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ON THE IMPLICATIONS OF THE *n*TH-ORDER VIRIAL EQUATIONS FOR HETEROGENEOUS AND CONCENTRIC JACOBI, DEDEKIND, AND RIEMANN ELLIPSOIDS

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ABSTRACT

The implications of the *n*th-order virial equations are analyzed for concentric heterogeneous ellipsoids with a density distribution of the form $\rho = \rho_c f(m^2)$, where $m^2 = \sum_{i=1}^{3} x_i^2/a_i^2$, $0 \leq m^2 \leq 1$, and a_i are the semiaxes of the external ellipsoid corresponding to $m^2 = 1$. Solutions analogous to Jacobi ellipsoids (with constant angular velocity $\boldsymbol{\Omega}$, without vorticity), to Dedekind ellipsoids (with nonuniform vorticity $\boldsymbol{Z}$ and zero angular velocity), and to Riemann ellipsoids (with constant angular velocity and nonuniform vorticity) are explored. It is shown that only the second- and fourth-order virial equations give nontrivial results: all the odd-order virial equations are identically satisfied for ellipsoids rotating around a principal axis of symmetry. The even-order virial equations (sixth, eighth, etc.) are shown to be a consequence of the lowest order equations. The entire family of homogeneous and heterogeneous concentric ellipsoids allowed by the virial equations is presented, confronted, and contrasted with the known cases in the literature.

Subject headings: stars: interiors — stars: rotation

1. INTRODUCTION

The fundamental problem of stellar dynamics is the construction of a phase-space distribution function f that satisfies the collisionless Boltzmann equation. In general, this problem has been solved for few realistic cases (Fridman & Polyachenko 1984, pp. 246–322). In fact, the stellar motions in axisymmetric or triaxial galaxies have integrals of motion which are not known explicitly.

Self-consistent models for gravitating collisionless systems can be studied by taking moments of the collisionless Boltzmann equation. The first moment gives the analog of Euler's equation for a self-gravitating isotropic fluid mass. If the fluid is not perfect or if one deals with a stellar system, the Euler's equation is generalized to the Jeans equation in which the term of pressure force is substituted by a stress tensor that describes an anisotropic pressure. Various authors have applied this method to model kinematic observations of elliptical galaxies (Binney, Davies, & Illingworth 1990; Cinzano & van der Marel 1994).

The higher moments of the Boltzmann equation originate the virial equations of various orders: tensor equations relating global properties of stellar system, such as total kinetic energy and mean-square streaming velocity (Binney & Tremaine 1987, pp. 211–219). The fulfillment of these equations does not assure that there exists a positive definite distribution function f describing the physical system. The tensor virial equations are integral relations, consequences of the equations of stellar hydrodynamics, and they yield necessary conditions than can furnish useful insights for the construction of self-consistent ellipsoidal models (Vandervoort & Welty 1981, 1982).

The virial method developed in Chandrasekhar (1987) shows that only in the case of homogeneous self-gravitating masses having a linear velocity field are the virial equations of second order equivalent to the complete set of hydrodynamic equations. In the general case of heterogeneous density and a nonlinear velocity field, this equivalence *does not exist*, and the *n*th-order virial equations represent *necessary* global conditions to be satisfied by any equilibrium configuration.

This is the last paper (Paper IX) in a series (Papers I–VIII) devoted to the generalization of the theory of ellipsoidal figures of equilibrium, endowed both with rotation ($\boldsymbol{\Omega}$) and vorticity ($\boldsymbol{Z}$) obtained for the homogeneous case in the classic works of Maclaurin (see Todhunter 1873), Jacobi (1834), Dedekind (1860), Dirichlet (1860), and Riemann (1860), and treated in a unified manner by Chandrasekhar (1987) in the virial equations formalism in his book " Ellipsoidal Figures of Equilibrium."

This series of papers has followed a variety of tentative approaches, consisting of successive generalizations of known results: looking at more general density distributions, nonlinear velocity fields, selected forms of the pressure tensor, and finally analyzing the constraints imposed by the *n*th-order virial equations. Clearly we have proceeded tentatively step by step. The complete set of virial equations of the *n*th order is now given in the present article. As we will see in the following, only the second- and fourth-order virial equations are independent, while those of higher even order and all odd orders are identically satisfied. These final results limit further the ranges of possible solutions and contain as special cases all the previous results of the series.

The first step forward in the theory of ellipsoidal figures of equilibrium was the introduction in Pacheco, Pucacco, & Ruffini (1986a, hereafter Paper I) of (*a*) a heterogeneous density distribution $\rho = \rho_c(1 - m^2)^n$ with $m^2 = \sum_{i=1}^{3} x_i^2/a_i^2$ and (*b*) an anisotropic pressure. Using only the second-order virial equations, the equilibrium and stability of heterogeneous generalized Riemann ellipsoids was analyzed for the case of a linear velocity field with a corresponding uniform vorticity. The stability against odd modes of second harmonic perturbations of these equilibrium solutions was also analyzed. As a simple analytical

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application, explicit quasi-spheroidal stable configurations were given for selected values of the angular velocity $\boldsymbol{\Omega}$ of the figure of the vorticity $\boldsymbol{Z}$ and of the anisotropic pressure.

In Pacheco, Pucacco, & Ruffini (1986b, hereafter Paper II), additional special solutions of the equations introduced in Paper I were considered: some generalized Maclaurin-Dedekind spheroids with anisotropic pressure and their stability properties were analyzed. It was shown how the presence of anisotropic pressure extends the region of stability toward greater values of the eccentricity, which is similar to the behavior of the homogeneous case considered by Wiegandt (1980).

In Busarello, Filippi, & Ruffini (1988, hereafter Paper III), a second step was made toward the generalization of the solutions by introducing a fully general stratified density distribution of the form $\rho = \rho(m^2)$, where ρ is an arbitrary function of the equidensity surfaces. As in the previous papers, the pressure is still anisotropic and the velocity field linear. The equilibrium and stability properties of anisotropic and heterogeneous generalized Maclaurin spheroids and Jacobi and Dedekind ellipsoids were studied. Particularly noteworthy is the fact that the Dedekind theorem, originally proved for homogeneous and isotropic configurations, is still valid for this more general case.

In Pacheco et al. (1989, hereafter Paper IV), several applications were presented of the previous treatment of the generalized Riemann sequences. Special attention was given to the axial ratios of the equilibrium figures, compatible with given values of the anisotropy. A stability analysis of the equilibrium was performed against odd modes of second harmonic perturbations.

In Busarello, Filippi, & Ruffini (1989, hereafter Paper V), the heterogeneous and anisotropic ellipsoidal Riemann configurations of equilibrium, obtained in the previous paper and characterized by both nonzero angular velocity $\boldsymbol{\Omega}$ of the figure and vorticity $\boldsymbol{Z}$, were used to model a class of elliptical galaxies. Their geometrical and physical properties were discussed in terms of the anisotropy, the uniform figure rotation, and the internal streaming motion.

In Busarello, Filippi, & Ruffini (1990, hereafter Paper VI), the equilibrium, stability, and some physical properties of a special class of oblate spheroidal configurations which rotate perpendicularly to the symmetry axis were analyzed, still within the framework of the second-order virial equations.

In Filippi, Ruffini, & Sepulveda (1990, hereafter Paper VII), we made an additional fundamental generalization by introducing a *nonlinear* velocity field with a cylindrical structure and a density distribution originally adopted in Paper I of the form $\rho = \rho_c(1 - m^2)^n$. The generalized anisotropic Riemann sequences coming from second-order virial equations were studied. Some of the results obtained in that article have been modified by the consideration of the virial equations of nth order, especially the claim regarding the validity of the Dedekind theorem, made on the basis of an unfortunate definition of certain coefficients.

In Filippi, Ruffini, Sepulveda (1991, hereafter Paper VIII), following the theoretical approach of its precedecessor, the nonlinear velocity field was extended to cover the most general directions of the vorticity and angular velocity. The more general form for the density $\rho = \rho(m^2)$ was adopted. Equilibrium sequences were determined, and their stability was analyzed against odd and even modes of second harmonic perturbations.

Finally, in the present and ninth paper of the series we consider a heterogeneous, rotating, self-gravitating fluid mass with anisotropic pressure and internal motions that are nonlinear functions of the coordinates in an inertial frame. We present the complete results for the virial equations of nth order, and we discuss the constraints for the equilibrium of spherical, spheroidal, and ellipsoidal configurations imposed by the higher order virial equations. In this context, the classical results of Hamy (1887) and Dive (1930) are also confirmed and generalized. In particular, (*a*) the Dedekind theorem is proved to be invalid in this more general case: the Dedekind figure with $\boldsymbol{\Omega} = 0$ and $\boldsymbol{Z} \neq 0$ *cannot* be obtained by transposition of the Jacobi figure, endowed with $\boldsymbol{\Omega} \neq 0$ and $\boldsymbol{Z} = 0$; (*b*) the considerations contained in the previous eight papers on the series, concerning spherical or spheroidal configurations, are generalized and recovered as special cases; (*c*) the nth-order virial equations severely constrain all heterogeneous ellipsoidal figures: as shown from Tables 1–3, all the heterogeneous ellipsoidal figures *cannot exist*.

In § 2 the nth-order virial equations are written in a very general form, using useful and compact definitions of the meaningful coefficients. Section 3 considers the most general velocity field possible for an ellipsoidal figure which preserves its form, as seen from a frame of reference in which the ellipsoid is at rest (Paper VIII), producing internal fluid motions of nonuniform vorticity. Here the set of virial equations is specialized to the case of generalized, S-type Riemann ellipsoids. In this case it is demonstrated that all the odd-order virial equations are identically zero. In § 4 the even-order virial equations are analyzed and applied to generalized, S-type Riemann ellipsoids. In § 5 the Dedekind's theorem is analyzed in the most general case. In § 6 we consider a classification of the allowed uniformly rotating equilibrium figures. In § 7 we consider a classification of the allowed equilibrium figures having a differential rotation. In § 8 the allowed generalized Riemann figures are classified. Conclusions are drawn in § 9. Mathematical details and useful definitions are given in the Appendix.

Besides the general solutions of the nth-order virial equations which contain all previously obtained results as special cases, this paper opens a new fundamental question on the nonexistence of triaxial ellipsoidal figures of equilibrium. One important question we will analyze elsewhere is the issue of whether or not such nonexistence follows from imposing *strictly ellipsoidal* configurations of equilibrium, and if configurations with small departures from ellipsoidal symmetry do indeed allow triaxial figures of equilibrium to exist. Recent work (Di Nunzio & Ruffini 1996) directly integrating the Vlasov equations numerically for distribution functions depending both on the energy and angular momentum has shown the existence of a new family of solutions which are ellipsoidal in the outer figure of equilibrium and toroidal in the inner region. Therefore, it appears that the "paradigm" pursued for almost a century of emphasizing solely the role of ellipsoidal figures should be extended further to a new and more general class of nonellipsoidal equilibrium configurations.

This issue and the ones mentioned above will not be resolved here but will be addressed in forthcoming papers.

2. THE nTH-ORDER VIRIAL EQUATIONS

We consider an ideal self-gravitating fluid of density $\rho(x, t)$ and diagonal, anisotropic pressure $P_{li} = \delta_{li} P_i$, rotating with a constant angular velocity $\boldsymbol{\Omega}$ in an inertial frame. Here and in the following, latin indices indicate the Cartesian coordinates.

The hydrodynamic equation governing the motions, referred to a frame of reference rotating with this angular velocity $\mathbf{\Omega}$, is given by (Goldstein 1980)

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \cdot P + \rho \nabla \mathfrak{v} + \frac{1}{2} \rho \nabla |\mathbf{\Omega} \times \mathbf{x}|^2 + 2\rho \mathbf{u} \times \mathbf{\Omega} - \rho \dot{\mathbf{\Omega}} \times \mathbf{x} , \tag{1}$$

where $\frac{1}{2} |\mathbf{\Omega} \times \mathbf{x}|^2$ and $2\mathbf{u} \times \mathbf{\Omega}$ represent the centrifugal potential and the Coriolis acceleration, respectively. While $\dot{\mathbf{\Omega}} = 0$, this equation reduces to that given in Chandrasekhar 1987, p. 24–28, [62]). In components equation (1) is expressed by

$$\rho \frac{du_i}{dt} = -\rho \sum_{lmpq} \epsilon_{ilm} \epsilon_{mpq} \Omega_l \Omega_p x_q + 2\rho \sum_{lm} \epsilon_{ilm} u_l \Omega_m - \rho \sum_{lm} \epsilon_{ilm} \dot{\Omega}_l x_m - \sum_l \partial_l P_{li} + \rho \partial_i \mathfrak{v} . \tag{2}$$

As usual, the gravitational potential $\mathfrak{v}$ satisfies the Poisson equation

$$\nabla^2 \mathfrak{v} = -4\pi G \rho . \tag{3}$$

The boundary of the configuration is defined by $P_{li} = 0$. We now generalize the form of the virial equations, given for the second and fourth orders by Chandrasekhar (1987), to the generic nth-order case. In a rotating frame, in complete analogy with the treatment given by Chandrasekhar (1987), the nth-order virial equations may be generated by multiplying the hydrodynamic equation by $x_i^{a-1} x_j^b x_k^c$ and integrating over the volume V. The index i appears $a-1$ times, the indices j, k appear b and c times, respectively, and $i \neq j \neq k$, $a \geq 1$, $b \geq 0$, $c \geq 0$, and $a + b + c = n$. Thus,

$$\begin{aligned} \int \rho \frac{du_i}{dt} x_i^{a-1} x_j^b x_k^c \, dV = &- \sum_{lmpq} \epsilon_{ilm} \epsilon_{mpq} \Omega_l \Omega_p \int \rho x_i^{a-1} x_j^b x_k^c \, dV + 2 \sum_{lm} \epsilon_{ilm} \Omega_m \int \rho u_l x_i^{a-1} x_j^b x_k^c \, dV \\ &- \sum_{lm} \epsilon_{ilm} \dot{\Omega}_l \int \rho x_i^{a-1} x_j^b x_k^c \, dV - \sum_l \int \partial_l P_{li} x_i^{a-1} x_j^b x_k^c \, dV + \int \rho \, \partial_i \mathfrak{v} x_i^{a-1} x_j^b x_k^c \, dV . \end{aligned} \tag{4}$$

We also generalize the moments of the distribution of density, pressure, velocity, and gravitational potential to the nth order. Then we have the following expressions for the nth moment of inertia tensor,

$$I_{i,j,k}^{a,b,c} = \int \rho x_i^a x_j^b x_k^c \, dV , \tag{5}$$

the nth moment of the kinetic energy tensor,

$$2T_{i,j,k}^{a,b,c} = \int \rho u_i u_j x_i^a x_j^b x_k^c \, dV , \tag{6}$$

the nth moment of the potential energy tensor,

$$W_{i,j,k}^{a,b,c} = \int \rho \, \partial_i \mathfrak{v} x_i^{a-1} x_j^b x_k^c \, dV , \tag{7}$$

and finally the $(n-2)$th moment of the pressure tensor,

$$\Pi_{i,j,k}^{a-2,b,c} = \int P_i x_i^{a-2} x_j^b x_k^c \, dV . \tag{8}$$

Following the previous definitions, the term on the left-hand side of equation (4) may be expressed as

$$\int \rho \frac{du_i}{dt} x_i^{a-1} x_j^b x_k^c \, dV = \frac{d}{dt} \int \rho u_i x_i^{a-1} x_j^b x_k^c \, dV - 2[(a-1) T_{i,j,k}^{a-2,b,c} + b T_{i,j,k}^{a-1,b-1,c} + c T_{i,j,k}^{a-1,b,c-1}] . \tag{9}$$

The term involving the pressure on the right-hand side of the same equation, after an integration by parts, gives

$$\sum_l \int \partial_l P_{li} x_i^{a-1} x_j^b x_k^c \, dV = -(a-1) \int P_i x_i^{a-2} x_j^b x_k^c \, dV = -(a-1) \Pi_{i,j,k}^{a-2,b,c} . \tag{10}$$

Finally, the virial equations of order n may therefore be written as

$$\begin{aligned} \frac{d}{dt} \int \rho u_i x_i^{a-1} x_j^b x_k^c \, dV = &\, 2[(a-1) T_{i,j,k}^{a-2,b,c} + b T_{i,j,k}^{a-1,b-1,c} + c T_{i,j,k}^{a-1,b,c-1}] + W_{i,j,k}^{a,b,c} + \Omega^2 I_{i,j,k}^{a,b,c} - \Omega_i \sum_{p=1}^{3} \Omega_p I_{p,i,j,k}^{a-1,b,c} \\ &+ 2 \sum_{lm} \epsilon_{ilm} \Omega_m \int \rho u_l x_i^{a-1} x_j^b x_k^c \, dV - \sum_{lm} \epsilon_{ilm} \dot{\Omega}_l I_{i,j,k}^{a-1,b,c} + (a-1) \Pi_{i,j,k}^{a-2,b,c} , \end{aligned} \tag{11}$$

with $I_{p,i,j,k}^{a-1,b,c}$ defined in equation (18). For a stationary state, both the left-hand side term and $\dot{\Omega}_l$ are equal to zero.

3. THE nTH-ORDER VIRIAL EQUATIONS FOR HETEROGENEOUS ELLIPSOIDS

We now examine the nth-order virial equations for a class of heterogeneous ellipsoids: the strata of equal density are similar to and concentric with the bounding ellipsoid $[\sum_{i=1}^{3} x_i^2/a_i^2 = m^2 = 1, \rho = \rho_c f(m^2)]$, a_i (semi-axes of the bounding ellipsoid).

3.1. *The Generalized Internal Potential of Heterogeneous Ellipsoids*

The Newtonian potential at an internal point x_i of a heterogeneous ellipsoid (of the kind we have been considering), following theorem 12 in Chandrasekhar (1987), is given by

$$\mathfrak{v} = \pi G a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta} F[m^2(u)] = (\pi G \rho_c) a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta} \tilde{F}[m^2(u)], \tag{12}$$

where $\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u)$,

$$F[m^2(u)] = \int_{m^2(u)}^1 \rho(m^2) dm^2 = \rho_c \tilde{F}[m^2(u)] \text{ and } m^2(u) = \sum_{i=1}^3 \frac{x_i^2}{a_i^2 + u} .$$

The gradient of $\mathfrak{v}$ is

$$\partial_i \mathfrak{v} = 2(\pi G \rho_c) a_1 a_2 a_3 x_i \int_0^\infty \frac{du}{\Delta(a_i^2 + u)} \frac{d\tilde{F}[m^2(u)]}{dm^2(u)} = -2(\pi G \rho_c) a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)} f[m^2(u)] = -2(\pi G \rho_c) x_i C_i(\boldsymbol{x}); \tag{13}$$

here $C_i(\boldsymbol{x})$ are the elements of a diagonal matrix, explicitly

$$C_i(\boldsymbol{x}) = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)} f[m^2(u)] , \tag{14}$$

and C_i reduces to the index symbol defined in Chandrasekhar (1987, pp. 53–55), in the homogeneous case.

3.2. *The Velocity Field Describing the Internal Motions*

The velocity field considered in Chandrasekhar (1987, p. 69) is a linear function of the coordinates. Following Paper VIII, we now propose a more general form of the velocity field within a self-similar ellipsoidal figure, assuming:

1. the preservation of the ellipsoidal form,
2. the continuity equation,
3. the existence of a constant unit vector $\hat{\boldsymbol{n}}$ fixed in the rest frame of the ellipsoid such that the velocity field circulates in planes perpendicular to $\hat{\boldsymbol{n}}$, and having the same direction as $\boldsymbol{\Omega}$. With these conditions, the velocity field in the rotating frame is

$$\boldsymbol{u} = \hat{\boldsymbol{n}} \times (\mathscr{M} \boldsymbol{r}) \tilde{\phi} .$$

The dimensionless function $\tilde{\phi}$ describes the character of the velocity field (linear or not), $m^2 = \boldsymbol{r} \cdot \mathscr{M} \boldsymbol{r}$ is the equation of the ellipsoid, with $\mathscr{M} = \sum_{ij=1}^3 \hat{e}_i \hat{e}_i M_{ij} = \sum_{i=1}^3 (\hat{e}_i \hat{e}_i)/a_i^2$, $\boldsymbol{r} = \sum_{i=1}^3 \hat{e}_i x_i$, and $N_{ij} = \sum_{k=1}^3 \epsilon_{ijk} n_k$, where n_k are the components of $\hat{\boldsymbol{n}}$. With these definitions, the components of the velocity field become

$$u_i = -\sum_{ij=1}^3 (NM)_{ij} x_j \tilde{\phi} \hat{e}_i ,$$

or equivalently

$$u_i = -\tilde{\phi} \sum_{j=1}^3 (AZA^{-1})_{ij} x_j , \tag{15}$$

the diagonal matrix A has the values of semi-axes a_i as elements, and $Z_{ij} = C(N_{ij}/a_i a_j) = \sum_{k=1}^3 \epsilon_{ijk} Z_k$. Here Z_{ij} is defined in such a way that the factor C gives to $\boldsymbol{Z}$ the same dimension as $\boldsymbol{\Omega}$. N_{ij} is the dual of the vector $\hat{\boldsymbol{n}}$ which determines the common direction of $\boldsymbol{\Omega}$ and $\boldsymbol{Z}$.

3.3. *The Virial Equations for Heterogeneous Ellipsoids*

Considering now the case of a steady state regime, and using the expressions (109), (112), (115), (117), and (118) from the appendix for the tensors I, J, K, Π, and W, it follows from equation (11) and the relation $(AZA^{-1})_{ij} = S_{ij}$ that

$$\sum_{pq=1}^3 [(a-1) S_{ip} S_{iq} K_{pq,i,j,k}^{a-2,b,c} + b S_{ip} S_{jq} K_{pq,i,j,k}^{a-1,b-1,c} + c S_{ip} S_{kq} K_{pq,i,j,k}^{a-1,b,c-1}] + W_{i,j,k}^{a,b,c} + \Omega^2 I_{i,j,k}^{a,b,c} - \Omega_i \sum_{p=1}^3 \Omega_p I_{p,i,j,k}^{a-1,b,c}$$
$$- 2 \sqrt{\frac{\gamma}{\alpha}} \sum_{lmp} \epsilon_{ilm} \Omega_m S_{lp} J_{p,i,j,k}^{a-1,b,c} = -(a-1) \Pi_{i,j,k}^{a-2,b,c} , \tag{16}$$

where

$$K_{pq,i,j,k}^{a-1,b-1,c} = \int \rho \tilde{\phi}^2 x_p x_q x_i^{a-2} x_j^b x_k^c dV , \tag{17}$$

$$I_{p,i,j,k}^{a-1,b,c} = \int \rho x_p x_i^{a-1} x_j^b x_k^c dV , \tag{18}$$

$$J_{p,i,j,k}^{a-1,b,c} = \int \rho \tilde{\phi} x_p x_i^{a-1} x_j^b x_k^c dV , \tag{19}$$

and p, q take values from the set $\{i, j, k\}$ with $i \neq j \neq k$.

Note first that these virial equations with odd values of $n = a + b + c$ are identically zero. In fact, if we consider the form of the integrals (17)–(19), and the analogous integral forms (117) and (118) in the appendix for $W_{i,j,k}^{a,b,c}$ and $\Pi_{i,j,k}^{a-2,b,c}$, we see that the function $\tilde{\phi}$ determining the velocity always appears multiplied by odd powers of each coordinate. Since every integral containing odd powers of the coordinates is zero because of the symmetry of the integration volume, these integrals all vanish. Note also that $\partial_i v$ is proportional to x_i so that the kernel of W is odd. Consequently, all the terms on the right-hand side of equation (16) are zero, so that $\Pi_{i,j,k}^{a-2,b,c}$ must be zero. This means that the function P_i appearing in the integral form of $\Pi_{i,j,k}^{a-2,bc}$ (see expression [117] in the Appendix) must contain terms like x_1x_2, x_1x_3, x_2x_3, x_1^2, x_2^2, x_3^2 and its powers.

4. THE EVEN nTH-ORDER VIRIAL EQUATIONS AND THEIR APPLICATION TO GENERALIZED S-TYPE RIEMANN ELLIPSOIDS

We now turn our attention to the virial equations with even values of n. The analysis may be performed easily by classifying the powers of the coordinates. In fact, there are only the following possibilities: (1) a = even, b, c = odd; (2) b = even, a, c = odd; (3) c = even, a, b = odd; (4) a, b, c = even. Cases (2) and (3) are equivalent because of the interchangeable positions of j and k in equations (16). Cases (1) and (2) are nonequivalent owing to the privileged position of the index i. Thus, using the Appendix, the steady state virial equations (16) can be classified as follows.

Case (1): a = even, b, c = odd.—It is easy to prove that the terms containing the moments of T_{ij} cancel out between themselves, whereas the rest of the terms take the form

$$Z_j Z_k(2\alpha_{i,j,k}^{a-2,b+1,c+1} - \alpha_{i,j,k}^{a,b-1,c+1} - \alpha_{i,j,k}^{a,b+1,c-1}) = 0 . \tag{20}$$

Case (2): b = even, a, c = odd. Equivalent to case (3), with c = even, a, b = odd:

$$\frac{Z_k Z_i}{\gamma^{(n)}}[b\alpha_{i,j,k}^{a-1,b,c+1} - (b+1)\alpha_{i,j,k}^{a-1,b+2,c-1}] - \Omega_i\Omega_k\frac{a_k}{a_i} - 2\frac{a_j}{a_i\gamma^{(n)}}\Omega_k Z_i = 0 . \tag{21}$$

Case (4): a, b, c = even. Equation (16) becomes

$$Z_j^2\left[(a-1)\frac{a_i^2}{a_k^2}K_{i,j,k}^{a-2,b,c+2} - cK_{i,j,k}^{a,b,c}\right] + Z_k^2\left[(a-1)\frac{a_i^2}{a_j^2}K_{i,j,k}^{a-2,b+2,c} - bK_{i,j,k}^{a,b,c}\right]$$
$$+ (\Omega_j^2 + \Omega_k^2)I_{i,j,k}^{a,b,c} + 2\left(\Omega_j Z_j\frac{a_k}{a_i} + \Omega_k Z_k\frac{a_j}{a_i}\right)J_{i,j,k}^{a,b,c} + W_{i,j,k}^{a,b,c} + (a-1)\Pi_{i,j,k}^{a-2,b,c} = 0 . \tag{22}$$

Using the reduced (dimensionless) cylindrical coordinates $\tilde{x}_1 = \tilde{r}\cos\theta$, $\tilde{x}_2 = \tilde{r}\sin\theta$, and $\tilde{\phi} = \tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, it is easy to show that

$$(a-1)\frac{a_i^2}{a_k^2}K_{i,j,k}^{a-2,b,c+2}\Big/K_{i,j,k}^{a,b,c} = c + 1 ,$$

and

$$(a-1)\frac{a_i^2}{a_j^2}K_{i,j,k}^{a-2,b+2,c}\Big/K_{i,j,k}^{a,b,c} = b + 1 ,$$

so that equation (22) takes the form

$$(Z_j^2 + Z_k^2)K_{i,j,k}^{a,b,c} + (\Omega_j^2 + \Omega_k^2)I_{i,j,k}^{a,b,c} + 2\left(\Omega_j Z_j\frac{a_k}{a_i} + \Omega_k Z_k\frac{a_j}{a_i}\right)J_{i,j,k}^{a,b,c} + W_{i,j,k}^{a,b,c} + (a-1)\Pi_{i,j,k}^{a-2,b,c} = 0 . \tag{23}$$

In this work, we extend the classical Maclaurin spheroids, Jacobi, Dedekind, and Riemann ellipsoids (Chandrasekhar 1987), to cover heterogeneous systems with nonuniform vorticity and anisotropic pressure that we denote as generalized Maclaurin spheroids, generalized Jacobi, Dedekind, and Riemann ellipsoids.

We examine in the following the generalized S-type Riemann ellipsoids (homogeneous and heterogeneous systems with uniform figure rotation $\boldsymbol{\Omega}$ parallel to the vorticity $\boldsymbol{Z}$, and isotropic or anisotropic pressure). These configurations encompass as special cases the generalized Dedekind ellipsoids (homogeneous and heterogeneous systems with $\boldsymbol{\Omega} = 0$, $\boldsymbol{Z} \neq 0$, and isotropic or anisotropic pressure), the generalized Maclaurin spheroids and the generalized Jacobi ellipsoids (homogeneous and heterogeneous axisymmetric or ellipsoidal systems, respectively, having $\boldsymbol{\Omega} \neq 0$, $\boldsymbol{Z} = 0$, with isotropic or anisotropic pressure). It is interesting to note that the usual Dedekind theorem, which transforms Dedekind ellipsoids into Jacobi ellipsoids and vice versa, no longer applies in the nonlinear velocity regime (see § 5). These two families will therefore be treated independently.

For generalized S-type Riemann ellipsoids, $\hat{n} = (0, 0, 1)$, $\boldsymbol{\Omega} = (0, 0, \Omega)$, $\boldsymbol{Z} = (0, 0, Z)$. With this choice, the virial equations (20) and (21) vanish identically. Only equations (23) of case (4) are different from zero. We have explicitly

$$Z^2K_{1,2,3}^{a,b,c} + \Omega^2I_{1,2,3}^{a,b,c} + 2\Omega Z\frac{a_2}{a_1} + J_{1,2,3}^{a,b,c} + W_{1,2,3}^{a,b,c} + (a-1)\Pi_{1,2,3}^{a-2,b,c} = 0 , \tag{24}$$

$$Z^2K_{2,1,3}^{a,b,c} + \Omega^2I_{2,1,3}^{a,b,c} + 2\Omega Z\frac{a_1}{a_2} + J_{2,1,3}^{a,b,c} + W_{2,1,3}^{a,b,c} + (a-1)\Pi_{2,1,3}^{a-2,b,c} = 0 , \tag{25}$$

$$W_{3,1,2}^{a,b,c} + (a-1)\Pi_{3,1,2}^{a-2,b,c} = 0 . \tag{26}$$

It is easy to show that only in the linear and homogeneous case [$\tilde{\phi} = f = 1$, $P_i = P_{ic}(1 - m^2)$], this infinite set reduces to just three equations which coincide with the hydrodynamic equations (Chandrasekhar 1987, pp. 74–75). In fact, in this special case, using the integrals (127) and (128), we have $K^{a,b,c}_{i,j,k} = I^{a,b,c}_{i,j,k} = J^{a,b,c}_{i,j,k}$, $C_i = A_i$, and

$$\frac{W^{a,b,c}_{i,j,k} + (a-1)\Pi^{a-2,b,c}_{i,j,k}}{I^{a,b,c}_{i,j,k}} = 2(\pi G \rho_c)A_i + 2\,\frac{P_{ic}}{\rho_c a_i^2}\,,$$

so that equation (22) becomes

$$Z_j^2 + Z_k^2 + \Omega_j^2 + \Omega_k^2 + 2\left(\Omega_j Z_j \frac{a_k}{a_i} + \Omega_k Z_k \frac{a_j}{a_i}\right) + 2(\pi G \rho_c)A_i + 2\,\frac{P_{ic}}{\rho_c a_i^2} = 0\,, \tag{27}$$

and this set coincides with the hydrodynamic equations for the linear and homogeneous case.

5. DEDEKIND'S THEOREM

In the linear and homogeneous case, it is well known that Dedekind's theorem establishes, for any state of motion that preserves a constant ellipsoidal figure, the existence of an "adjoint" state of motion that preserves the same ellipsoidal figure. This can be maintained by uniform rotation or by internal motion of uniform vorticity, the geometry being determined by the same second-order virial equations (Chandrasekhar 1987, pp. 71–73). Here we try to generalize the Dedekind's theorem and the two adjoint sequences of Jacobi ("pure" rotating systems) and Dedekind ellipsoids (nonrotating systems, maintaining their figure by internal streaming only) to the case of heterogeneous density. In terms of α, η, γ (see the Appendix), equation (23) may be expressed as

$$\frac{\alpha^{i,j,k}_{a,b,c}}{\gamma^{(n)}}(Z_j^2 + Z_k^2) + \Omega_j^2 + \Omega_k^2 + 2\,\frac{\eta^{i,j,k}_{a,b,c}}{\gamma^{(n)}}\left(\frac{a_k}{a_i}\,\Omega_j Z_j + \frac{a_j}{a_i}\,\Omega_k Z_k\right) + \frac{W^{a,b,c}_{i,j,k} + (a-1)\Pi^{a-2,b,c}_{i,j,k}}{I^{a,b,c}_{i,j,k}} = 0\,. \tag{28}$$

An important class of equilibrium solutions are the self-adjoint (Dedekind's theorem): $\Omega\dagger = Z$, $Z\dagger = \Omega$; ($\dagger$ indicates Dedekind conjugation: matrix transposition and interchange of Ω and Z). This operation does not change the form of equations (28) so that they are self-adjoint if $\alpha^{i,j,k}_{a,b,c}/\gamma^{(n)} = 1$. According to the definitions given in the Appendix, we conclude that this condition is valid just in the linear case.

As a consequence, *Dedekind's theorem about the existence of a generalized Dedekind ellipsoid, corresponding to a generalized Jacobi ellipsoid or, in general, the existence of self-adjoint configurations, is limited to the linear case. The density may be inhomogeneous, and the pressure may be anisotropic.*

This result was obtained independently by Chambat (1994) by means of the hydrodynamic equations. In Paper VIII the validity of Dedekind's theorem in the nonlinear case was asserted using only the second-order virial equations. Now it has been demonstrated that the validity of the theorem is limited solely to the linear case, for any order of the virial equations.

6. GENERALIZED MACLAURIN SPHEROIDS AND JACOBI ELLIPSOIDS

We further specialize our analysis to the homogeneous and heterogeneous figures of S-type, spherically, spheroidally, and ellipsoidally stratified, in the case of uniform rotation ($\boldsymbol{\Omega}$ = const, $\boldsymbol{Z} = 0$). We include here as particular cases the classical Maclaurin spheroids and Jacobi ellipsoids (Chandrasekhar 1987). We generalize these equilibrum sequences to heterogeneous density and anisotropic pressure. The virial equations of nth order are, from equations (24)–(26),

$$\Omega^2 I^{a,b,c}_{1,2,3} + W^{a,b,c}_{1,2,3} + (a-1)\Pi^{a-2,b,c}_{1,2,3} = 0\,, \tag{29}$$

$$\Omega^2 I^{a,b,c}_{2,1,3} + W^{a,b,c}_{2,1,3} + (a-1)\Pi^{a-2,b,c}_{2,1,3} = 0\,, \tag{30}$$

$$W^{a,b,c}_{3,1,2} + (a-1)\Pi^{a-2,b,c}_{3,1,2} = 0, \tag{31}$$

where $a \geq 2$, b, $c \geq 0$. We will consider the second- and fourth-order virial equations; the higher orders give the same information.

Explicitly, the second- and fourth-order virial equations are

$$\Omega^2 I_{11} + W_{11} + \Pi_{11} = 0\,, \tag{32}$$

$$\Omega^2 I_{22} + W_{22} + \Pi_{22} = 0\,, \tag{33}$$

$$\Omega^2 I_{1111} + W_{1111} + 3\Pi_{1111} = 0\,, \tag{34}$$

$$\Omega^2 I_{1122} + W_{1122} + \Pi_{1122} = 0\,, \tag{35}$$

$$\Omega^2 I_{2211} + W_{2211} + \Pi_{2211} = 0\,, \tag{36}$$

$$\Omega^2 I_{2222} + W_{2222} + 3\Pi_{2222} = 0\,, \tag{37}$$

$$\Omega^2 I_{1133} + W_{1133} + \Pi_{1133} = 0\,, \tag{38}$$

$$\Omega^2 I_{2233} + W_{2233} + \Pi_{2233} = 0\,, \tag{39}$$

$$W_{3311} + \Pi_{3311} = 0\,, \tag{40}$$

$$W_{3322} + \Pi_{3322} = 0\,, \tag{41}$$

$$W_{3333} + \Pi_{3333} = 0\,, \tag{42}$$

$$W_{33} + \Pi_{33} = 0\,. \tag{43}$$

By eliminating Ω^2 from the previous equations and using the Appendix, we obtain

$$\int\left[(2\pi G\rho_c)f(-C_1\tilde{x}_1^2 + C_2\tilde{x}_2^2) + \left(\frac{P_1}{a_1^2} - \frac{P_2}{a_2^2}\right)\right]d\tilde{V} = 0\,, \tag{44}$$

$$\int\left[(2\pi G\rho_c)fC_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2\tilde{x}_2^2) + 3\,\frac{P_1}{a_1^2}(\tilde{x}_1^2 - \tilde{x}_3^2)\right]d\tilde{V} = 0\,, \tag{45}$$

$$\int\left[(2\pi G\rho_c)f(-C_1\tilde{x}_1^4 + 3C_2\tilde{x}_2^2\tilde{x}_1^2) + 3\left(\frac{P_1}{a_1^2} - \frac{P_2}{a_2^2}\right)\tilde{x}_1^2\right]d\tilde{V} = 0\,, \tag{46}$$

$$\int\left[(2\pi G\rho_c)fC_1(-\tilde{x}_1^2\tilde{x}_2^2 + \tilde{x}_1^2\tilde{x}_3^2) + \frac{P_1}{a_1^2}(\tilde{x}_2^2 - \tilde{x}_3^2)\right]d\tilde{V} = 0\,, \tag{47}$$

$$\int\left[(2\pi G\rho_c)fC_2(-\tilde{x}_2^4 + 3\tilde{x}_2^2\tilde{x}_3^2) + 3\,\frac{P_2}{a_2^2}(\tilde{x}_2^2 - \tilde{x}_3^2)\right]d\tilde{V} = 0\,, \tag{48}$$

$$\int\left[(2\pi G\rho_c)fC_2(-\tilde{x}_2^2\tilde{x}_3^2 + \tilde{x}_2^2\tilde{x}_1^2) + 3\,\frac{P_2}{a_2^2}(\tilde{x}_3^2 - \tilde{x}_1^2)\right]d\tilde{V} = 0\,, \tag{49}$$

$$\int\left[(2\pi G\rho_c)fC_2(-\tilde{x}_1^4 + 3\tilde{x}_1^2\tilde{x}_2^2) + 3\,\frac{P_1}{a_1^2}(\tilde{x}_1^2 - \tilde{x}_2^2)\right]d\tilde{V} = 0\,, \tag{50}$$

$$\int\left[(2\pi G\rho_c)fC_2(-\tilde{x}_2^4 + 3\tilde{x}_1^2\tilde{x}_2^2) + 3\,\frac{P_2}{a_2^2}(\tilde{x}_2^2 - \tilde{x}_1^2)\right]d\tilde{V} = 0\,, \tag{51}$$

$$\int\left[(2\pi G\rho_c)fC_1\left(-\frac{7}{3}\frac{\tilde{x}_1^4}{\gamma^{(4)}} + \frac{\tilde{x}_1^2}{\gamma}\right) + \frac{P_1}{a_1^2}\left(7\,\frac{\tilde{x}_1^2}{\gamma^{(4)}} - \frac{1}{\gamma}\right)\right]d\tilde{V} = 0\,, \tag{52}$$

From equations (40)–(42), we have

$$\int fC_3\left(-\tilde{x}_3^2\beta_{31}\,\frac{a_3^2}{a_1^2} + 3\tilde{x}_1^2\right)\tilde{x}_3^2\,d\tilde{V} = 0\,, \tag{53}$$

$$\int fC_3\left(\tilde{x}_1^2 - \frac{\beta_{31}}{\beta_{32}}\frac{a_2^2}{a_1^2}\,\tilde{x}_2^2\right)\tilde{x}_3^2\,d\tilde{V} = 0\,, \tag{54}$$

For the fourth-order virial equations, the coefficients of anisotropic pressure β_{ij} are defined by $\beta_{ij} = \Pi_{iijj}/\Pi_{3333}$, with $i \neq j$, corresponding for the second-order virial equations to $\beta_i = \Pi_{ii}/\Pi_{33}$.

6.1. *Sphere*

We can perform the analysis beginning with the *sphere*. In this case $a_1 = a_2 = a_3$ and consequently $C_1 = C_2 = C_3 = C(r)$. In both the homogeneous and heterogeneous cases, from equations (44)–(51), since the first term in each of the integrals is zero, we conclude that $P_1 = P_2$ and that P_1 must be barotropic (spherically stratified) because it is invariant under the interchange of $\tilde{x}_1^2$ and $\tilde{x}_2^2$, of $\tilde{x}_2^2$ and $\tilde{x}_3^2$, of $\tilde{x}_3^2$ and $\tilde{x}_1^2$. Moreover, equations (53) and (54) require P_3 to be barotropic. We emphasize that in the homogeneous case, when improperly used, the term "barotropic" means "stratified as the boundary surface." It is important to note that from equation (52), in the homogeneous case [$\gamma^{(4)} = \gamma = 1$], since the first term in the integral is zero, we obtain $\int P_1(7\tilde{x}_1^2 - 1)d\tilde{V} = 0$. Since the pressure is spherically stratified and vanishes at the boundary, we can write $P_1 = P_{1c}(1 - \tilde{r}^2)^K$, and substituting this into the previous equation, it follows necessarily that $k = 1$.

If we next make use of equations (32) and (43), using the definition of β_1, we have

$$\Omega^2 I_{11} + (\beta_1 - 1)|W_{33}| = 0\,, \tag{55}$$

consequently $\beta_1 \leq 1$, so that $P_1 \leq P_3$. The conclusion is that *the uniformly rotating sphere, homogeneous or heterogeneous, must be anisotropic, and the pressure must be spherically stratified.* The same results come from the fourth-order virial equations:

$$\Omega^2 I_{1111} + |W_{3333}|(\beta_{11} - 1) = 0\,, \tag{56}$$

$$\Omega^2 I_{1133} + |W_{3311}|(\beta_{13} - 1) = 0\,, \tag{57}$$

$$\Omega^2 I_{1122} + W_{1122} - \frac{\beta_{12}}{\beta_{31}}\,W_{3311} = 0\,, \tag{58}$$

$$3W_{3311} - \beta_{31}\,W_{3333} = 0\,, \tag{59}$$

from which $\beta_{11} < 1$, $\beta_{13} < 1$, $\beta_{12} < \beta_{31}$, $\beta_{31} = 1$; the pressure must be stratified spherically, and $P_1 < P_3$.

The higher order virial equations are identically satisfied and do not add any new information.

6.2. *Spheroids*

In this case, $f = f(\tilde{r}^2 + \tilde{x}_3^2)$, $C_1 = C_2 = C_1(\tilde{r}^2, \tilde{x}_3^2)$. Using reduced cylindrical coordinates ($\tilde{x}_1 = \tilde{r}\cos\theta$, $\tilde{x}_2 = \tilde{r}\sin\theta$), we can demonstrate that

$$\int fC_1(-\tilde{x}_1^2 + \tilde{x}_2^2)d\tilde{V} = 0\,, \tag{60}$$

$$\int fC_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2\tilde{x}_2^2)d\tilde{V} = 0\,, \tag{61}$$

from which using the equations (44)–(51), it follows that $P_1 = P_2$ and

$$\int\left[(2\pi G\rho_c)fC_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2\tilde{x}_3^2) + 3\,\frac{P_1}{a_1^2}\,(\tilde{x}_1^2 - \tilde{x}_3^2)\right]d\tilde{V} = 0\,, \tag{62}$$

$$\int\left[(2\pi G\rho_c)fC_1(-\tilde{x}_1^2\tilde{x}_2^2 + \tilde{x}_1^2\tilde{x}_3^2) + \frac{P_1}{a_1^2}\,(\tilde{x}_1^2 - \tilde{x}_3^2)\right]d\tilde{V} = 0\,, \tag{63}$$

$$\int P_1(\tilde{x}_1^2 - \tilde{x}_2^2)d\tilde{V} = 0\,, \tag{64}$$

Consequently the pressure component P_1 must contain $\tilde{r}^2$ and in the homogeneous case (since the first term in each of the integrals [62] and [63] is zero) the component P_1 must be spheroidally stratified. From equation (53), it follows that $\beta_{31} = \beta_{32} = a_1^2/a_3^2$ so that P_3 must be spheroidally stratified. Equation (52) requires further that $P_1 = P_{1c}(1 - \tilde{r}^2 - \tilde{x}_3^2)$. To complete the analysis in the homogeneous case, considering equations (32) and (43) we have

$$\Omega^2 I_{11} + W_{11} - \beta_1 W_{33} = 0\,, \tag{65}$$

from which it follows that

$$W_{11} - \beta_1 W_{33} < 0\,, \tag{66}$$

or

$$\int f(-C_1\tilde{x}_1^2 + \beta_1 C_3\tilde{x}_3^2)d\tilde{V} < 0\,, \tag{67}$$

and since $A_i = C_i$, we obtain

$$-A_1^2 a_1^2 + \beta_1 A_3 a_3^2 < 0\,. \tag{68}$$

Then using the properties of the index symbols in Chandrasekhar (1987), $A_1^2 a_1^2 - A_3 a_3^2 = (a_1^2 - a_3^2)B_{13}$, equation (68) becomes

$$(a_3^2 - a_1^2)B_{13} + A_3(\beta_1 - 1)a_3^2 < 0\,. \tag{69}$$

1. In the case of homogeneous oblate spheroids ($a_1 > a_3$), the term $\beta_1 - 1$ can be greater than, less than or equal to zero: *homogeneous, oblate Maclaurin spheroids exist and can be isotropic* ($P_1 = P_2 = P_3$) *or anisotropic* ($P_1 = P_2 \neq P_3$), *with the pressure spheroidally stratified*. This last result generalizes the solution of Chandrasekhar (1987, pp. 77–80).

2. In the case of homogeneous prolate spheroids ($a_3 > a_1$), the anisotropy coefficient must be $\beta_1 < 1$, namely, $P_1 < P_3$; then, *homogeneous, prolate Maclaurin spheroids exist and must be anisotropic* ($P_3 > P_1$) *and the pressure spheroidally stratified.*

Finally, we consider the heterogeneous case; equations (53) and (54) are compatible with a baroclinic form of P_3 ($\beta_{31} = \beta_{32}$). The equations (62) and (63) do not permit P_1 to be barotropic; in fact, from equation (64) it must be baroclinic. Thus *heterogeneous, isotropic or anisotropic, barotropic Maclaurin spheroids cannot exist.* This conclusion generalizes Dive's results (Dive 1930) to the anisotropic case. Instead, *heterogeneous, baroclinic Maclaurin spheroids are allowed.*

6.3. *Ellipsoids*

From equations (44)–(51) we conclude that heterogenous, barotropic ellipsoids are nonexistent. Instead, these equations are satisfied in the homogeneous case with ellipsoidal stratifications of the isotropic pressure (the Jacobi ellipsoids in Chandrasekhar 1987, pp. 101–103). In addition, there exist homogeneous ellipsoids having the pressure ellipsoidally stratified and anisotropic ($P_1 \neq P_2 \neq P_3$). From equations (53) and (54), we find that P_3 is ellipsoidally stratified and from equation (52), P_1 and P_3 must have the form $P = P(1 - \tilde{x}_1^2 - \tilde{x}_2^2 - \tilde{x}_3^2)$. From equations (32)–(43), we demonstrate the existence of oblate ellipsoids with $a_1 > a_2 > a_3$ and $a_1 > a_3 > a_2$; for the prolate ellipsoids, we observe that $a_3 > a_1 > a_2$, and the pressure must be anisotropic with $P_1 < P_3$ and $P_2 < P_3$. In both the homogeneous and heterogeneous cases, not all the equations (44)–(51) can be satisfied if the pressure is baroclinic; thus, *the generalized Jacobi ellipsoids cannot exist with baroclinic form of the pressure, in both the homogeneous and heterogeneous cases, isotropic or not.*

7. GENERALIZED DEDEKIND ELLIPSOIDS

We study now the heterogeneous figures of S-type, spherically, spheroidally, and ellipsoidally stratified, in the case of nonuniform vorticity ($\boldsymbol{Z} \neq 0$) (differential rotation), without figure rotation ($\boldsymbol{\Omega} = 0$). The nth-order virial equations are in this case

$$Z^2 K^{a,b,c}_{1,2,3} + W^{a,b,c}_{1,2,3} + (a-1)\Pi^{a-2,b,c}_{1,2,3} = 0 , \tag{70}$$

$$Z^2 K^{a,b,c}_{2,1,3} + W^{a,b,c}_{2,1,3} + (a-1)\Pi^{a-2,b,c}_{2,1,3} = 0 , \tag{71}$$

$$W^{a,b,c}_{3,1,2} + (a-1)\Pi^{a-2,b,c}_{3,1,2} = 0 . \tag{72}$$

As in the previous case, we will consider only the second- and fourth-order virial equations. The higher order virial equations are identically satisfied and do not give any new information.

In the following we consider two independent cases in which the form of velocity field is respectively $\tilde{\phi} = \tilde{\phi}(m^2)$ and $\tilde{\phi} = \tilde{\phi}^*$, indicating the general forms $\tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(m^2, \tilde{r}^2)$, $\tilde{\phi}(m^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2)$, with $m^2 = \tilde{r}^2 + \tilde{x}_3^2$, $\tilde{r}^2 = x_1^2/a_1^2 + x_2^2/a_2^2$.

(*a*) $\tilde{\phi} = \tilde{\phi}(m^2)$.—Equations (70)–(72) are formally the same as the analogues ones, in the case of uniformly rotating generalized Maclaurin spheroids and Jacobi ellipsoids (eqs. [29]–[31]). We have just replaced $I^{a,b,c}_{i,j,k}$ by $K^{a,b,c}_{i,j,k}$. Consequently the conclusions of the Dedekind case with $\tilde{\phi}(m^2)$, valid also in the linear case $\tilde{\phi} = 1$, are the same as in the Jacobi case. Summarizing:

1. *A sphere having nonuniform vorticity (differential rotation), homogeneous or heterogeneous, must be anisotropic with $P_3 > P_1$.*
2. *There exists a homogeneous or heterogeneous sphere having $\tilde{\phi} = \tilde{\phi}(\tilde{r}^2)$ and the pressure radially stratified, $P_1(r) \neq P_3(r)$.*
3. *There exist homogeneous spheroids having $\tilde{\phi} = \tilde{\phi}(m^2)$: the oblate ones can be isotropic or anisotropic, and the prolate ones must be anisotropic with $P_1 < P_3$, the pressure being spheroidally stratified.*
4. *The heterogeneous spheroids must be baroclinic* (see Dive 1930).
5. *The homogeneous Dedekind's ellipsoids with pressure ellipsoidally stratified isotropic or anisotropic are classifiable as oblate, isotropic and anisotropic, or prolate and anisotropic with $P_1 < P_3$, $P_2 < P_3$.* These results generalize the oblate, isotropic configurations given in Chandrasekhar (1987, pp. 124–125).
6. *Generalized heterogeneous Dedekind ellipsoids cannot exist.* This conclusion agrees with the recent results obtained by Chambat (1994), using the hydrodynamic equations.

(*b*) We now assume the others forms of $\tilde{\phi}$, denoted as $\tilde{\phi}^*$: from equations (70) and (71), we have

$$\frac{W_{11} + \Pi_{11}}{K_{11}} = \frac{W_{22} + \Pi_{22}}{K_{22}} = \frac{W_{1111} + 3\Pi_{1111}}{K_{1111}} = \frac{W_{1122} + \Pi_{1122}}{K_{1122}} = \frac{W_{1133} + \Pi_{1133}}{K_{1133}} = \frac{W_{2233} + \Pi_{2233}}{K_{2233}} = \frac{W_{2211} + \Pi_{2211}}{K_{2211}} = \frac{W_{2222} + 3\Pi_{2222}}{K_{2222}} . \tag{73}$$

We define the coefficient λ as a relation between the denominators of equation (73) so that

$$\frac{K_{1111}}{K_{1133}} = 3\,\frac{a_1^2}{a_3^2}\,\lambda , \tag{74}$$

$$\frac{K_{1111}}{K_{1122}} = 3\,\frac{a_1^2}{a_2^2} , \tag{75}$$

and so on, with

$$\lambda = \frac{\int f\tilde{\phi}^2\tilde{r}^5\, d\tilde{r}\, d\tilde{x}_3}{\int f\tilde{\phi}^2\tilde{r}^3\, \tilde{x}_3^2\, d\tilde{r}\, d\tilde{x}_3} . \tag{76}$$

As a consequence of equation (73),

$$(W_{11} + \Pi_{11})\,\frac{K_{1111}}{K_{11}} = W_{1111} + 3\Pi_{1111} = (W_{1122} + \Pi_{1122})3\,\frac{a_1^2}{a_2^2} = (W_{1133} + \Pi_{1133})3\,\frac{a_1^2\lambda}{a_3^2} = (W_{2233} + \Pi_{2233})3\,\frac{a_1^4\lambda}{a_2^2 a_3^2} = (W_{2211} + \Pi_{2211})3\,\frac{a_1^2}{a_2^2} = (W_{2222} + 3\Pi_{2222})\,\frac{a_1^4}{a_2^4} , \tag{77}$$

from which we obtain the following equations:

$$\int \left[(2\pi G\rho_c)f(-C_1\tilde{x}_1^4 + 3C_2\,\tilde{x}_1^2\tilde{x}_2^2) + 3\left(\frac{P_1}{a_1^2} - \frac{P_2}{a_2^2}\right)\tilde{x}_1^2\right] d\tilde{V} = 0 , \tag{78}$$

$$\int \left[(2\pi G\rho_c)fC_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2\,\tilde{x}_2^2) + 3\,\frac{P_1}{a_1^2}\,(\tilde{x}_1^2 - \tilde{x}_2^2)\right] d\tilde{V} = 0 , \tag{79}$$

$$\int \left[(2\pi G\rho_c)fC_2(-\tilde{x}_2^4 + 3\tilde{x}_1^2\tilde{x}_2^2) + 3\,\frac{P_2}{a_2^2}\,(\tilde{x}_2^2 - \tilde{x}_1^2)\right] d\tilde{V} = 0 , \tag{80}$$

$$\int \left[(2\pi G\rho_c)f(-C_1\tilde{x}_1^2 + C_2\,\tilde{x}_2^2) + \left(\frac{P_1}{a_1^2} - \frac{P_2}{a_2^2}\right)\tilde{x}_3^2\right] d\tilde{V} = 0 , \tag{81}$$

$$\int\left[(2\pi G\rho_c)fC_1\left(-\frac{7}{3}\frac{\tilde{x}_1^4}{\alpha^{(4)}}+\frac{\tilde{x}_1^2}{\alpha}\right)+\frac{P_1}{\alpha_1^2}\left(7\frac{\tilde{x}_1^2}{\alpha^{(4)}}-\frac{1}{\alpha}\right)\right]d\tilde{V}=0\,, \tag{82}$$

The equations involving the λ term are identically satisfied. Equations (78)–(81) are identically satisfied for a homogeneous and heterogeneous sphere, with P_1 baroclinic or barotropic. According to equations (53) and (54), P_3 is barotropic. From the second-order virial equations $\beta_1 < 1$, namely, $P_1 < P_3$, we conclude that *there exist homogeneous and heterogeneous anisotropic spheres with nonuniform vorticity* (*differential rotation*) *of the form* $\tilde{\phi}^*$.

We consider now the spheroidal case. Equations (78)–(81) allow homogeneous and heterogeneous barotropic and baroclinic spheroids, but equations (53) and (54) allow these configurations only for the homogeneous case and P_3 barotropic, or for the heterogeneous case and P_3 baroclinic. Equation (82) imposes that in the homogeneous and linear case one must $P_1 = P_{1c}(1 - \tilde{r}^2 - \tilde{x}_3^2)$. In the isotropic case, the heterogeneous spheroid has a baroclinic pressure or the homogeneous spheroid is barotropic. Therefore, we recover the special case considered by Lebovitz (1979), who gives a solution of the hydrodynamic equations with a density $f = 1 - \tilde{r}^2 + \tilde{x}_3^2$, which is isotropic and baroclinic. In conclusion, *the homogeneous spheroids with nonuniform vorticity* (*differential rotation*) *of the form* $\tilde{\phi}^*$ *exist, and their pressure is spheroidally stratified; the heterogeneous spheroids with differential rotation of the form* $\tilde{\phi}^*$ *exist and must be baroclinic*. These results generalize the conclusions obtained by Dive (1930).

In the case of the ellipsoids, equations (79), (80), (53), and (54) allow only homogeneous and barotropic solutions with $\tilde{\phi} = \tilde{\phi}^*$, both isotropic or anisotropic. These figures are classifiable using the same procedure as in the previous case of uniform rotation. Thus *the heterogeneous ellipsoids with* $\tilde{\phi} = \tilde{\phi}^*$ *do not exist*, and this is again in agreement with Chambat (1994).

8. GENERALIZED RIEMANN ELLIPSOIDS

We now turn to the case of the generalized S-type Riemann ellipsoids, with uniform figure rotation ($\boldsymbol{\Omega} \neq 0$) and nonuniform vorticity ($\boldsymbol{Z} \neq 0$). The virial equations of nth order are

$$Z^2K_{1,2,3}^{a,b,c}+\Omega^2I_{1,2,3}^{a,b,c}+2\Omega Z\frac{a_2}{a_1}J_{1,2,3}^{a,b,c}+W_{1,2,3}^{a,b,c}+(a-1)\Pi_{1,2,3}^{a-2,b,c}=0\,, \tag{83}$$

$$Z^2K_{2,1,3}^{a,b,c}+\Omega^2I_{2,1,3}^{a,b,c}+2\Omega Z\frac{a_1}{a_2}J_{2,1,3}^{a,b,c}+W_{2,1,3}^{a,b,c}+(a-1)\Pi_{2,1,3}^{a-2,b,c}=0\,, \tag{84}$$

$$W_{3,1,2}^{a,b,c}+(a-1)\Pi_{3,1,2}^{a-2,b,c}=0\,, \tag{85}$$

with λ defined in equations (74)–(76). Taking into account that $K_{1122}/K_{1111} = a_2^2/3a_1^2$, $K_{1133}/K_{1111} = a_3^2/3a_1^2\lambda$, and introducing the dimensionless coefficient μ in the form

$$\frac{J_{1111}}{K_{1133}}=\frac{3a_1^2\mu}{a_3^2}\,, \tag{86}$$

where

$$\mu=\frac{\int f\tilde{\phi}\tilde{r}^5\,d\tilde{r}\,d\tilde{x}_3}{\int f\tilde{\phi}\tilde{r}^3\tilde{x}_3^2\,d\tilde{r}\,d\tilde{x}_3}\,, \tag{87}$$

the second- and fourth-order virial equations are

$$Z^2K_{11}+\Omega^2I_{11}+2\Omega Z\frac{a_2}{a_1}J_{11}+W_{11}+\Pi_{11}=0\,, \tag{88}$$

$$Z^2K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_2}{a_1}J_{1111}+W_{1111}+3\Pi_{1111}=0\,, \tag{89}$$

$$Z^2K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_2}{a_1}J_{1111}+(W_{1122}+\Pi_{1122})\frac{3a_1^2}{a_2^2}=0\,, \tag{90}$$

$$\frac{Z^2}{\lambda}K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_2}{a_1}\frac{J_{1111}}{\mu}+(W_{1133}+\Pi_{1133})\frac{3a_1^2}{a_3^2}=0\,, \tag{91}$$

$$\frac{Z^2}{\lambda}K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_1}{a_2}\frac{J_{1111}}{\mu}+(W_{2233}+\Pi_{2233})\frac{3a_1^4}{a_2^2a_3^2}=0\,, \tag{92}$$

$$\frac{Z^2}{\lambda}K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_1}{a_2}\frac{J_{1111}}{\mu}+(W_{2211}+\Pi_{2211})\frac{3a_1^2}{a_2^2}=0\,, \tag{93}$$

$$\frac{Z^2}{\lambda}K_{1111}+\Omega^2I_{1111}+2\Omega Z\frac{a_1}{a_2}\frac{J_{1111}}{\mu}+(W_{2222}+3\Pi_{2222})\frac{a_1^4}{a_2^4}=0\,. \tag{94}$$

The higher order virial equations are identically satisfied.

In the case $\tilde{\phi} = \tilde{\phi}(m^2)$, then $\lambda = 1$, so that for spheres and spheroids

$$W_{1111} + 3\Pi_{1111} = (W_{1122} + \Pi_{1122}) \frac{3a_1^2}{a_2^2} = (W_{1133} + \Pi_{1133}) \frac{3a_1^2}{a_3^2}$$

$$= (W_{2233} + \Pi_{2233}) \frac{3a_1^4}{a_2^2 a_3^2} = (W_{2211} + \Pi_{2211}) \frac{3a_1^2}{a_2^2} = (W_{2222} + 3\Pi_{2222}) \frac{a_1^4}{a_2^4} . \tag{95}$$

For ellipsoids

$$W_{1111} + 3\Pi_{1111} = (W_{1122} + \Pi_{1122}) \frac{3a_1^2}{a_2^2} = (W_{1133} + \Pi_{1133}) \frac{3a_1^2}{a_3^2} , \tag{96}$$

$$W_{2222} + 3\Pi_{2222} = (W_{2211} + \Pi_{2211}) \frac{3a_2^2}{a_1^2} = (W_{2233} + \Pi_{2233}) \frac{3a_2^2}{a_3^2} , \tag{97}$$

and including equations (85), we have

$$\int \left[(2\pi G\rho_c) f C_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2 \tilde{x}_2^2) + 3 \frac{P_1}{a_1^2} (\tilde{x}_1^2 - \tilde{x}_2^2) \right] d\tilde{V} = 0 , \tag{98}$$

$$\int [(2\pi G\rho_c) f C_1(-\tilde{x}_1^4 + 3\tilde{x}_1^2 \tilde{x}_3^2) + 3 \frac{P_1}{a_1^2} (\tilde{x}_1^2 - \tilde{x}_3^2)] d\tilde{V} = 0 , \tag{99}$$

$$\int [(2\pi G\rho_c) f C_2(-\tilde{x}_2^2 + 3\tilde{x}_1^2 \tilde{x}_2^2) + 3 \frac{P_2}{a_2^2} (\tilde{x}_2^2 - \tilde{x}_1^2)] d\tilde{V} = 0 , \tag{100}$$

$$\int \left[(2\pi G\rho_c) f C_2(-\tilde{x}_2^4 + 3\tilde{x}_2^2 \tilde{x}_3^2) + 3 \frac{P_2}{a_2^2} (\tilde{x}_2^2 - \tilde{x}_3^2) \right] d\tilde{V} = 0 , \tag{101}$$

$$\int f C_3 \left(3\tilde{x}_1^2 - \tilde{x}_3^2 \beta_{31} \frac{a_3^2}{a_1^2} \right) \tilde{x}_3^2 \, d\tilde{V} = 0 , \tag{102}$$

$$\int f C_3 \left(\tilde{x}_1^2 - \tilde{x}_2^2 \frac{\beta_{31}}{\beta_{32}} \frac{a_2^2}{a_1^2} \right) \tilde{x}_3^2 \, d\tilde{V} = 0 . \tag{103}$$

These equations are satisfied for homogeneous ellipsoids and with P_1, P_2, P_3 barotropic. According to equation (88) with $\boldsymbol{\Omega}$ and $\boldsymbol{Z}$ parallel, we have $W_{11} + \Pi_{11} < 0$, namely, $W_{11} - \beta_1 W_{33} < 0$, from which if $a_1 < a_3$, $\beta_1 < 1$, $P_1 < P_3$, and $\beta_2 < 1$, $P_2 < P_3$. If $a_1 > a_3$, then P_1 can be greater than, less than, or equal to P_3. We have proved that *the homogeneous prolate ellipsoids with* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *parallel must be anisotropic:* $P_1 < P_3$, $P_2 < P_3$. *If* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *are antiparallel, the ellipsoids can be isotropic or anisotropic.*

According to equations (98)–(103), the heterogeneous ellipsoids cannot be barotropic or baroclinic: *there do not exist heterogeneous generalized Riemann ellipsoids with* $\tilde{\phi} = \tilde{\phi}(m^2)$. This is a generalization of the result obtained by Chambat (1994) in the linear case ($\tilde{\phi} = 1$).

For spheres and spheroids, since the system of equations (95) is the same as equations (44)–(52), and taking into account equations (53) and (54), we have the result that *a homogeneous or heterogeneous sphere must be anisotropic if* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *are parallel, with* P_1, P_3 *spherically stratified. If* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *are antiparallel, the sphere can be isotropic. A homogeneous spheroid with* $\tilde{\phi} = \tilde{\phi}(m^2)$ *must be barotropic. If* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *are parallel, then one must have* $P_1 < P_3$. *If* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *are antiparallel, the spheroid can be isotropic or anisotropic. A heterogeneous spheroid must be baroclinic and can be isotropic or anisotropic.*

The case $\tilde{\phi} = \tilde{\phi}^*$ gives for spheres and spheroids

$$W_{1111} + 3\Pi_{1111} = 3(W_{1122} + \Pi_{1122}) . \tag{104}$$

For ellipsoids, we have

$$W_{1111} + 3\Pi_{1111} = (W_{1122} + \Pi_{1122}) \frac{3a_1^2}{a_2^2} , \tag{105}$$

$$W_{2222} + 3\Pi_{2222} = (W_{2211} + \Pi_{2211}) \frac{3a_2^2}{a_1^2} , \tag{106}$$

$$\int \left[(2\pi G\rho_c) f C_1(\tilde{x}_1^4 - 3\tilde{x}_1^2 \tilde{x}_2^2) + 3 \frac{P_1}{a_1^2} (\tilde{x}_1^2 - \tilde{x}_2^2) \right] d\tilde{V} = 0 , \tag{107}$$

$$\int \left[(2\pi G\rho_c) f C_2(\tilde{x}_2^4 - 3\tilde{x}_1^2 \tilde{x}_2^2) + 3 \frac{P_2}{a_2^2} (\tilde{x}_2^2 - \tilde{x}_1^2) \right] d\tilde{V} = 0 . \tag{108}$$

For spheres and spheroids, equation (107) is valid, and it is identically satisfied in the homogeneous and heterogeneous cases. Therefore, *homogeneous and heterogeneous spheres and spheroids exist, with* $\boldsymbol{\Omega}$ *and* $\boldsymbol{Z}$ *parallel or antiparallel.*

TABLE 1

EQUILIBRIUM CONFIGURATIONS WITH $\boldsymbol{\Omega} \neq 0$ AND $\boldsymbol{Z} = 0$[a]

Configurations	Density	Shape	Pressure
Spheres	Homogeneous	...	$P_1 < P_3$, B
	Heterogeneous	...	$P_1 < P_3$, B
Generalized Maclaurin spheroids	Homogeneous	Oblate	isotropic ($P_1 = P_2 = P_3$), B
			anisotropic ($P_1 = P_2 \neq P_3$), B
		Prolate	anisotropic ($P_1 < P_3$), B
	Heterogeneous	..	isotropic, BC
			anisotropic ($P_1 = P_2 \neq P_3$), BC
Generalized Jacobi ellipsoids	Homogeneous	Oblate	isotropic $P_1 = P_2 = P_3$), B
			anisotropic ($P_1 \neq P_2 \neq P_3$, B
		Prolate	anisotropic ($P_1 < P_3$; $P_2 < P_3$), B
	Heterogeneous	...	...

[a] Homogeneous and heterogeneous figures of equilibrium having a uniform angular velocity $\boldsymbol{\Omega}$, with isotropic or anisotropic pressure (the different components P_1, P_2, and P_3 can be barotropic or stratified as the density [denoted by B], or baroclinic, $P_i = P_i(\tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_3^2)$ [denoted by BC]).

As in the previous case, the spheroids are classifiable as oblate and prolate and are barotropic. Finally, let us consider the case of ellipsoids: *the homogeneous, isotropic or anisotropic Riemann ellipsoids exist and can be barotropic and baroclinic in* P_1, P_2, *but* P_3 *must be barotropic. The heterogeneous, generalized Riemann ellipsoids having* $\tilde{\phi}^*$ *cannot exist.*

9. CONCLUSIONS

The tensor virial equations for a self-gravitating, rotating fluid mass are generalized to the nth order using useful and compact definitions of the meaningful coefficients. The only hypothesis on the density distribution is that the equidensity surfaces are similar concentric ellipsoids. The most general velocity field within an ellipsoidal figure, preserving its form as seen from a frame of reference in which the ellipsoid is at rest and producing internal fluid motions of nonuniform vorticity, has been considered, following the treatment of Paper VIII. The assumption that the velocity is tangent to the equidensity surfaces is equivalent to assuming an incompressible flow (justified by the discussion of Ipser & Managan 1981). The necessary equilibrium conditions coming from the second- and fourth-order virial equations have been obtained for the case of generalized S-type Riemann ellipsoids. In all cases, the higher order virial equations do not produce new information with respect to that gained from only the second- and fourth-order virial equations. The main results concerning (*a*) the *generalized Maclaurin spheroids and Jacobi ellipsoids*, (*b*) the *generalized Dedekind ellipsoids*, and (*c*) the *generalized Riemann ellipsoids* are summarized in Tables 1–3, respectively.

In this work we have recovered some classical results obtained by Dive (1930) (a stratified heterogeneous spheroid, rotating and without differential rotation, cannot be a barotrope) and generalized to ellipsoidal, anisotropic configurations. The Hamy theorem (a mass ellipsoidally stratified cannot have a uniform rotation) is confirmed also for the anisotropic case. All the results in Chandrasekhar (1987) for homogeneous configurations are generalized to the anisotropic case.

The virial equations of nth order prove the nonexistence of triaxial, stratified, heterogeneous equilibrium ellipsoids. Only a certain class of axisymmetric equilibrium figures with differential rotation (having or not) rigid rotation can exist.

TABLE 2

EQUILIBRIUM CONFIGURATIONS WITH $\boldsymbol{\Omega} = 0$ AND $\boldsymbol{Z} \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity
Spheres	Homogeneous	...	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}$
			anisotropic, P_1 (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous	...	$P_1 < P_3$, B	$\tilde{\phi}$
			$P_1 < P_3$, B	$\tilde{\phi}^*$
Spheroids	Homogeneous	Oblate	isotropic, B	$\tilde{\phi}$
			anisotropic ($P_1 \neq P_3$), B	$\tilde{\phi}$
		Prolate	$P_1 = P_2 < P_3$ (B)	$\tilde{\phi}$
		Oblate	isotropic, B	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$, (B, BC), P_3 (B)	$\tilde{\phi}^*$
		Prolate	$P_1 = P_2$, (B, BC) $< P_3$ (B)	$\tilde{\phi}^*$
	Heterogeneous		isotropic (BC)	$\tilde{\phi}$
			anisotropic ($P_1 = P_2 \neq P_3$), BC	$\tilde{\phi}$
			isotropic, BC	$\tilde{\phi}^*$
			anisotropic $P_1 = P_2$ (B, BC), P_3 (BC)	$\tilde{\phi}^*$
Generalized Dedekind ellipsoids	Homogeneous	Oblate	isotropic and anisotropic, B	$\tilde{\phi}$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}$
		Oblate	isotropic and anisotropic, B	$\tilde{\phi}^*$
		Prolate	anisotropic ($P_1 < P_3$), B	$\tilde{\phi}^*$
	Heterogeneous	...	...	

[a] Homogeneous and heterogeneous figures of equilibrium, isotropic or anisotropic cases are shown, and the different components of the pressure, barotropic (B) or baroclinic (BC) are noted, with differential rotation $\boldsymbol{Z}$, and a velocity field defined by the functional form $\tilde{\phi} = \tilde{\phi}(m^2)$; $\tilde{\phi}^*$ includes $\tilde{\phi}(m^2, \tilde{x}_3^2)$, $\tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2)$, $\tilde{\phi}(\tilde{x}_3^2)$.

TABLE 3

EQUILIBRIUM CONFIGURATIONS WITH $\Omega \neq Z \neq 0$[a]

Configuration	Density	Shape	Pressure	Velocity	Ω, Z
Spheres	Homogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
	Heterogeneous	...	anisotropic $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
			isotropic and anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
Spheroids	Homogeneous	Oblate	isotropic, anisotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
		Prolate	anisotropic, $P_1 < P_3$, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
	Heterogeneous	Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}$	↑↑
		Prolate		...	↑↓
		Oblate	isotropic (BC) and anisotropic	$\tilde{\phi}^*$	↑↑
		Prolate	$P_1 = P_2$(B, BC), P_3(BC)	...	↑↓
Generalized Riemann ellipsoids	Homogeneous	Prolate	anisotropic ($P_1 < P_3$, $P_2 < P_3$), B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↑
		Oblate	isotropic, B	$\tilde{\phi}$, $\tilde{\phi}^*$	↑↓
		Oblate	anisotropic, P_1(B), P_2(BC), P_3(B)	$\tilde{\phi}^*$	↑↓
	Heterogeneous	...	...	...	...

[a] Homogeneous and heterogeneous figures of equilibrium in which the direction Ω and Z are parallel or antiparallel and lie along the rotation axis x_3 (Ω, Z parallel = ↑↑, Ω, Z antiparallel = ↑↓); isotropic and anisotropic cases are noted, and the different components of the pressure, barotropic (B) or baroclinic (BC), are shown. The various forms of the velocity field $\tilde{\phi}$, $\tilde{\phi}^*$ are considered.

The approximate hydrostatic equilibrium solutions for rotating polytropes (Lai, Rasio, & Shapiro 1993) can be reconsidered in the framework of our results, recovering the same conclusions given in Chambat (1994).

Tensor virial equations of higher orders can be used for the construction of equilibrium configurations for the description of stellar systems or galaxies in the oblate or prolate cases. Methods recently formulated by various authors require that the conditions of mechanical equilibrium be satisfied in the average sense of the virial equations (Nelson & Papaloizou 1993). All the above conditions obtained from the virial equations are necessary but not sufficient for the determination of the equilibrium configurations. It will now be important to implement these global averaged conditions for specific distribution functions in order to obtain the result that the dynamic equations are satisfied pointwise by any configuration of equilibrium.

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APPENDIX

Here we calculate explicitly the nth-order tensors, using the general forms for the velocity field $\tilde{\phi} = \tilde{\phi}(m^2)$ and $\tilde{\phi}^*$ as specified in § 7 and the density distribution $f = f(m^2)$. In this paper $\alpha^{a,b,c}_{i,j,k}$, $\eta^{a,b,c}_{i,j,k}$, $\gamma^{(n)}$, f, g_i and all the symbols defined with " ~ " are dimensionless. Also, $\alpha^{(2)} = \alpha$, $\eta^{(2)} = \eta$, $\gamma^{(2)} = \gamma$. We introduce the following definitions:

1. The n-order tensor of inertia $I^{a,b,c}_{i,j,k}$, where the indices i, j, k appear a, b, c times, respectively:

$$I^{a,b,c}_{i,j,k} = \int \rho x_i^a x_j^b x_k^c \, dV = \rho_c \, a_i^{a+1} a_j^{b+1} a_k^{c+1} \int f \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} = \rho_c \, a_i^{a+1} a_j^{b+1} a_k^{c+1} \, \frac{4\pi(a-1)!!(b-1)!!(c-1)!!}{(n+3)!!} \gamma^{(n)} , \tag{109}$$

where $\rho = \rho_c f(m^2)$, $\tilde{x}_i = x_i/a_i$, $d\tilde{V} = d\tilde{x}_1 \, d\tilde{x}_2 \, d\tilde{x}_3$; $i \neq j \neq k$; $n = a + b + c$. In order to have nonzero integrals, a, b, and c must be even, so that n is even. We have defined

$$\gamma^{(n)} = \frac{(n+3)!!}{4\pi(a-1)!!(b-1)!!(c-1)!!} \int f \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} , \tag{110}$$

where $\gamma^{(n)}$ is normalized to 1 for homogeneous ellipsoids. By transformation to reduced (dimensionless) spherical coordinates $\tilde{x}_1 = m \sin\theta \cos\phi$, $\tilde{x}_2 = m \sin\theta \sin\phi$, $\tilde{x}_3 = m \cos\theta$ and taking into account the ellipsoidal symmetry in $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, we obtain

$$\gamma^{(n)} = (n+3) \int_0^1 f m^{n+2} \, dm . \tag{111}$$

This expression depends only on the value of n, and not on the individual values of a, b, c, so it is valid just for ellipsoidal profiles of density.

2. The nth-order Coriolis tensor is

$$J^{a,b,c}_{i,j,k} = \int \rho \tilde{\phi} x_i^a x_j^b x_k^c \, dV = \rho_c \, a_i^{a+1} a_j^{b+1} a_k^{c+1} \int f \tilde{\phi} \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} = \rho_c \, a_i^{a+1} a_j^{b+1} a_k^{c+1} \, \frac{4\pi(a-1)!!(b-1)!!(c-1)!!}{(n+3)!!} \eta^{a,b,c}_{i,j,k} . \tag{112}$$

With the definition

$$\eta^{a,b,c}_{i,j,k} = \frac{(n+3)!!}{4\pi(a-1)!!(b-1)!!(c-1)!!} \int f \tilde{\phi} \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} , \tag{113}$$

where $\eta^{a,b,c}_{i,j,k}$ depends on the specific values of a, b, c and is normalized to 1 in the linear homogeneous case, the integral may be calculated explicitly once we know the functional form of $\tilde{\phi}$.

3. The $(n+2)$th- and nth-order tensors associated with the kinetic term are, respectively,

$$2T^{a,b,c}_{i,j,k} = \int \rho u_i u_j x_i^a x_j^b x_k^c \, dV = \sum_{lm} (AZA^{-1})_{il} (AZA^{-1})_{jm} \int \rho \tilde{\phi}^2 x_l x_m x_i^a x_j^b x_k^c \, dV , \tag{114}$$

and

$$K^{a,b,c}_{i,j,k} = \int \rho \tilde{\phi}^2 x_i^a x_j^b x_k^c \, dV = \rho_c a_i^{a+1} a_k^{c+1} \int f \tilde{\phi}^2 \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} = \rho_c a_i^{a+1} a_j^{b+1} a_k^{c+1} \frac{4\pi(a-1)!!(b-1)!!(c-1)!!}{(n+3)!!} \alpha^{a,b,c}_{i,j,k} , \tag{115}$$

where we introduce the definition

$$\alpha^{a,b,c}_{i,j,k} = \frac{(n+3)!!}{4\pi(a-1)!!(b-1)!!(c-1)!!} \int f \tilde{\phi}^2 \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} , \tag{116}$$

with $\alpha^{a,b,c}_{i,j,k}$ normalized to 1 in the linear homogeneous case.

4. The $(n-2)$th-order diagonal tensor, associated with the kinetic energy of the internal streaming motion (pressure tensor), is

$$\Pi^{a-2,b,c}_{i,j,k} = \int P_i x_i^{a-2} x_j^b x_k^c \, dV = a_i^{a-1} a_j^{b+1} a_k^{c+1} \int \tilde{P}_i \tilde{x}_i^{a-2} \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} , \tag{117}$$

where for the linear homogeneous case $P_i = P_{ic}(1 - m^2)$.

5. The nth-order tensor associated with the potential is

$$W^{a,b,c}_{i,j,k} = \int \rho \, \partial_i \nu x_i^{a-1} x_j^b x_k^c \, dV = -2(\pi G \rho_c^2) \int f(m^2) C_i(x) x_i^a x_j^b x_k^c \, dV = -2(\pi G \rho_c^2) x_i^{a+1} x_j^{b+1} x_k^{c+1} \int f(m^2) C_i(x) \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} . \tag{118}$$

In the following, we consider generalized S-type Riemann ellipsoids were $\tilde{\phi} = \tilde{\phi}^*$. As follows from equation (109), a, b, c in $I^{a,b,c}_{i,j,k}$ must be even. Owing to the presence of x_3^2 in $\tilde{\phi}$, the tensors J, K, Π, W must also have a, b, c even.

For the second- and fourth-order virial equations, we have

$$\alpha^{(2)}_{ii} = \frac{15}{4\pi} \int f \tilde{\phi}^{*2} \tilde{x}_i^2 \, d\tilde{V} , \tag{119}$$

$$\eta^{(2)}_{ii} = \frac{15}{4\pi} \int f \tilde{\phi}^* \tilde{x}_i^2 \, d\tilde{V} , \tag{120}$$

$$\gamma^{(2)} = 5 \int_0^1 f m^4 \, dm . \tag{121}$$

The parameters of α, η, γ as defined here do not coincide with α, β, γ in Paper VIII, where $\tilde{\phi}$ was of the form $\tilde{\phi}(m^2)$.

According to the definitions (116), for fourth-order virial equations one has

$$\alpha^{2,2,0}_{1,2,3} = \frac{7!!}{4\pi} \int f \tilde{\phi}^{*2} \tilde{x}_1^2 \tilde{x}_2^2 \, d\tilde{V} , \tag{122}$$

$$\alpha^{2,0,2}_{1,2,3} = \frac{7!!}{4\pi} \int f \tilde{\phi}^{*2} \tilde{x}_1^2 \tilde{x}_3^2 \, d\tilde{V} , \tag{123}$$

$$\alpha^{4,0,0}_{1,2,3} = \frac{7!!}{12\pi} \int f \tilde{\phi}^{*2} \tilde{x}_1^4 \, d\tilde{V} , \tag{124}$$

and anaogous definitions for η.

The simplest forms may be obtained if $\tilde{\phi} = \tilde{\phi}(m^2)$. Also in this case a, b, c = even. From equations (113) and (116), we have for the nth-order virial equations

$$\eta^{a,b,c}_{i,j,k} = \eta^{(n)} = (n+3) \int_0^1 f \tilde{\phi}(m^2) m^{n+2} \, dm , \tag{125}$$

$$\alpha^{a,b,c}_{i,j,k} = \alpha^{(n)} = (n+3) \int_0^1 f \tilde{\phi}^2(m^2) m^{n+2} \, dm . \tag{126}$$

We note that $\alpha^{(n)}$ and $\eta^{(n)}$ depend only on n.

It is useful to introduce the two integrals

$$I_1 = \int \tilde{x}_i^a \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} = \frac{4\pi(a-1)!!(b-1)!!(c-1)!!}{(n+3)!!} \tag{127}$$

and

$$I_2 = \int (1 - m^2) \tilde{x}_i^{a-2} \tilde{x}_j^b \tilde{x}_k^c \, d\tilde{V} = \frac{8\pi(a-3)!!(b-1)!!(c-1)!!}{(n+3)!!} \, . \tag{128}$$

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