Astron. Astrophys. 322, 1018–1025 (1997)

ASTRONOMY
AND
ASTROPHYSICS

Scale relativity and quantization of the solar system

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Received 2 April 1996 / Accepted 10 October 1996

Abstract. The scale relativity theory, by giving up the differentiability of space-time coordinates at very large time-scales, describes the solar system in terms of fractal trajectories governed by a Schrödinger-like equation. The predictions of the theory are expressed in terms of probability densities, that we interpret as a tendency for the system to make structures. Planets can no longer orbit at any distance from the Sun, but instead at preferential distances given at lowest order by: $a_n = (GM/w_0^2)n^2$. In this formula, M is the mass of the Sun and $w_0 \approx 145$ km/s is a fundamental constant which is observed from the planetary scales to the extragalactic scales. Our theoretical predictions agree very well with the observed values of the actual planetary orbital parameters, including those of the asteroid belts. In addition, since Mercury ranks $n = 3$ in the above formula, there is good reason to anticipate a small planet or two between the Sun and Mercury. We propose to check the theory by searching for such an object, on the second orbit which has a semi major axis of ≈ 0.18 AU.

Key words: chaos – diffusion – gravitation – planets and satellites: general – solar system: general

1. Introduction

Recently Nottale (1993a,b; 1995a,b) has developed a new model of the solar system structuring, on the basis of the "scale relativistic" approach. The theory is based on an extension of Einstein's principle of relativity to scale laws. Up to now, relativity had only been applied to motion laws. Scale relativity consists of two points:
(i) reinterpreting the Space-Time resolutions as essential variables that define the "state of scale" of reference systems
(ii) setting a "principle of scale relativity", according to which the laws of physics apply to any coordinate system, whatever its state. In other words, the equations of physics must undergo "scale covariance" under *resolution* transformations.

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The scale relativity theory allows to recover standard quantum mechanics and can be applied to cosmology (Nottale 1989,1993a). The theory applies also to strongly chaotic systems, i.e. systems with very large time-scales relative to their inverse Lyapunov exponent. The scale relativity theory operates on the basis that the differentiability of Space-Time is abandoned. Nottale (1993a, 1994) demonstrated that a non-differentiable Space-Time continuum is equivalent to a fractal, i.e., that it shows structures at all scales and it is explicitly resolution dependent. An important property of chaotic trajectories is precisely that they appear as non-differentiable for large time-scales. This suggests that the scale relativity methods should be applied to chaotic systems. In this way, we expect to investigate beyond the horizon of predictability, where classical methods fail.

The importance of chaos in understanding the solar system has been stressed by several authors: for the formation of the solar system (Hills 1970; Brahic 1982), and for the evolution of the solar system (Hénon & Heiles 1964; Petit & Hénon 1986; Wisdom 1987; Sussman & Wisdom 1988, 1991; Conway & Elsner 1988; Laskar 1989, 1990; Sussman & Wisdom 1992). We assume that, at the end of the solar system formation, the motion of all bodies was highly chaotic. The distribution of matter we find by applying our model, satisfies a law which has the same form as the solutions of the Schrödinger equation for the hydrogen atom. The orbits of the planets are then equivalent to the Bohr orbits characterized by two quantum numbers n and l. The new theory is much more relevant than the Titius-Bode law because it has a theoretical background, its predictions agree perfectly well with observations, and it includes the orbits of Pluto, Neptune and the asteroids (Nottale 1993a). Moreover, the predicted structures depend on a unique free parameter w_0, which has recently been shown to be universal (Nottale 1996a,b). Once this parameter is set e.g. from extragalactic data (Tifft 1977), the theory is totally constrained.

The aim of the present study is to point out that, in addition to its successes, the theory is able to make new predictions. It has already been demonstrated (Nottale 1996a,b) that it applies to the recently discovered extra-solar planetary systems. In our own solar system, Mercury has the quantum number n = 3, which means that there is space for two other orbits between

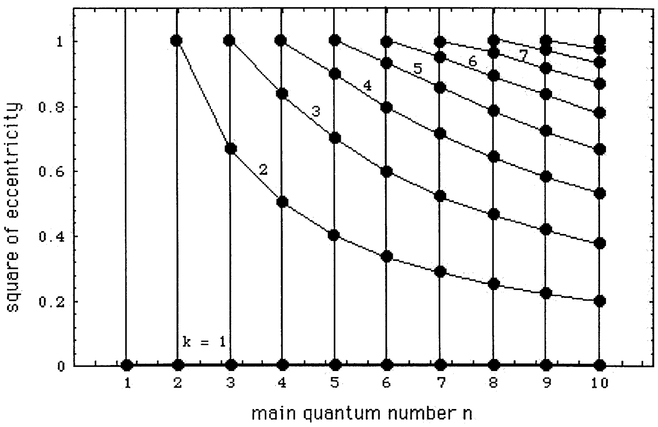

Fig. 1. Possible values of the eccentricity in terms of the principal quantum number n (recall that $a_n \propto n^2$). The lines of constant $k = n - l$ are shown. $k = 1$ corresponds to circular orbits. For small values of n, the first non circular orbit ($k = 2$), is already highly eccentric.

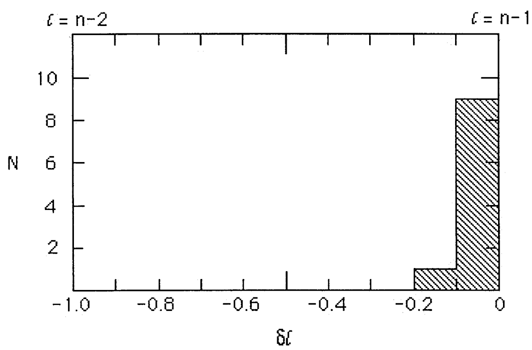

Fig. 2. Histogram of the observed values of $\delta l = l - (n-1)$ for planets in the solar system, showing the angular momentum quantization.

the Sun and Mercury (Fig. 3). We analyze the possibility of existence of such new planets, and we describe an observational strategy for searching the corresponding objects.

2. Theoretical background

Our method consists in first giving up the concept of well defined trajectory on large time scales. Then we introduce families of virtual trajectories which, being continuous but nondifferentiable when seen with large time-resolution, own fractal properties. The "real" trajectory is now only one random realization among the infinite number of trajectories of the family. Since we are at very large time resolutions (say, $\Delta t >> 20\tau$, where $1/\tau$ is a Lyapunov exponent) the information on the initial conditions becomes completely lost and the individual increments on the trajectory become Markovian. We can describe them in terms of a mean dx_i, and a fluctuation $d\xi_i$, i.e., $dX_i = dx_i + d\xi_i$. We expect the $d\xi_i(t)$ to be Gaussian with mean zero, mutually independent and such that, $< d\xi_i d\xi_j >= 2\mathscr{D}\delta_{ij}dt$, where $\mathscr{D}$ can be interpreted as a diffusion coefficient. Such a law corresponds to a fractal dimension $D = 2$ (see, e.g., Mandelbrot 1982), and can be shown as the simplest law satisfying the principle of scale relativity. Its main consequence is the appearance of new second order terms in differential equations, since $< d\xi_i d\xi_j > /dt$ is now finite rather than being an infinitesimal.

The second fundamental consequence of nondifferentiability is the *breaking of time reversibility* at the level of the individual space increments. This means that the reversed process ($dt \longrightarrow -dt$) is a *priori* different from the direct one, while it must be equally valid for the description of the temporal evolution, since the "future" and the "past" are not defined at this level of the description. Up to now, all fundamental equations of physics are locally reversible, i.e., unchanged in the reflection $dt \longrightarrow -dt$. In our approach, we consider *both* direct and reverse processes in parallel. That leads to the introduction of a twin Wiener (backward and forward) process, that we describe in terms of a single complex process (Nottale 1993a). Then, in terms of this new global tool, reversibility is recovered.

Following Nelson (1966), we introduce mean forward and backward derivatives:

$$\frac{d_\pm}{dt} y(t) = \pm \lim_{\Delta t \longrightarrow 0\pm} \langle \frac{y(t+\Delta t) - y(t)}{\Delta t} \rangle \tag{1}$$

From these quantities we introduce a complex derivative operator (Nottale 1993a):

$$\frac{\not{d}}{dt} = \frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \tag{2}$$

When applied to the position vector, it yields a complex velocity:

$$\mathscr{V} = \frac{\not{d} x}{dt} \tag{3}$$

From the properties of our "double-Wiener" process, the complex derivative writes (Nottale 1993a):

$$\frac{\not{d}}{dt} = \frac{\partial}{\partial t} + \mathscr{V}. \nabla - i\mathscr{D}\Delta \tag{4}$$

This operator plays the role of a scale-covariant derivative in the framework of the scale relativity theory.

Consider now a particle moving in a gravitational field and subjected to strong chaos. For large time-scales, we can write its equation of motion in terms of a complex generalization of Newton's equation:

$$m\frac{\not{d}^2 x}{dt^2} + \nabla\Phi = 0, \tag{5}$$

where Φ is the Newtonian potential, such that $\Delta\Phi = -4\pi G\rho$.

Now Eq. (5) may also be derived from a generalization of the principle of stationary action. Indeed, we can implement scale covariance simply by replacing $\frac{d}{dt}$ by $\frac{\not{d}}{dt}$ in the whole of classical mechanics. Therefore, we introduce a complex Lagrange function $\mathscr{L}(x, \mathscr{V}, t)$, then a complex action, $\$ = \int \mathscr{L}(x, \mathscr{V}, t)dt$. If we write now $\delta\$ = 0$, we obtain Euler-Lagrange equations that are precisely Eq. (5) in the Newtonian case, $\mathscr{L} = \frac{1}{2}m\mathscr{V}^2 - \Phi$.

Another well-known result of classical mechanics is also easily generalized. A complex momentum $\mathscr{P} = \frac{\partial\mathscr{L}}{\partial\mathscr{V}}$ can be

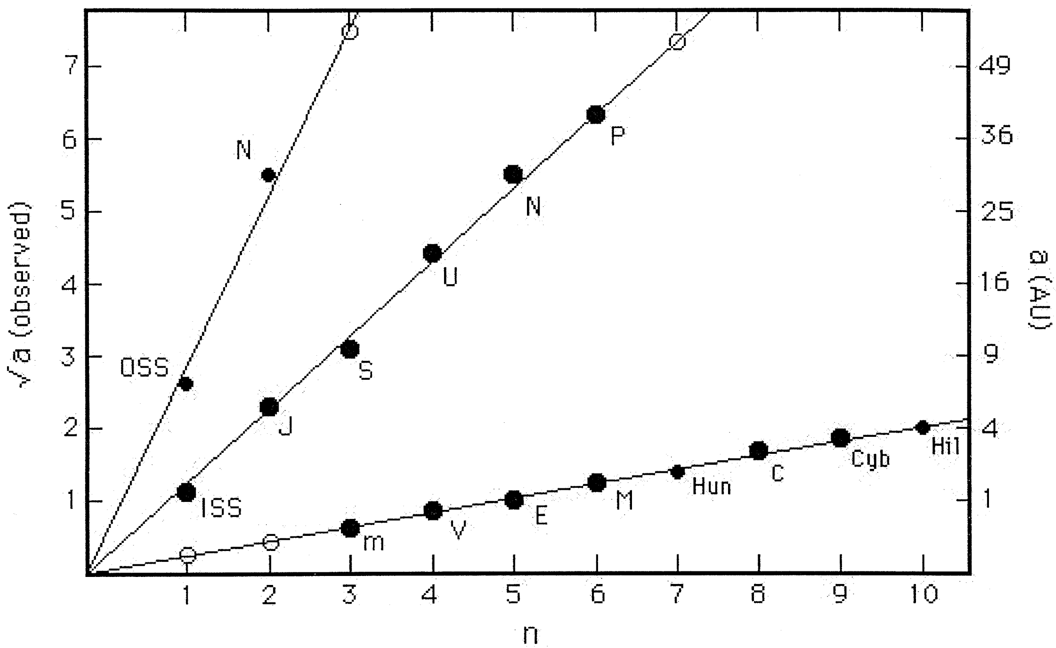

Fig. 3. Comparison of the observed average distances of planets from the Sun with our theoretical values. On the inner system, one has Mercury (m), Venus (V), the Earth (E), Mars (M), and the main mass peaks of the asteroids belt: Hungarias (Hun), Ceres (C), Hygeia (Hyg) and Hildas (Hil). Two additional possible intra-mercurial small planets (open circles) are predicted at $n = 1$ (0.05 AU) and $n = 2$ (0.18 AU). On the outer system, one has Jupiter (J), Saturn (S), Uranus (U), Neptune (N), Pluto (P), and the inner system as a whole on orbital $n = 1$ (ISS). The outer solar system as a whole (OSS), itself stands out as the fundamental orbital of a larger system. Neptune, which ranks $n = 5$ in the outer system, would also rank $n = 2$ of this new system, which may explain its mass excess (see Fig. 7). Farther than Pluto, one finds a new probability peak at ≈ 60 AU, which ranks both $n = 7$ of the outer system and $n = 3$ of the larger system (open circles). We then expect that the Kuyper belt be also quantized.

defined, such that $\mathscr{P} = m\mathscr{V}$ in the Newtonian case. Now when considering the action as a function of the upper limit of integration, we recover the relation $\mathscr{P} = \nabla \$$. Let us now define the "wave function" Ψ. It is nothing but a re-expression in log form of the complex action:

$$\Psi = e^{\frac{i\$}{2m\mathscr{D}}} \Rightarrow \mathscr{V} = -2i\mathscr{D} \nabla (\ln\Psi) \tag{6}$$

Accounting for Eqs. (4) and (6), and being aware that $d\!\!/$ and ∇ do not commute, Eq. (5) can be integrated after a short calculation (Nottale 1993a) as:

$$\mathscr{D}^2 \Delta\Psi + i\mathscr{D} \frac{\partial}{\partial t}\Psi = \frac{\Phi}{2m}\Psi \tag{7}$$

The meaning of Ψ can be finally understood by setting $\rho = \Psi\Psi^*$. In terms of this new variable, the imaginary part of Eq. (7) writes $\frac{\partial\rho}{dt} + div(\rho V) = 0$. One recognizes here the equation of continuity: this leads to a statistical interpretation of Eq. (7) in which ρ is the probability density of finding the particle at some given position. In the case of $\mathscr{D} = \frac{\hbar}{2m}$, and when assuming strict nondifferentiability, Eq. (7) is Schrödinger's equation and we get a new interpretation and generalization of quantum mechanics (Nottale 1992, 1993a, 1996c). Applied to chaotic systems observed on very large time scales, this method provides us with equations that imply the occurrence of preferential positions (given by the peaks of the probability density), that one may interpret as a tendency for the system to make structures.

3. Application to the solar system

Let us apply these methods to the problem of the formation and evolution of the solar planetary system, knowing that we concentrate on the prediction of new possible orbits. In this case we write Eq. (7) with the potential being the Keplerian gravitational potential $\Phi = -\frac{GmM}{r}$, where M is the mass of the Sun, and m the mass of the planet under consideration.

In the case of stationary motion with conservative energy $E = 2i\mathscr{D}m\frac{\partial}{\partial t}$, Eq. (7) becomes:

$$2\mathscr{D}^2\Delta\Psi + \left[\frac{E}{m} + \frac{GM}{r}\right]\Psi = 0 \tag{8}$$

Eq. (8) is similar to the Schrödinger equation for the hydrogen atom, up to the substitution: $\frac{\hbar}{2m} \longrightarrow \mathscr{D}$ and $e^2 \longrightarrow GmM$ so that the natural unit of length, which corresponds to the Bohr radius is:

$$a_0 = \frac{4\mathscr{D}^2}{GM} \tag{9}$$

Let us be more specific about the value of our unique free parameter $\mathscr{D}$ (i.e., of the fundamental length-scale a_0). In microphysics, it must be inversely proportional to the mass of the particle (Nottale 1993a, 1996c). In order to derive its form in the macroscopic gravitational case considered here, recall that the scale-covariant derivative has been constructed from two contributions, that of the fractal fluctuation and that of the breaking of local time reversibility described in terms of complex numbers. If we now include only this last contribution, we can define an incomplete covariant derivative, $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathscr{V}\cdot\nabla$ where $\mathscr{V}$ is the complex velocity. The equation of fractal "free" motion now takes the form of Newton's equation of dynamics, $\frac{d\mathscr{V}}{dt} = i\mathscr{D}\Delta\mathscr{V}$. In this equation, the effect of the fractal fluctuation is now expressed in terms of a complex "fractal force":

$$\mathscr{F} = im\mathscr{D}\Delta\mathscr{V}. \tag{10}$$

The principle of equivalence must still hold for our quantized Newton equation. This means that the inertial mass of the test-particle must disappear from Eq. (8), as it does in its classical counterpart (Hamilton-Jacobi equation). Therefore $\mathscr{D}$ must be independent of m. Moreover, in the situation that is considered here, the fluctuations remain of pure gravitational origin, so that the force in Eq. (10) must be proportional to the product mM. Then $\mathscr{D}$ is proportional to M and it can be written in terms of a universal constant w_0 having the dimension of a velocity:

$$\mathscr{D} = \frac{GM}{2w_0} \tag{11}$$

We can now use the well-known quantum mechanics results for the Coulomb potential to obtain the solutions of Eq. (8). We are interested in the solutions with well determined values of E, l^2 and l_z, since their classical counterpart are expected to have kept a well-defined angular momentum perpendicular to the initial disk. These solutions are the spherical wave functions Ψ_{nl} (the "magnetic" quantum number plays no role here). We thus find that the E/m ratios of planets are "quantized" as $\frac{E_n}{m_n} = -\frac{G^2M^2}{8\mathscr{D}^2n^2}$ where $n = 1, 2, 3...$ We also expect angular momenta to scale as $L_z = 2m_n\mathscr{D}l$ with $l = 0, 1, 2..., n-1$. The average distance to the Sun and the eccentricity e are given, in terms of the two quantum numbers numbers n and l, by the following relations:

$$a_{nl} = \left\{\frac{3}{2}n^2 - \frac{1}{2}l(l+1)\right\} a_0 \tag{12}$$

$$e^2 = 1 - \frac{l(l+1)}{n(n-1)}. \tag{13}$$

Let us briefly compare these predictions to the observed structures in the solar system. The difference of physical and chemical composition of the inner and outer solar systems suggests that they can be treated as two different systems, i.e., that we expect two different diffusion coefficients for them. The main results are summarized hereafter (see Nottale 1993a, 1995a for more details).

3.1. Circularity of the orbits

Our first result is that we expect nearly circular orbits for the planets. Indeed, Eq. (13) implies that, after the purely circular state $l = n-1$, the first non circular state $l = n-2$ yields eccentricities larger than 0.58 for $n \leq 6$ (see Fig. 1). Precisely, n remains smaller than 6 in the solar system, except for the asteroid belts (see below). Such a large value of the eccentricity would imply strong chaos and orbit crossing, and cannot correspond to a stable configuration on large time scales. Then only the quasi-circular orbits remain acceptable solutions. Observations confirm this prediction (Fig. 2), since even the largest eccentricities correspond to small values of e^2 (Pluto, $e^2 = 0.065$; Mercury, $e^2 = 0.042$).

Recall also that the ratio $\frac{L_z}{m}$ is quantized rather than L_z. This solves the well-known and difficult problem of the distribution of angular momentum in the solar system. Indeed, the angular momentum is mainly carried on by the large planets (Jupiter 60%, Saturn 25%), rather than decreasing from the Sun outward as required by most formation models. Since the principal "quantum" number n remains small ($n \leq 6$), the distribution of angular momentum is expected to mainly mirror the distribution of mass, as observed.

3.2. Distribution of the distances of planets

For circular orbits ($l = n-1$), the predicted mean distance (Eq. 12) takes the simple form:

$$\sqrt{a} = n\left(1 + \frac{1}{2n}\right)^{\frac{1}{2}}\sqrt{a_0} \approx \left(n + \frac{1}{4}\right)\frac{\sqrt{GM}}{w_0} \tag{14}$$

while the peak of probability density identifies with the lowest order formula:

$$\sqrt{a} = n\frac{\sqrt{GM}}{w_0} \tag{15}$$

The velocities of planets are also expected to be quantized, from Kepler's third law, as

$$v_n = \frac{w_0}{n} \tag{16}$$

The observed semi-major axes of the planets compare very well with these formulae for the inner and the outer systems respectively (Nottale 1993a and Figs. 3 and 4). Jupiter, Saturn, Uranus, Neptune and Pluto rank $n = 2, 3, 4, 5, 6$ in the outer system. The average distance of the inner solar system is in very good agreement with $n = 1$ of the outer system (this may be the result of a process of successive fragmentation, see below). Note also the agreement of Neptune and especially Pluto with the outer relation: recall that they did not fit the original Titius-Bode law, and very rarely other empirical laws (Nieto 1972; Neuhäuser & Feitzinger 1986).

Mercury, Venus, Earth and Mars take respectively ranks $n = 3, 4, 5, 6$ in the inner system. The central peak of the asteroid belt (Ceres "group", 2.64 AU) agrees also remarkably well with $n = 8$ of the inner system, and the main peak (Hygeia "group", 3.16 AU) with $n = 9$. Despite resonances with Jupiter, small secondary peaks agree with $n = 7$ (Hungarias, 1.94 AU) and 10 (Hildas, 3.96 AU).

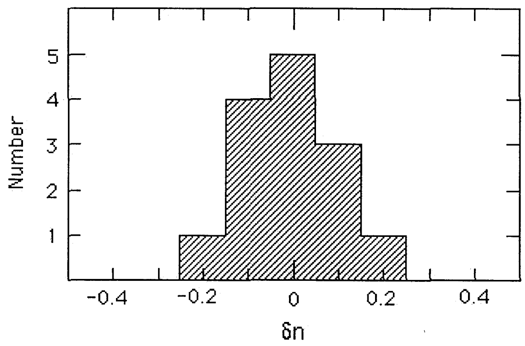

Fig. 4. Observed distribution of $\delta n = w_0\sqrt{\frac{a_n}{GM}} - n$ for planets in the solar system (best fit). The distribution is peaked at zero rather than uniform in the interval [-0.5, 0.5], which shows that our distance law is a genuine quantization.

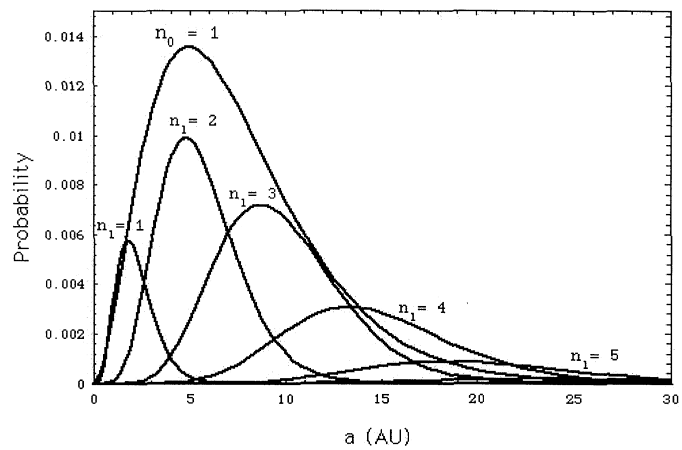

Fig. 5. Schematic representation of the hierarchy process. The orbital $n_0 = 1$ is divided into sub-orbitals $n_1 = 1$ (inner system), $n_1 = 2$ (Jupiter), $n_1 = 3$ (Saturn).... The same is true for the inner system that fragments itself into orbitals $n_2 = 1, 2, 3$ (Mercury), 4 (Venus).... This implies that the inner and outer systems are similar, as observed.

The value of the constant w_0 can now be calculated. We find $w_0 = 144.3 \pm 1.2$ km/s in the inner system from the lowest order formula. As we shall see below, the inner and outer systems are related since the whole inner system corresponds to the orbital $n = 1$ of the outer system (Nottale 1995a). We therefore expect the outer constant to be $w_{\rm out} = w_0/5$, since the mass distribution of the inner system is found to peak at the Earth, which ranks $n = 5$. This prediction is well verified by the data. We find $w_0 = 5w_{\rm out} = 140 \pm 3$ km/s. The agreement is still better accounting for the second order correction in Eq. 14. We find $w_0 = 144.8 \pm 2.6$ km/s. This result agrees remarkably well with the quantization in units of 145 km/s (and its multiples and submultiples) observed by Tifft (1977a) in the velocity differences of galaxy pairs. This effect has been subsequently confirmed by different authors in several other gravitational systems (galaxy pairs: Sulentic 1984; Schneider & Salpeter 1992; Cocke 1992; galaxy groups and clusters: Tifft 1980; Arp 1986; large scale galaxy distribution: Tifft 1978; Tifft & Cocke 1984; Guthrie & Napier 1991, 1996; galactic dynamics: Tifft 1977b; double stars and stars radii: Nottale et al. 1996). The scale relativity theory has been successfully applied to these various gravitational systems (Nottale 1996b,c). It has also been recently demonstrated by Nottale (1996a,b) that the same quantization applies, in terms of the same constant, to the recently discovered extra-solar planetary systems. Therefore the constant w_0 seems to stand out as a universal constant of nature which is already observed for scales ranging from 10^6 km to 30 Mpc i.e. on 15 decades.

3.3. Distribution of the masses

Our theoretical approach provides a model for the mass distribution of matter in the solar system. Indeed, as well the observed distribution of the whole system as that of the inner system (which stands globally as the first orbital of the outer one) are in agreement with the law of probability density derived from Eq. (8). It writes for the various values of n (circular orbits, $l = n - 1$, and $\int P(r)dr = 1$):

$$P(r) = \frac{1}{2n!}\left(\frac{2}{na}\right)^{2n+1} r^{2n} e^{\frac{-2r}{na}}. \qquad (17)$$

This suggests a hierarchic mechanism for the distribution of planetesimals and their final accretion into planets (Nottale 1995a and Fig. 5): the density of the whole initial disk is first given by the orbital (Eq. 15) with $n_0 = 1$, corresponding to a coefficient $\mathscr{D}_0$. Then the disk is fragmented in terms of several new orbitals $n_1 = 1, 2, 3...6$, with a new coefficient $\mathscr{D}_1$. While $n_1 = 2, 3...$ give single planets (Jupiter, Saturn etc...), the first orbital $n_1 = 1$ is once again fragmented into new orbitals $n_2 = 1, 2, 3$ (Mercury), 4 (Venus) etc..., with a new coefficient $\mathscr{D}_2$.

Note that, the peak of the $n_1 = 1$ orbital being the Earth at $n_2 = 5$, our distance law can therefore be expressed in a unique form using the numeration 3, 4, 5 ,6 (inner system), and 10, 15, 20, 25, 30 (outer system, only multiple of 5).

This mechanism allows one to construct a model for the resulting mass distribution. At the first level of this process, the matter density is proportional to the probability distribution $P_1(a_1) \propto r^2 e^{\frac{-r}{a_1}}$. At the second level, the new probability density will act on this density distribution, yielding a new distribution $P_1(a_1).P_1(a_2)$, etc... Finally, the planets will result from the fragmentation of the final mass distribution into intervals lying at distances $r \propto n_2$, of width $\delta r \propto 2n + 1 \propto r^{\frac{1}{2}}$. Then we expect the mass of planets to be distributed as $M(r) \propto r^{\frac{1}{2}} \prod_{i=1}^{p} P_1(a_i) \propto r^k e^{-br}$ with $k = 2p + \frac{1}{2}$. This can also be expressed introducing the distance $r_{\rm max}$ of the peak of this mass distribution:

$$M(r) \propto (re^{-(\frac{r}{r_{\rm max}})})^k \qquad (18)$$

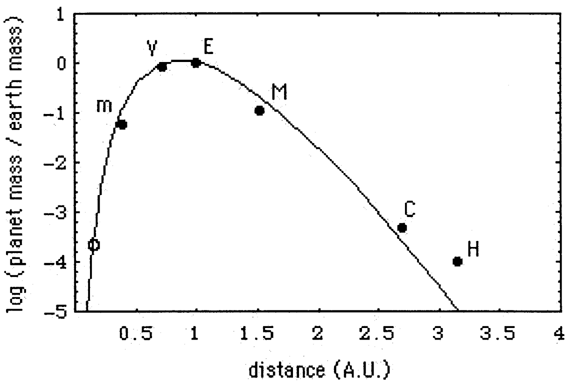

Fig. 6. Comparison of the predicted and observed masses of planets for the inner solar system. C and H stand for the mass peaks in the asteroid belt (Ceres and Hygeia). The possible additional planet (open circle) is expected to have a mass of about $10^{-4} m_{\oplus}$.

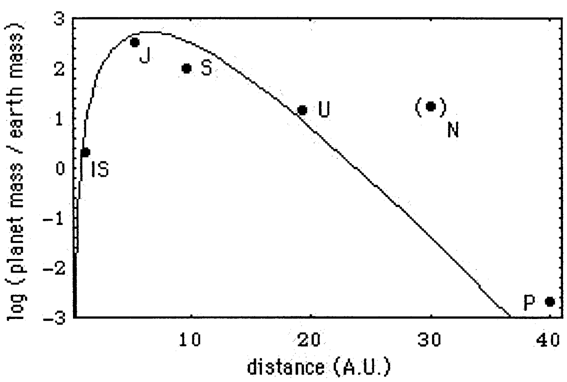

Fig. 7. Comparison of the predicted and observed masses of planets for the outer solar system. IS stand for the inner system as a whole. Only Neptune is discrepant, which could be the signature of another hierarchic structure at larger scale.

The observed distributions of the mass of planets in the inner and outer solar systems are in good agreement with such a model, as can be seen in Figs. 6 and 7. Only the mass of Neptune is much higher than expected. A possible explanation for this discrepancy could be the existence of a larger system in which Neptune ranks n = 2 (see Fig. 3).

Note however that the above model remains very rough, since it is constructed in the framework of the two-body equation. A more correct treatment, that we shall consider in future works, would be to apply the theory to a disk potential, or, even better, to solve directly for the coupled Poisson and Newton complex equations (Nottale 1996c).

Remark finally that our theory will certainly help solving another problem encountered by models of planetary formation. The accretion time of planetesimals, though correct for tellurian planets, becomes too large for giant planets. For only the cores of Jupiter and Saturn, it is already > 1 Gyr, and even far larger for Uranus and Neptune (Lissauer 1993). In our framework, the initial distribution of planetesimals is no longer flat, but already peaked at the final value of the planet positions, a mechanism which should decrease the accretion time.

4. Possible existence of an intramercurial small planet

As we can see in Fig. 3, Mercury has the quantum number n = 3 in the inner system. The quantum numbers n = 1 and 2 are not occupied in our solar system, and we suggest that there could be small planets on the corresponding orbits. The radius of the first orbit (n = 1), if its eccentricity is assumed to be zero, is 0.05 AU. The surface temperature of a body at this distance from the Sun is about 1700 K. The second orbit (n = 2) has a calculated radius of 0.18 AU and the corresponding surface temperature is about 900 K. Concerning the mass of such expected objects, there is an upper limit given by the error bar of the observed advance of the perihelion of Mercury (43.11 arcsec/cy $\pm$ 0.45, Weinberg 1972), compared with Einstein's prediction (43.03 arcsec/cy). If the mass of the objects exceeds a certain value, the error bar of the observed values will no more surround the prediction. We have calculated, based on this principle (see Appendix A), that for the orbit n = 2, the mass of the object cannot exceed $10^{-3} m_{\oplus}$, so that its diameter is smaller than 1700 km assuming a density of 3 g/cm^3, or 1400 km with 5 g/cm^3. The last parameter we can calculate from Kepler's law is the synodical period of those objects. For the second orbit (n = 2), with a = 0.18 AU, we find a period of 31.6 days.

Last century, Le Verrier (1849) had suggested the existence of a planet lying between Mercury and the Sun, to explain the advance of the Mercury perihelion he had measured. From Le Verrier's calculations, such a planet on the orbit we consider here (0.18 AU) should have about the mass of Mercury. However, the existence of such a massive planet is excluded since the advance of Mercury's perihelion is now satisfactorily explained by Einstein's general relativity (see e.g. discussion in Weinberg 1972). Nevertheless we have shown (Appendix A) that there is still place for a body of about 1500 km in diameter, because of the uncertainty of the observed value of the perihelion advance. Le Verrier has also reported 24 observations of small objects crossing in front of the Sun between 1761 and 1876, and especially the observation of Lescarbault in 1845.

During the 20th century, the search of intramercurial bodies has been carried on, and the theoretical limits of their possible orbits have been specified. Campins et al. (1996) have found that dynamical (Mercury perturbations) and thermodynamical (evaporation) constraints restrict the search area in the range of 0.1 to 0.25 AU from the Sun. These arguments rule out the orbit n = 1 at 0.05 AU but put the orbit n = 2 at 0.18 UA just in the middle of the survival zone. Two types of observations have been performed: photographic searches during total solar eclipses (see references in Leake et al. 1987, and Campins 1996), and more recently IR observations have been carried on by Leake et al. The incomplete character of all these observations (a single image only in the photographic eclipse search, and a very small area covered in the the IR search) does not allow a definitive conclusion. Leake et al. conclude that there is no appreciable population of large bodies ($d > 100$ km) orbiting interior to Mercury. The possibility that either a belt of smaller bodies or a single larger object do exist remains open.

Before concluding, let us be more specific about the nature of our prediction. At the level of description in this paper, the theory does not strictly predict that there must be a planet orbiting the Sun at 0.18 AU. Indeed, it gives a description of the orbiting average properties of the beam of all virtual trajectories of a test particle, moving in the gravitational field of the Sun, considered at very large time scale. We rather predict that, if bodies orbit the Sun closer than Mercury, they must, with highest probability, lie at 0.05 and 0.18 AU. But one can also remark that the general shape of the probability distribution (Eq. 18), agrees with the observed mass distribution in the inner solar system, namely, mass increases from $r = 0$ to the probability peak achieved by the Earth at $n = 5$, then decreases and becomes very low at the asteroid belt distances (Fig. 6). This suggests that this probability distribution constrains the shape of the initial distribution of planetesimals. Fitting Eq. (18) to the observed masses in the inner system, we find, provided no other physical effects has pushed out matter from these orbitals, that the masses of the small planets on $n = 1$ and $n = 2$ must be $m_1 = 10^{-10}$ and $m_2 = 10^{-4}$ terrestrial mass, corresponding respectively to 10 km and 500 km in diameter. Such predicted masses agree with the upper limit given by the uncertainty of the measurement of Mercury's perihelion advance. However, the existence of the $n = 1$ planet is definitively ruled out by the evaporation constraint (Campins et al. 1996), while the mass and distance of the $n = 2$ planet allows its survival.

5. Conclusion

We are currently considering two possible ways for the search of a small object located on orbit $n = 2$. The first way is a ground based experiment in the infrared at 2.2μm. This wavelength corresponds to a window between the thermal background and the light diffused by the aerosols of the atmosphere, taking into account the expected temperature of the searched body (900 K). The second way consists of analyzing the data of SOHO. The coronagraphic instrument LASCO has a mode (C3) whose field is about 16 deg, compatible with the size of the searched orbit. The sensibility of the camera allows to detect objects up to the magnitude 9 in the visible, corresponding to a diameter of 30 km.

Concerning our theoretical law, it has already been remarked (e.g. Pecker & Schatzman 1959) that distance laws in n^2 give a far better fit to the solar planetary system than other proposed empirical laws (which are most of the time scale laws, see Graner & Dubrulle 1994). One can also recall the early attempt of Jehle (1938), who proposed to apply the newly constructed quantum mechanics to the solar system, and that of Blanchard et al. (1984), who used Nelson's stochastic mechanics as a diffusion description of the initial disk.

Our prediction of a new planet holds only provided no other physical effects has pushed out matter from its orbital. Nevertheless, the theory has already been confirmed (Nottale 1996a) by the recent discovery at 0.05 AU from their star of three extra-solar planets, including 51 Peg B (Mayor & Queloz 1995). This distance corresponds to the orbit $n = 1$. Two more planets have been found on $n = 2$. Moreover, the agreement, with very high accuracy, between the theory (including second order terms) and the observed position of the three planets orbiting the pulsar PSR B1257+12 (Nottale 1996a,b), gives clearly substance to our theory.

The discovery of a new small planet on the orbit $n = 2$ would be of high interest for the solar system knowledge. It could also yield a new test of general relativity, the expected perihelion advance of this planet being $273''$/cy. Such a discovery would reinforce the validity of the scale relativity theory, and give a fresh boost to the study of its consequences to the whole universe. It would be the revelation that since Galileo and Kepler, since more than four centuries, we have before our very eyes a genuine macroscopic quantum system: our own solar system.

Acknowledgements. This work has been supported by CNRS. We would like to thank Drs. U. Frisch, F. Crifo and B. Chauvineau for helpful discussions, and the referee G. Ord for his valuable suggestions of several improvements to this paper.

Appendix A: upper limit of the mass due to the uncertainty of Mercury perihelion's advance

The orbit of a planet moving in a central field is given by the Binet formula:

$$\frac{d^2}{d\Theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{r^2}{C^2}U'(r) \qquad \text{(A1)}$$

where (r, Θ) are the polar coordinates. U is the potential function and $C = r^2\dot{\Theta}$ the area constant. For a potential of the form:

$$U(r) = \frac{GM}{r}\left(1 + \left(\frac{\varepsilon}{r}\right)^2\right) \qquad \text{(A2)}$$

where ε is a distance small compared to r, the solution is a precessing elliptic orbit given by:

$$\frac{1}{r} = \frac{GM}{C^2}(1 + e\cos\alpha\Theta) \qquad \text{(A3)}$$

with $C^2 = Gma(1 - e^2)$. α, close to unity, is given by:

$$\alpha = 1 - 3e^2\left(\frac{GM}{C^2}\right)^2 \qquad \text{(A4)}$$

The angular shift of the perihelion by period is then:

$$\delta\Theta = \frac{6\pi e^2}{\alpha^2(1 - e^2)^2} \qquad \text{(A5)}$$

Consider, dispersing the intra-mercurial planet on its orbit, that the Sun-planet system is equivalent to an oblate object, the ε term in Eq. A5 writes: $\varepsilon^2 = \frac{\delta A}{2M}$ where δA is the difference of the polar and equatorial inertial moments of the oblate object. Then we have $\delta A = \frac{mD^2}{2}$ where m and D are the mass and orbital radius of the planet, supposed moving on a circular orbit in the same plane as Mercury. Taking $D = 0.19$ AU, one finds for the perihelion motion:

$$\delta\Theta = (0.7''/\text{century}) \left(\frac{\rho}{3\text{g/cm}^3}\right) \left(\frac{R}{1000\text{km}}\right)^3 \quad \text{(A6)}$$

where ρ and R are the planet's density and radius. Considering that this perihelion motion is known with an accuracy of $0.45''$/cy from Clemence's analysis (Clemence 1947; Weinberg 1972), and that a possible intra-mercurial object has no visible effect on it, one finds that the diameter of the planet is necessarily smaller than about 1700 km for a density of $\sim 3\text{g/cm}^3$.

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