-Roman-Crockett to only 17 days in the case of comet 52P/Harrington-Abell. These data are summarized in Table 2. The beginning and end times of the transitions are actually slightly contained within the times listed in Table 2. It turns out that the transitions discussed in this paper for all hops and most quasi-hops occur within Jupiter's classical sphere of influence CS, which has a radius of approximately 0.36 AU=54 million km. Because of the close proximity of the CS passage to the hop or quasi-hop start and end, the times the comets enter and leave the CS are used to mark the beginning and end of the transitions. This is done for the sake of convenience and uniformity only, since the CS has nothing to do with the actual dynamics. There are several quasi-hop cases where the comet does not enter Jupiter's CS. For these, only the time of closest approach to Jupiter is given in Table 2. These more distant transitions are discussed in more detail below.

The first nine comets shown in Table 1 perform hops or quasi-hops. All these cases have characteristic visual similarities during the transitions from which it is easy to spot that a hop or quasi-hop may be occurring. They are

(1) The plots of the cometary trajectories in inertial sun-centered coordinates show well-defined ellipses connected by chaotic paths.

(2) The plots of E_S show constant levels connected by fast, nonlinear oscillations.

(3) The plots of E_J show oscillations with large amplitudes and positive values, with relatively small intervals of time when the values become slightly negative (or positive and nearly zero), and when, during these times, the curves tend to flatten out.

A resonance is precisely established numerically by the following procedure: when an ST is observed for a comet at a given epoch, integrate the orbit forwards and backwards from this epoch. Integrate long enough in each direction that another "ST-like interaction" is observed. An ST-like interaction is one in which the above conditions 1, 2, and 3 are observed. In practice, conditions 2 and 3 are sufficient to use to locate these interactions. From the elapsed time in the backward direction to the next interaction, count the number

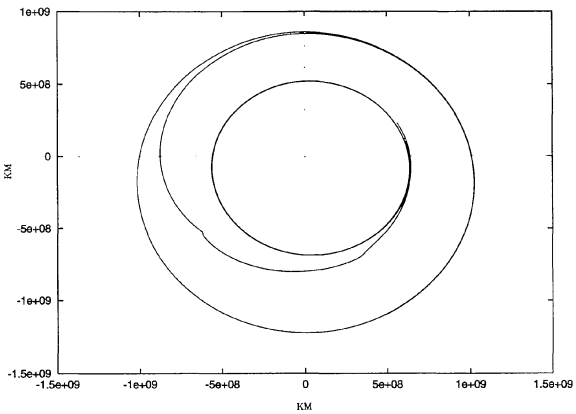

FIG. 2. Inertial plot of 82P/Gehrels 3 showing resonant ellipses connected by a chaotic path representing a hop 2:3→3:2. Projected on the J2000.0 ecliptic plane.

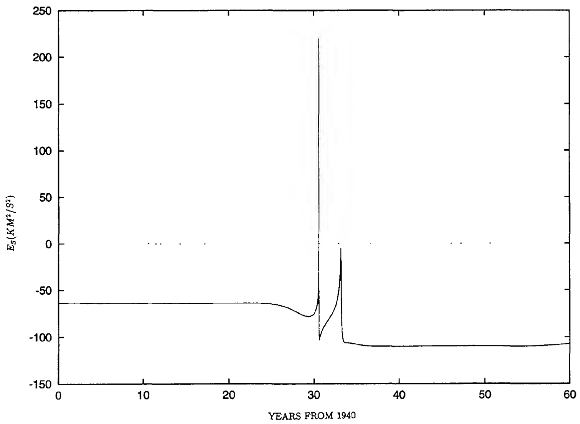

FIG. 3. Kepler energy wrt the sun. The sharp nonlinear oscillations indicate weak or quasi-weak capture.

m of revolutions of Jupiter, and count the number n of revolutions of the comet about the sun. This gives the ratio $m{:}n$. Follow the same procedure in the forward direction to obtain the corresponding numbers i and j. Thus one has the hop, or quasi-hop, $m{:}n \rightarrow i{:}j$. It is noted that, when performing this procedure, the quasi-hop interactions may be weak, and conditions (2) and (3) are not immediately obvious. For example, only one jump in E_S may occur, rather than multiple, sharp oscillations. Moreover, the jump may not be too clearly defined. However, there is always a characteristic cusp or sharp spike prior to being near an ST. The appearance of condition (3) is better behaved, and well-defined minima are generally observed relatively near zero in the case of E_J, even for a weak interaction. This is illustrated below in the case of 14P/Wolf, which displays intriguing dynamics.

In the weaker quasi-hop interactions, the comet may not enter Jupiter's CS, and, as mentioned above, the time of periapsis passage is only indicated. These weaker interactions occur for comets 14P/Wolf and 7P/Pons-Winnecke. To quantify precisely the two different types of interactions, hop and quasi-hop, labeled Q and QH, respectively, we look at the velocity V_p at the periapsis during these interactions. In order for the comet to be in an ST, a sufficiently small incremental velocity change DV in V_p is required for the comet to change its stability as defined in Sec. 2. We make this requirement only at the minima of all the relative periapses a comet may have during its transition. For an H we require that $|DV|$ does not exceed approximately 10% of the value of V_p. For a QH, the pericenter passage is more energetic, and this increases the relative magnitude of $|DV|$ compared with V_p. In this case, we require that $|DV|$ does not exceed approximately 30% of the magnitude of V_p.

The conditions (1), (2), and (3) are illustrated with comet 82P/Gehrels 3 in Figs. 2, 3, and 4, and the behavior described above is clearly seen; Fig. 3 is directly comparable with Fig. 2 of Rickman (1979). The shorter the duration of the transition, the more difficult it is to see these effects. This difficulty is illustrated in the case of comet 52P/Harrington-Abell, where $-0.2 \leq E_J \leq 0$ km^2/s^2. Comet 50P/Arend does not show these effects at all. As noted in the Introduction, this comet actually librates about the 3:2 resonance, but there is no approach within 0.4 AU of Jupiter (Marsden 1970) and, moreover, no ST is observed. That is, it is not performing a QH. This is not a resonance transition candidate since the comet seems to be locked into a 3:2 resonance. This locking seems too strong for an ST to occur. Although not as stable a case, comet 7P/Pons-Winnecke is temporarily librating about the 2:1 resonance (Marsden 1970). It makes somewhat closer approaches to Jupiter and is just barely performing an ST. In this sense, this comet is near the boundary of a QH and a locked resonance case.

Once it is seen that resonance behavior is involved, the actual verification of whether a comet is doing an H or a QH requires an estimate of the value of $|DV|$ at periapsis. This is straightforward to do from the definition of stable and unstable motion given in Sec. 2. For an example, we consider comet 39P/Oterma in its second resonance transition (3:2→2:3 during 1962–1964). It is verified that, within this time, $-2.5 \leq EJ \leq 0$. When the comet was closest to Jupiter (perijove distance 0.095 AU=14 million km) on 1963 April 12, $E_J = -1.56$, and the jovicentric velocity is 3.834 km/s. The magnified plot, Fig. 5, of the jovicentric motion of the comet in rotating coordinates shows that 39P/Oterma is un-

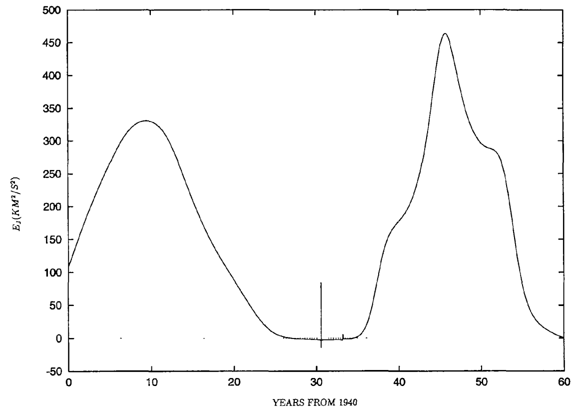

FIG. 4. Kepler energy wrt Jupiter. Negative, near-zero energy indicates weak capture may be occurring.

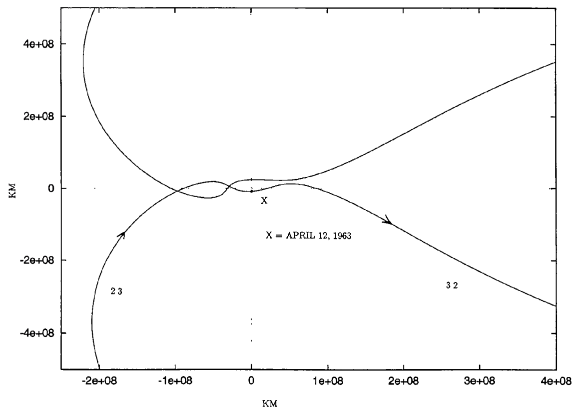

FIG. 5. Jovicentric plot of 39P/Oterma in an ecliptic J2000.0 rotating coordinate system, magnified. The indicated point is unstable. The plot shows both hops at different periods of time.

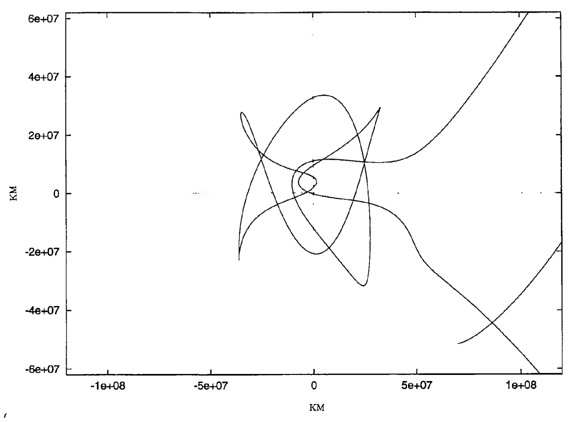

FIG. 6. Jovicentric plot of 111P/Helin-Roman-Crockett in an ecliptic J2000.0 rotating coordinate system. Shown is the complex motion while the comet is weakly captured and performing a hop.

stable with respect to the point corresponding to 1963 April 12, indicated on the graph. This is because, on leaving this point, and therefore the plane through it that is transverse to the velocity vector, the comet will not return to this plane without first moving around the sun. The comet can be made to become stable with respect to Jupiter, for $|DV|=0.16$ km/s. In the case of a QH, a similar calculation done for the comet 59P/Kearns-Kwee, a relatively energetic example considered in this paper, yields a value of $|DV|=2.5$ km/s for $V_p=9.4$ km/s to cause stability. A corresponding analysis shows that comet 50P/Arend is not near an ST. This comet simply flies by Jupiter (at a distance of 0.64 AU in 1969) and barely notices its perturbative effect.

The dynamics a comet performs while it is doing a hop can be very complicated. This is seen for comet 111P/Helin-Roman-Crockett in Fig. 6. Here the motion is clearly chaotic, as is also evident in the work of Tancredi *et al.* (1990). The reason for this is discussed in Sec. 5.

The pattern of resonances revealed in Table 1 is interesting and provides a lot of useful information. The hop can be thought of as a more stable process than the quasi-hop. This is because $E_J<0$, and the comet is more tightly bound than it is if $E_J>0$. The fact that the 2:3 and 3:2 resonances almost exclusively appear in the negative-energy case suggests that these resonances are the most stable. When the energy becomes positive, the comet is more energetic with respect to Jupiter and not bound so tightly. A wider variety of resonances, including some of much higher order, is therefore available. The fact that the first-order 2:1 resonance appears in a quasi-hop suggests that it is relatively unstable, as the case of comet D/1770 $L1$ is indeed known to be.

It may seem remarkable that a tenth-order resonance of 3:13 should appear, as it does in the case of comet 59P/Kearns-Kwee. But what it means is that, in its earlier larger trajectory, the comet passed nowhere near Jupiter during the 155 years before the 1961 encounter, and even then, the miss distance was as much as 3.6 AU. In 1961 the minimum distance from Jupiter was 0.033 AU. Because of the comet's chaotic motion, and the fact that nongravitational forces also play a role, one should therefore be cautious about taking the 3:13 resonance too seriously. Indeed, a similar computation using a preliminary orbit showed the pre-1961 resonance to be closer to 2:9. By chance, that resonance was so precise that Kazimirchak-Polonskaya (1967b) was prompted to identify an entirely fictitious further Jupiter encounter 107 years earlier; see also Marsden & Aksnes (1967). Likewise, the 6:13 resonance in the case of comet 74P/Smirnova-Chernykh keeps this comet away from Jupiter back to the year 1801.

Table 1 suggests that if a comet near ST has $E_J\approx 0$, i.e., is in weak or quasi-weak capture, this serves as an approximate way to separate cases involving the 2:3 and 3:2 resonances from those involving other resonances, at least for most of the examples considered in this paper. The comet 52P/Harrington-Abell is an exception.

Comet 14P/Wolf shows unusual resonance dynamics, in that, following the 7:4 quasi-hop of 1922, it performs an interchange of resonances between 3:2 and 4:3. The pattern is 7:4, 3:2, 4:3, 3:2, 4:3, 3:2, 4:3, 3:2, 4:3. This orbit was integrated to 2200. After 1922, the ST-like interactions with Jupiter are all QHs and outside Jupiter's CS, but the cusp-like behavior in E_S verifies their occurrence as seen in Fig. 7(a). From this figure alone, it is not clear what is happening. To understand this better, the plot for E_J is generated. Near where the resonances are interchanging, there are clear minima as seen in Fig. 7(b). When these two plots are overlaid, the demarcations between the resonance transitions are identified. They occur approximately midway between a jump in the E_S curve. Starting the first 3:2 on 1922 December 18 at the end of the 7:4→3:2 QH, the following consecutive time intervals are observed in years: 36, 22.5, 35, 23, 35, 24, 35, where these have been rounded off to the nearest year or half year. The Jupiter periapsis passages occur near the resonance transitions, and all occur outside Jupiter's CS. These time intervals are very close to two and three times Jupiter's period, and the comet is observed to perform three and four revolutions about the sun, respectively, thus yielding the 3:2, 4:3 interchanges. In Table 2, QHs after 1922 are not indicated because they occur outside of the CS and because of the preceding discussion. This comet may be termed a "double resonance oscillator" and suggests the potential complexity of resonance shifting. It is probable that many

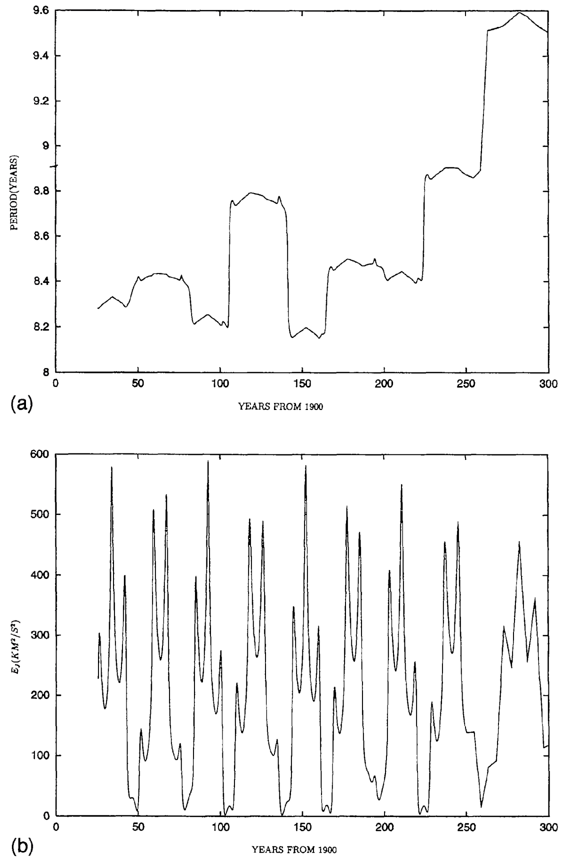

FIG. 7. (a) Period of 14P/Wolf showing cusps typical of a weak quasi-hop. (b) Kepler energy of 14P/Wolf wrt Jupiter showing minima that coincide with the cusps in (a). Taken together, these imply resonance interchange.

comets when analyzed in this way may reveal more complicated resonance interchanges. Because the comets never pass within Jupiter's CS, the interchanges are not noticed.

A concluding comment relates to the fact that comets can hop from ellipses that lie far from the orbit of the earth to ellipses that actually cross the orbit of the earth. When comet D/1770 L1 (Lexell) did a quasi-hop from a 4:3 resonance to 2:1, it became an earth crosser, the perihelion distance being reduced to only 0.67 AU. Prior to the quasi-hop the perihelion distance was 4.39 AU. After the second Jupiter encounter we find the aphelion to be at a distance of 91.50 AU, with the comet reaching that point in 1947. Whether or not this comes close to representing the comet's true motion, the computation does provide a well-defined path, via a quasi-hop, from an earth-crossing situation to a point in or beyond the Kuiper Belt.

4. A PHYSICAL EXPLANATION AND PROOF SKETCH FOR FAST RESONANCE SHIFTING

The phenomenon of a comet's quickly jumping from one resonance to another is not entirely understood. However, a physical explanation can be given. This is done in this section as a sketch of a proof. All the steps and assumptions are supported by the numerical simulation of all the comets in this paper, as well as in Belbruno (1990, 1995). This proof suggests that the hop is a very robust process, and it seems to imply that for a large class of comets, the hop is the preferred dynamics. A discussion of the techniques required to make this proof mathematically rigorous is given. A completely rigorous proof is beyond the scope of this paper and may involve the ideas of Mather (1982a, 1982b).

The main result for which we should like to sketch a proof is

Let C be in weak or quasi-weak capture at Jupiter for $t = t_0$. Then C escapes Jupiter for $t \geqslant t_1 > t_0$ onto an $m{:}n$ resonance, $m \neq n, m,n = 1,2,3,\ldots$

It is assumed that for $t \in [t_0, t_1]$, C is not near to initial conditions leading to collision with Jupiter. By escape, we mean that C moves beyond the fb of Jupiter in physical space.

4.1 *Sketch of Proof*

For the sake of argument, the model describing the motion of C is simplified. Instead of using DE403 as defined in Sec. 3 by Eq. (1) for all nine planets and the sun, only Jupiter and the sun are modeled. This is a valid simplification, since the perturbations by the other planets on C are negligible for the time spans considered in this paper, and it is verified that the hop persists. Thus, Eq. (1) is reduced to the three-dimensional three-body problem among the sun, Jupiter, and C, where the motion of the sun and Jupiter is prescribed by the ephemeris. In what follows, we shall make further simplifications to this model, however, for now it is sufficient to consider this case. We shall refer to this as the A^3 problem.

One condition that C is in weak or quasi-weak capture implies that $E_J(\mathrm{C}) \approx 0$. Thus,

$$E_S(\mathrm{C}) \approx E_S(\mathrm{J}). \qquad (2)$$

Since C is near an ST, this implies that it will escape Jupiter at time $t_1 = t_0 + \delta$, where δ is small and positive, and it remains in weak or quasi-weak capture for $t \in [t_0, t_1]$.

As seen from Table 2, δ ranges from 6 months to 14 years for the comets considered in this paper. Here, escape means that C moves beyond the fb of Jupiter. For the case of the hop in the earth-moon system, about the earth due to the moon (Belbruno 1990), δ is on the order of days.

Once C is beyond the fb, it is essentially not perturbed by Jupiter to a significant degree, since the fb is an approximate measure of the extent of the gravitational influence of Jupiter. As mentioned in Sec. 2, in physical space, the fb extends out to approximately 0.4 AU from the center of Jupiter and is approximately spheroidal. Thus, when beyond this distance, C moves in a near-Keplerian fashion about the sun.

C moves in an elliptical orbit about the sun for $t \geqslant t_1$. This is for several reasons. First, it cannot get a significantly large increase in $E_S(\mathrm{C})$ by moving very close to Jupiter for $t_0 \leqslant t \leqslant t_1$, since by our assumption, C will not move on or near a collision orbit. The only other way $E_S(\mathrm{C})$ could become significantly large is if t_1 became sufficiently large. However, it was seen that this is not the case. Thus, from Eq. 2, $E_S(\mathrm{C})$ cannot increase significantly beyond $E_S(\mathrm{J})$. This implies that for $t \geqslant t_1$, C moves in an elliptical orbit about the sun.

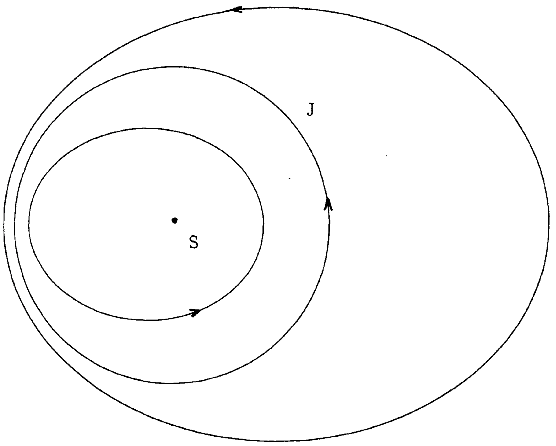

FIG. 8. Two types of comet trajectories after escaping weak capture. These are termed "inner" and "outer" orbits.

Let Per(C) be the period of the orbit of C. We shall now show that

$$\mathrm{Per(C)}=(n/m)Per(\mathrm{J}). \tag{3}$$

Let $\Psi(t)$ denote the elliptic orbit of C for $t \geqslant t_1$. Ψ has a key property that $\Psi(t_1)$ is an apoapsis or periapsis of the orbit. The reason for this is that for $t<t_1$ the comet is in an ST with $E_J(\mathrm{C}) \approx 0$. Thus, at the moment of escape at $t=t_1$, C is moving approximately tangential to Jupiter so that $dr/dt \approx 0$, where r is the distance of C to the sun. Since in general the eccentricity of Ψ is not equal to the eccentricity of Jupiter, then C must fall toward the sun for $t>t_1$. Thus, Ψ has an apoapsis or periapsis for $t=t_1$. Also, for $t \geqslant t_1$, C is beyond the fb of Jupiter, so that the orientation of the elliptical orbit does not change.

Therefore, assuming C is not significantly perturbed by Jupiter for $t_1 \leqslant t \leqslant \mathrm{Per(C)}$, C returns approximately to the same position and velocity with respect to the sun. That is,

$$\Psi[t_1+nPer(\mathrm{C})] \approx \Psi(t_1), \tag{4}$$

where, $n=1,2,3,\ldots$, and where Ψ is represented as a six-vector of position and velocity coordinates with respect to the sun. Since, at $t=t_1$, C is at the boundary of the fb, then $r(t_1) \approx r_J(t_1)$, where r_J is the distance of Jupiter from the sun. Thus, by Eq. (4),

$$r[t_1+n\ \mathrm{Per(C)}] \approx r_J(t_1),$$

for $n=0,1,2,3,\ldots$ Hence, C keeps periodically returning close to the location of a point on Jupiter's orbit, provided that C is not significantly perturbed by Jupiter. However, Jupiter need not be at this point when C returns.

There is a basic property of the elliptic orbit of C that forces C to be in resonance:

Two possible types of orbits for C are noted. If $r(t_1)$ is a periapsis, then the entire orbit lies outside of Jupiter's orbit, and if $r(t_1)$ is an apoapsis, then the entire orbit lies inside Jupiter's orbit; see Fig. 8. In both cases, we assume that C never moves within the fb distance from Jupiter's orbit, except near apoapsis or periapsis locations where $t=t_i=t_1+i\mathrm{Per(C)}$, $i=1,2,3,\ldots$ This assumption is supported by all the comets studied in this paper.

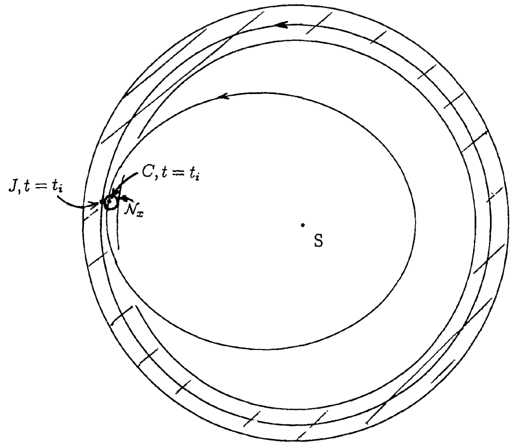

FIG. 9. Mechanism for resonance motion in the "inner" orbit case. The band about Jupiter's orbit indicates the region swept out by the fb of Jupiter. The comet can reencounter Jupiter only near the point x.

This implies that the only time C can get close to Jupiter is in a neighborhood of the point $\mathbf{x} \in \mathscr{R}^3$ denoting the apoapsis or periapsis point in physical space (with respect to the sun) of the elliptical orbit for C. From Eq. (4), C will return to approximately $\mathbf{x}$ at each time t_i. When t is near t_i, Jupiter will be at some point of its orbit, in general not close to $\mathbf{x}$. However, rather than keep track of Jupiter, it is key to keep track of the fb of Jupiter. If the fb gets sufficiently close to C for $t<t_i$, for some i, then C could be pulled within the fb and into a given neighborhood of $\mathbf{x}$ in the six-dimensional phase space, $\mathscr{N}_x$, for $t=t_i$; see Fig. 9. This would then show that C is an $i{:}n$ resonance, proving (4) for $i=m$.

To indicate that this can occur, two things must be shown. Pick a coordinate system with its origin at the sun, its xy plane that of Jupiter's orbit, and its x axis directed to a departure point preceding the ascending node on the ecliptic by the longitude of that node from the J2000.0 equinox. Let θ be

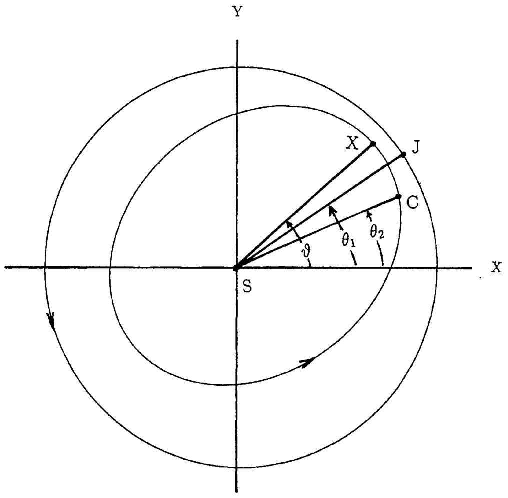

FIG. 10. Geometry and angles.

the polar angle in the plane. This is a suitable plane to use since C approximately moves in it. Then the first thing we must show is that for a given choice of $\theta_1 = \theta_1^*$ for Jupiter, C can achieve any other desired polar angle $\theta_2 = \theta_2^*$ for t sufficiently large. That is, there exists a sufficiently large $t = t^{**}$ such that $\theta_1(t^*) = \theta_1^*$, $\theta_2(t^*) = \theta_2^*$, to arbitrarily high precision. This must always be the case by the Poincaré Recurrence Theorem (Arnold 1989), or PRT, since θ_1, θ_2 can be viewed as angular variables on a two-dimensional torus, and as both C and Jupiter move, these variables wind around the torus, T^2. T^2 is defined by $(\theta_1, \theta_2) \equiv (\theta_1, \theta_2)$ mod 2π. Poincaré's theorem says that for irrational values of χ=Per(C)/Per(J), the resulting curve on the torus is dense. That is, all values of the point (θ_1, θ_2) are (approximately) covered. If χ were rational, then Eq. (3) would be proven. Now, let ϑ be the angle denoting the position of **x**. Choose $\theta_1 < \vartheta$, as shown in Fig. 10. Set $d = \vartheta - \theta_1$. Set $\theta_1^* = \theta_2^*$. Therefore, by the PRT, a value of $t = t^*$ exists such that

$$\theta_1(t^*) = \theta_2(t^*). \tag{5}$$

This means that C and Jupiter are radially lined up as shown in Fig. 9. Moreover, as $d \to 0$, then $t^* \to t_i$ for i sufficiently large. This follows from Eq. (5) since, for $t = t^*$ sufficiently large,

$$\theta_2(t^*) = \theta_1(t^*) = \vartheta, \tag{6}$$

and since ϑ is taken on for $t = t_i$. Therefore, C will be approximately at the boundary of the fb, which occurs for $t = t_i$, implying C is near to the point **x**.

The final thing to show is that C moves into $\mathscr{N}_x$ as $t \to t_i$. This should be possible by small adjustment of the parameters that would preserve the preceding steps. End of Proof.

4.2 Consequences and Discussion

This proof outline is seen to be valid for small variations in the orbits for C, and therefore the hop seems to be a robust, or structurally stable, process, and in the seven-dimensional extended phase space of position, velocity and time there is an open set of such orbits. This is numerically demonstrated for two representative comets, 82P/Gehrels 3 and 59P/Kearns-Kwee. 82P/Gehrels 3 is a typical 2:3, 3:2 hop case, and 59P/Kearns-Kwee is a typical QH case. At a chosen initial condition for each of these comets, prior to a hop, the range of variation of each of the initial coordinates is estimated so that the hop is preserved. Set $\mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z)$, and $\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) \equiv (dx, dy, dz)$. In the case of 82P/Gehrels 3, choose as the initial time 1977 April 7. Vary each coordinate separately, keeping the other five fixed to their initial values. Let $c > 0$ be the variation, and vary by $\pm c$ so that the hop remains preserved. This is done by recording the period at the start of the integration prior to the hop starting on 1973 December 5 in backwards time. Integrate through the hop until sufficiently far from its end. Here, we choose 1947 August 9. Record the period at the end time. It is required that the values of the periods be within 200 days of the initial period. The maximal variation of c is approximated so that this is satisfied. This is done for each coordinate. The results are summarized in Table 3. A large variation in position coordinates is observed, with the average value of approximately 500 000 km. The average value of the velocity variation is 0.04 km/s. These ranges are large and show that the hop in this case is robust. The same is true for 59P/Kearns-Kwee, however, with a relatively smaller variation. The initial time is chosen to be 1963 November 27. The quasi-hop, in backwards time, starts on 1962 February 4. The end time is chosen to be 1938 Jan. 1. The direction of the quasi-hop is 4:3 to 13:3. These results are also summarized in Table 3. Here the average value for the variation in the position is 10 000 km, and the velocity is 0.0006 km/s. The robustness of 82P/Gehrels 3 is seen to be considerably more than that of 59P/Kearns-Kwee. This is probably reflective of the property of the higher relative stability of the 2:3, 3:2 resonances.

TABLE 3. Robustness analysis.

Comet	x	y	z	dx	dy	dz
82P/Gehrels 3						
$+c$	1 000 000	100 000	500 000	0.001	0.08	0.01
$-c$	1 000 000	60 000	500 000	0.1	0.05	0.01
59P/Kearns-Kwee						
$+c$	10 000	10 000	10 000	0.0001	0.0001	0.001
$-c$	10 000	10 000	10 000	0.001	0.0001	0.001

Notes to Table 3.

Variation of each of the initial coordinates, x, y, z, dx, dy, dz by $\pm c$, $c > 0$. Maximal span of c is estimated so that hop or quasi-hop is preserved.

Thus, the hop does not appear to be a pathological process, but one that is not difficult to achieve. The proof sketch suggests that once a comet, or any other particle, is weakly captured at Jupiter, it falls into resonance in a straightforward manner. By the PRT, it is only a matter of time. The longer the time, the higher the order of the resonance. If the orbit of C were more circular and thus closer to the orbit of Jupiter, then the resonance should be of lower order.

The proof presented here is qualitative in nature, and making it completely rigorous is beyond the scope of this paper. For example, the actual existence of δ would have to be proven and estimated. Another important item to be made more rigorous is the assumption that as C moves about the sun for $t > t_1$, it does not pass within the fb distance to Jupiter's orbit, and that during this time, Jupiter does not significantly affect its path. Also, the existence of weak-capture conditions at Jupiter would have to be understood. These are deep mathematical issues, and techniques from KAM theory, and the work of Mather mentioned in the Introduction may be necessary. This proof was carried out in the A^3 problem, and trying to refine it in the more simplified planar circular restricted three-body problem would be easier to do. These types of issues are discussed in the next section.

5. GENERAL THEORETICAL CONSIDERATIONS

The deeper mathematical issues associated with the hop are discussed in this section. An understanding of these topics is probably necessary to make rigorous the proof presented in Sec. 4.

It is desirable to reduce the complexity of this system as much as possible and still preserve the dynamics of interest. The desired system is a three-body problem involving just the comet, the sun and Jupiter. The motion of Jupiter about the sun is initially assumed to be a uniform Kepler ellipse. The elements of Jupiter's orbit with respect to the ecliptic and mean J2000.0 equinox are: $a=778\,328\,370$ km, $e=0.048494$, $i=1°303286$, $\Omega=100°442387$, $\omega=273°718184$. The same integrator is used as in the original system. This model is the three-dimensional elliptic restricted three-body problem. For brevity, we shall refer to it as the E^3 problem.

The comet used to investigate the occurrence of the hop in the E^3 problem, and other models to follow, is 39P/Oterma. The second hop—the 3:2→2:3 hop during 1962–1964—is studied. The state of the comet in the original model on 1959 December 10 is also used as the initial state in the E^3 problem. The latter situation is integrated forward in time to 1966 March 16, and it is found that the hop also occurs. The periods at these times are 7.91 and 17.27 years, respectively, values that are actually significantly closer to the 3:2 and 2:3 resonances than in the general computation—so the hop is substantially improved. As with the original system, the E^3 problem is seven dimensional (or has $3\frac{1}{2}$ degrees of freedom). If the motion of C is constrained to lie in the plane of Jupiter's orbit, the simplified model reduces to the E^2 planar elliptic restricted three-body problem, requiring five dimensions for C (or $2\frac{1}{2}$ degrees of freedom). If one starts from the same initial conditions as before, the hop is lost. However, the hop can be recovered if the initial conditions are instead taken from the original eleven-body integration in the middle of the hop, at perijove on 1963 April 12. The initial conditions for the E^2 problem are obtained by projecting the position and velocity on this date onto the jovian orbit plane as given above. Integration forward and backward in time again yields the hop, the periods before and afterward being 8.46 and 17.72 years, respectively.

The next case to test is that of the C^3 three-dimensional circular restricted problem, where Jupiter's orbit is taken to be circular. Since a coordinate system tied to Jupiter now involves uniform rotation, the C^3 problem is independent of the time and hence has only three degrees of freedom. Moreover, in these coordinates, the model also admits the well-known Jacobi integral (Siegel & Moser 1971). This integral of the motion, J, is independent of time and is equivalent to the energy of C. If **X**,**V** are the three-dimensional vectors of position and velocity, we must have $J(\varphi)=\text{constant}$ along any solution $\varphi=\varphi(t)=[\mathbf{X}(t),\mathbf{V}(t)]$. Thus, all solutions must lie on the surfaces

$$J(\mathbf{X},\mathbf{V})=c,$$

where $c\in\mathscr{R}$ is any constant. These so-called energy surfaces are only five-dimensional since it is possible to solve for one coordinate of **X** or **V** in terms of the other five. Using J, we can therefore reduce the phase space for C to five dimensions. Dependent on the value of c, the analysis of these energy surfaces reveals well-defined Hill's regions where C can move. In this sense, the C^3 problem yields more information on the motion of C than the E^3 problem does. Unfortunately, this information has little bearing on the hop. However, the energy surfaces can be used to study the dynamics, as is described below.

If we use for the C^3 problem the same perijove state as in the E^2 problem, forward and backward integration also yields a hop. The periods before and after the hop are 8.11 and 18.25 years, respectively.

The last model to consider is C^2, the planar problem obtained by restricting the motion of C to Jupiter's orbital plane. This problem is therefore four-dimensional or has two degrees of freedom. It can be reduced to three dimensions on each energy surface. Forward and backward integration from the projected perijove state does not yield the hop. The forward integration gives a period of 18.93 years. Backward integration gives a solution of period 11.48 years, which is a 1:1 resonance. This represents a transition of 1:1→2:3. In this case, the reduction to two degrees of freedom removes the 3:2 resonance. It may be that the availability of only two degrees of freedom is too restrictive for a full hop easily to occur.

For a hop to occur, it is instructive to consider the motion of C in the six-dimensional space of position and velocity. In these coordinates, the fb is a four-dimensional region surrounding Jupiter. The comet starts in a resonance, $m{:}n$, at a point **a** outside the fb. It then passes near the fb and exits into another resonance, $i{:}j$, measured at another point **b**, again outside the fb. To understand this, it is necessary to examine two issues. The first is how C is able to move between two different locations without being obstructed. The second is how a resonance shift can happen. The first issue relates to use of the KAM theorem, and the second to results of Mather, both mentioned in the Introduction. The second issue also relates to the nature of the fb.

For the sake of argument, we consider first the C^3 problem. If the Jupiter-sun mass ratio μ were small enough, the KAM theorem would ensure that the elliptic motion of C about the sun was constrained to lie on three-dimensional tori. As the comet moves about the sun in a quasi-periodic way, it is really winding around a torus. These tori exist when the period of this comet is sufficiently nonresonant. Thus, since almost all initial conditions away from the sun yield nonresonant motions, this means that these so-called KAM tori are almost everywhere, one inside another, expanding out from the sun. Now, if one fixes the motion to an energy level $J=c$, this means that these three-dimensional tori are lying on a five-dimensional energy surface. If μ were small enough, these tori would exist almost right up to Jupiter—and continue beyond Jupiter. But this causes a problem for a path connecting **a** to **b**. If the initial condition at **a** is chosen to be an $m{:}n$ resonance, there is no KAM torus there. However, there will be many tori between **a** and the desired end point, **b**. These tori will get in the way of C as it tries to move. They will not prevent C from reaching **b**, because the difference in dimension between the tori and energy surface is two, allowing C to move around the tori. This motion can be very slow. As C moves slowly by these KAM tori and makes its way to **b**, the obstruction due to the tori can slow down the motion to a very small level. This is the phenomenon known as Arnold diffusion. The velocity is on the order of

$$e^{-1/\mu^{\lambda}},$$

where, $0<\lambda<1$, which in this three-body problem can yield timescales on the order of millions, even billions, of years for C to go from **a** to **b** (Nekoroshev 1971).

In the C^2 problem the situation is worse. For μ sufficiently small C can never reach **b**. This is purely because of the dimensions, the energy surface $J=c$ being three-dimensional and the tori two-dimensional. The tori will therefore completely obstruct the motion of C, and it will not reach **b**.

A way out of the C^2 dilemma and the trouble due to the very slow motion in the C^3 case has to do with the statement that μ is "sufficiently small." If μ is not sufficiently small, the KAM tori do not necessarily exist. As μ increases, more and more tori will, in general, vanish. In their place some complex dynamics, to be described later, will occur. If μ is large enough, enough tori can disappear that none lies between **a** and **b**. Thus, in the C^2 problem, the connection between the two resonances can exist. Moreover, in both the C^2 and C^3 problems, the motion is not slowed down by the presence of nearby tori. The rate of transition can therefore be very fast—substantially less than a million years. Transition rates of only a few years, as for the comets discussed in this paper, could be possible, as is explained more fully below. This is only a qualitative argument and does not represent a proof. It seems that fast shifting in the absence of invariant tori is not an understood dynamic process, and it is not yet mathematically proven to exist.

So far, we have addressed only the ability of C to be connected to two widely separated points in the phase space. We now consider why resonances would occur. This has to do with the question of what kind of mathematical objects are around when the tori disappear. In the following argument, the C^2 problem is considered unless otherwise specified.

The basic work of Mather addresses, in part, the structure of the phase space when the invariant tori vanish. In this case the resulting dynamics are complicated. The remnants of the tori are the so-called Aubrey-Mather sets (Katok 1982). The Aubrey-Mather sets of relevance here are so-called Birkhoff periodic orbits. These are resonant periodic orbits that are unstable. Associated with each periodic orbit there are four surfaces called stable and unstable hyperbolic manifolds. The two stable manifolds lead toward the periodic orbit, and the two unstable ones lead away. The motion near these surfaces has a velocity proportional to $e^{\sigma t}$, where t is time and σ is an eigenvalue of the variational matrix relative to the periodic orbit. The stable manifold of one Birkhoff periodic orbit may be the unstable manifold of another. The Birkhoff orbits are connected in this manner. If any of the invariant manifolds intersects another, this leads to an infinitely complicated network of invariant manifolds called a hyperbolic tangle. A comet C moving in a region of phase space where these Birkhoff periodic orbits exist will move close to, but not exactly on, a resonant periodic orbit. It will then move away from the periodic orbit on an unstable manifold. The transition from one manifold to another causes the motion, in general, to be quite complicated, since it will be within the hyperbolic tangle. The comet will then be guided to another resonant periodic orbit via a stable manifold. Although this process has been described for the C^2 problem, it is likely that it also occurs in the C^3 problem.

In order for the resonance shifting to be fast, the comet can not come too close to the Birkhoff periodic orbits, and hence to the invariant manifolds. This closeness could be prevented by having a region where the tangle is not too dense and where the resonant periodic orbits are fairly sparse and predominantly of first order. In addition, these orbits should lie near those locations where the comet should enter or exit Jupiter's neighborhood. A mathematical proof of fast resonance shifting by this mechanism may require reliance on global arguments.

It order to locate where resonant shifting can occur, it is therefore important to find a region where Birkhoff periodic orbits exist. In general, it is very difficult to find such regions in practice. A clue is to locate a hyperbolic tangle, but again, this is very hard to do. A more general question is to discern the motion of a comet near a hyperbolic tangle. Obviously, this motion must be very sensitive and chaotic in nature. The fb is conjectured to be near a hyperbolic tangle about Jupiter (Belbruno 1995). The techniques developed for finding fb's are relatively easy to apply, and there may therefore be techniques for easily finding hyperbolic tangles, without recourse to the very time-intensive method of constructing Poincaré sections (Arnold 1989). It is remarked that numerical work indicates that in the C^2 problem a hyperbolic tangle about Jupiter can be formed from the intersections of the stable and unstable manifolds associated with unstable retrograde periodic orbits about the locations of the Lagrange points L_1 and L_2 (Llibre *et al.* 1985). It seems probable that one could also form a hyperbolic tangle near these so-called Lyapunov orbits in the C^3 case.

As nature would have it, the comets studied in this paper that are found to hop have the correct energy to lie near Jupiter's fb. They therefore have no choice but to jump from resonance to resonance when they get near this region. The resonant orbit that C has as it moves about the sun is probably near a Birkhoff periodic orbit that passes through Jupiter's fb, thereby allowing C to switch from one resonance to another. The fb can therefore be viewed as a switching location for comets with the correct energy. Although much of this discussion has been involved with the C^2 problem, it should carry over to the more realistic modeling using DE403, where these dynamics should be seen up to first order.

6. CONCLUSIONS

Casting the comets studied here into the framework of fuzzy boundary theory provides a way to understand their behavior for resonance shifting that happens over short time spans on the order of a few years. It also provides a means to relate this area to some current areas of interest in mathematics. These examples provide many examples of an interesting dynamic process called "the hop," which is not theoretically understood. However, the hop can be explained dynamically, and a sketch of a proof is given. The comets studied in this paper reveal a pattern to the different resonance classes that occur. Moreover, a relatively simple criterion can be used to describe this pattern. Many interesting dynamics are discussed, and comet 14P/Wolf is found to exhibit a surprising double resonance. The hop is vastly faster than Arnold diffusion, which can take millions or billions of years for an object to move between resonances, as demonstrated by Wisdom (1980).

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