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Burrau's three-body problem in the post-Newtonian approximation

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ABSTRACT
We study the three-body problem, first introduced by Burrau in 1913, where three bodies start from rest at the corners of a Pythagorean right-angled triangle. The bodies are assumed to be black holes and therefore post-Newtonian corrections to the Newtonian equations of motion are applied at close encounters. We find that the nature of the solution changes gradually from the breakup of the system to a complete collapse of the system as the masses of the black holes are gradually increased. For a system of 5 pc in maximum initial diameter, the total collapse into a single black hole becomes dominant at the systematic mass of about $10^8\ M_\odot$. The maximum ejection speed of single bodies is about 1500 km s^{-1} and it occurs when the total mass is slightly above $10^8\ M_\odot$.

Key words: relativity – celestial mechanics, stellar dynamics – galaxies: nuclei.

1 INTRODUCTION

In 1913, Burrau published the orbits of three bodies starting at rest from the corners of a right-angled triangle. The sides of the triangle were 3, 4 and 5 distance units, and the masses of the bodies opposite to each side were 3, 4 and 5 mass units, respectively. With no computers available to perform numerical orbit integration at that time, Burrau was able to follow the orbits only through a limited time-span.

Burrau's problem was again attacked with the aid of modern computers by Szebehely & Peters (1967). They followed the evolution of the system until it finally disintegrated into a binary and a single body which were in a hyperbolic relative orbit. The orbits are described in detail by Szebehely & Peters (1967) and an illustration of the initial and final segments of the orbits is also found in Valtonen & Mikkola (1991).

Since 1967, much numerical work has been carried out on the orbits of three-body systems (see, e.g., Valtonen 1988). The significance of Burrau's problem is today seen in the way it illustrates a typical solution to a bound three-body system of zero angular momentum. The dynamical instability leading to a breakup is a general property of the three-body system. The speed of escape of the single body from the binary is generally somewhat below the mean orbital speeds of the temporary binaries which exist for a short time in a bound triple system. In astrophysics, three-body systems are common, but the results from the point-mass Newtonian solutions are not always applicable. This is especially true for tight three-black-hole systems. Not only must collisions be taken into account, but also the orbits are modified according to the differences between Einstein's theory of general relativity and the Newtonian theory. In this work we study the effects of the post-Newtonian corrections to the orbits of three bodies. We choose Burrau's problem again as an illustrative example. It is not a single problem any more, however, since it is not scale-free in the post-Newtonian approximation (PNA), but we get a one-parameter family of orbits. One of the important points to notice about the orbit of Burrau's problem, and three-body orbits in general, is that they are made up of a series of close encounters between different bodies in turn. In the solution by Szebehely & Peters (1967), for example, the closest approach distance is about 10^{-4} times the original distance of the bodies. Thus the relativistic corrections are generally important only for two bodies at a time, and the motion of the distant third body may be treated as usual in the Newtonian manner. This state of affairs allows us to simplify the calculation and to use the PNA corrections previously worked out in the two-body problem.

2 THE ORBIT INTEGRATOR

In the PNA of order 2.5, the acceleration $\boldsymbol{a}$ of the first body in the two-body system can be written as follows (Damour & Deruelle 1981a,b; Damour & Schäfer 1987; Soffel 1989):

$$\boldsymbol{a}+\boldsymbol{a}_0+c^{-2}\boldsymbol{a}_2+c^{-4}\boldsymbol{a}_4+c^{-5}\boldsymbol{a}_5, \tag{1}$$

where c is the speed of light and the various $\boldsymbol{a}$ terms are given by

$$\boldsymbol{a}_0=-Mn_2\boldsymbol{n}/r^2,$$

$$\boldsymbol{a}_2=(Gm_2/r^2)[\boldsymbol{n}(-\boldsymbol{v}_1^2-2\boldsymbol{v}_2^2+4S_{12}+\tfrac{3}{2}S_2^2+5Gm_1/r+4Gm_2/r)+\boldsymbol{v}(4S_1-3S_2)], \tag{2}$$

$$
\begin{aligned}
\boldsymbol{a}_4 = (Gm_2/r^2)\{\boldsymbol{n}[&-2\boldsymbol{v}_2^4 + 4\boldsymbol{v}_2^2 S_{12} - 2S_{12}^2 + \tfrac{3}{2}\boldsymbol{v}_1^2 S_2^2 \\
&+ \tfrac{9}{2}\boldsymbol{v}_2^2 S_2^2 - 6S_{12}S_2^2 - \tfrac{15}{8}S_2^4 \\
&+ (Gm_1/r)(-\tfrac{15}{4}\boldsymbol{v}_1^4 + \tfrac{5}{4}\boldsymbol{v}_2^2 - \tfrac{5}{2}S_{12} + \tfrac{39}{2}S_1^2 - 39\,S_1S_2 + \tfrac{17}{2}S_2^2) \\
&+ (Gm_2/r)(4\boldsymbol{v}_2^2 - 8S_{12} + 2S_1^2 - 4S_1S_2 - 6S_2^2)] \\
&+ \boldsymbol{v}[\boldsymbol{v}_1^2 S_2 + 4\boldsymbol{v}_2^2 S_2 - 4S_{12}S_1 \\
&+ 4S_{12}S_2 - 6S_1S_2^2 + \tfrac{9}{2}S_2^3 \\
&+ (Gm_1/r)(-\tfrac{63}{4}S_1 + \tfrac{55}{4}S_2) + (Gm_2/r)(-2S_1 - 2S_2)]\} \\
&+ (G^3 m_2/r^4)\boldsymbol{n}[-\tfrac{57}{4}m_1^2 - 9m_2^2 - \tfrac{69}{2}m_1 m_2]
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\boldsymbol{a}_5 = \tfrac{4}{5}(G^2 m_1 m_2/r^3)[&\boldsymbol{v}(-\boldsymbol{v}^2 + 2Gm_1/r - 8Gm - 2/r) \\
&+ \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{v})(3\boldsymbol{v}^2 - 6Gm_1/r + \tfrac{52}{3}Gm_2/r)].
\end{aligned}
\tag{4}
$$

Here the unit vector $\boldsymbol{n} = (\boldsymbol{x}_1 - \boldsymbol{x}_2)/r$, where the position vectors of the bodies are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and their mutual distance is r. The masses of the two bodies are m_1 and m_2, and G is the gravitational constant. The velocities of the two bodies are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and their difference $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$. The various S symbols are abbreviations for the scalar products $S_1 = \boldsymbol{n}\cdot\boldsymbol{v}_1$, $S_2 = \boldsymbol{n}\cdot\boldsymbol{v}_2$ and $S_{12} = \boldsymbol{v}_1\cdot\boldsymbol{v}_2$.

The actual numerical integration was carried out using the equations of motion derivable from the time-transformed Hamiltonian

$$
\tilde{H} = [H - E + \textstyle\sum \boldsymbol{r}_k\cdot\boldsymbol{F}_k(t)]/L\,, \tag{5}
$$

where H is the Newtonian Hamiltonian, E is its value (obtained by numerical integration) and L is the corresponding Lagrangian (Zare & Szebehely 1974). The external force terms $\boldsymbol{F}_k(t)$, which are considered to be functions of time only when deriving the equations of motion, consist of the relativistic corrections due to the $\boldsymbol{a}_2$, $\boldsymbol{a}_4$ and $\boldsymbol{a}_5$ contributions.

Experimentation has convinced us that the use of (perturbed) regularization (Mikkola & Aarseth 1993) is not of much help in this problem. The reason is that the $\boldsymbol{a}_4$ term in particular is so much more strongly singular than the Newtonian problem. Instead we chose to use the above Hamiltonian and a Bulirsch–Stoer (1966) integrator with a very small (10^{-14}) one-step error tolerance.

The most obvious effect of the post-Newtonian corrections $\boldsymbol{a}_2$ and $\boldsymbol{a}_4$ is the relativistic precession of the two-body orbit (see Fig. 1 for an example). The 2.5th-order correction, i.e. $\boldsymbol{a}_5$, causes loss of orbital energy, and is related to the gravitational radiation emitted by the system. At this order the radiation is considered isotropic. Higher order terms lead to anisotropy, which may be important in some situations (Fitchett 1983; Redmount & Rees 1989). The influence of these higher order terms is not considered here. Fig. 2 illustrates the final evolution of the binary system of Fig. 1. The orbits of the two black holes become nearly circular, and there is a catastrophic plunge together in half the orbital revolution, when their separation r becomes equal to $6G(m_1 + m_2)/c^2$ (Fitchett 1983).

We emphasize that the validity of the PNA 2.5th-order approximation is questionable at the end of the orbit calculation, but it is interesting that the approximation reproduces qualitatively the 'last stable orbit' behaviour of the merging of two black holes.

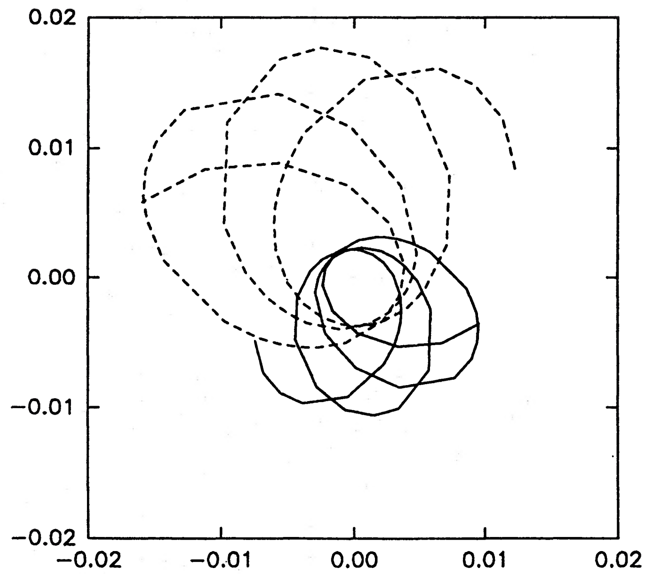

Figure 1. The orbits of two black holes of masses 2×10^9 (broken line) and 4×10^9 $M_\odot$ when the semimajor axis of their relative orbit is initially about $a = 0.0175$ pc and eccentricity $e \simeq 0.6$ pc. Distance unit = 1 pc.

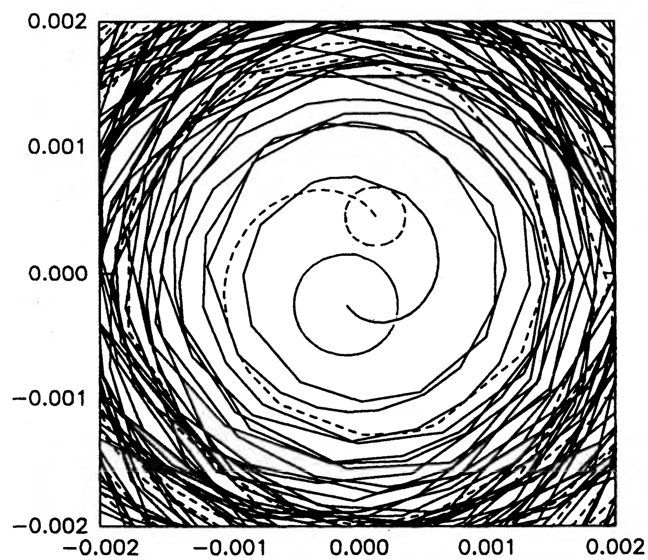

Figure 2. The orbits of two black holes of masses 2×10^9 (broken line) and 4×10^9 $M_\odot$ just before they merge together. The approximate sizes of the Schwarzschild radii of the two black holes are drawn as circles at the points where the orbital integration was terminated. Distance unit = 1 pc.

3 ORBITS ON DIFFERENT SCALES

Our starting point is a near-Newtonian system. We choose a mass unit of 10^5 $M_\odot$ and a distance unit of 1 pc. Then even the closest encounter in Burrau's original problem at 4×10^{-4} distance units is more than 10^3 times greater than the merger distance. This closest encounter occurred at time $t = 15.83$ units in the system of Szebehely & Peters (1967). For simplicity, to which the times we refer in the following discussion are of this system of units.

Our three-black-hole orbit follows the Newtonian solution fairly well until $t \simeq 30$ units. After that there is a divergence from the Newtonian result, even though qualitatively the solution is the same: the system breaks up into a binary and an escaping third body at $t = 55$ units (see Fig. 3). The escape speeds are 2 km s^{-1} for the binary and 4 km s^{-1} for the single body.

The second case has a mass unit of 10^6 M$_\odot$ and a distance unit of 1 pc. Now the closest approach distance of Burrau's original problem is still more than 10^2 times the merger distance, and thus no mergers take place. The final result is again the break up into a binary with escape speed of 6 km s^{-1} and a single body of escape speed 18 km s^{-1}. The system breaks up at $t = 50$ units. The orbit is illustrated in Fig. 4.

In the third case, the mass unit is increased to 10^7 M$_\odot$, while the distance unit is still 1 pc. The original close encounter distance is now only somewhat over 10 times the merger distance and the relativistic effects become noticeable. The black holes still go past the critical approach at $t = 15.83$ time units, but the subsequent orbit is different from the Newtonian solution. Fig. 5 illustrates the orbit between $t = 16.5$ and 33 units. The final breakup occurs at $t = 43$ units, with escape speeds of 918 and 306 km s^{-1}

When we increase the black hole mass unit to 10^8 M$_\odot$, keeping the distance unit at 1 pc, it becomes obvious that we will not get past the collision barrier at $t = 15.83$ units. The first collision takes place at $t = 15.57$ and the second at $t = 17.1$. A single black hole forms at the centre of mass of the system. The orbits are shown in Fig. 6.

Going further to a mass unit of 10^9 M$_\odot$, the mergers happen even sooner at $t = 1.9$ and $t = 3.0$ units (Fig. 7). In this scale the black holes obviously have no chance of getting past each other without mergers.

We notice that there is a limiting mass-scale somewhere around 10^7 M$_\odot$, at which the system still breaks up into two parts. When we come towards this limit from below, the breakup speed increases. Going over this limit, collisions start to dominate. An interesting question is what is the maximum ejection speed that is possible. The answer would be of interest in applications to galactic nulcei where three

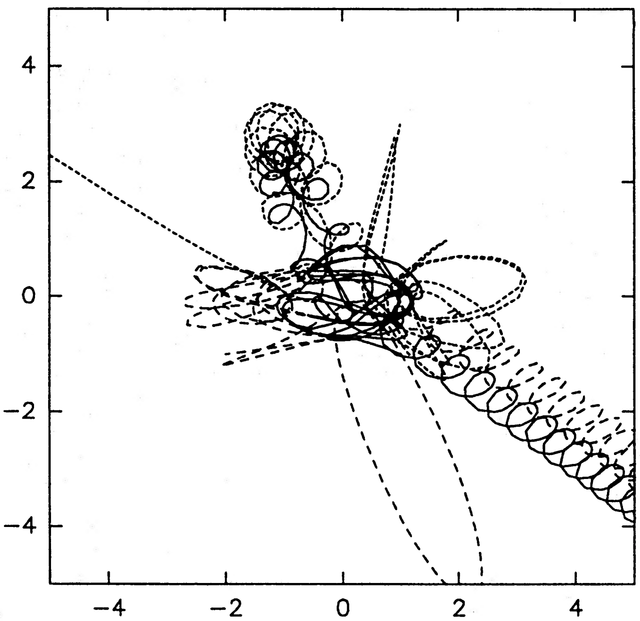

Figure 4. Burrau's problem with mass unit 10^6 M$_\odot$. The orbits are shown from $t = 0$ onwards and the ejection takes at $t = 50.4$ units. Distance unit = 1 pc.

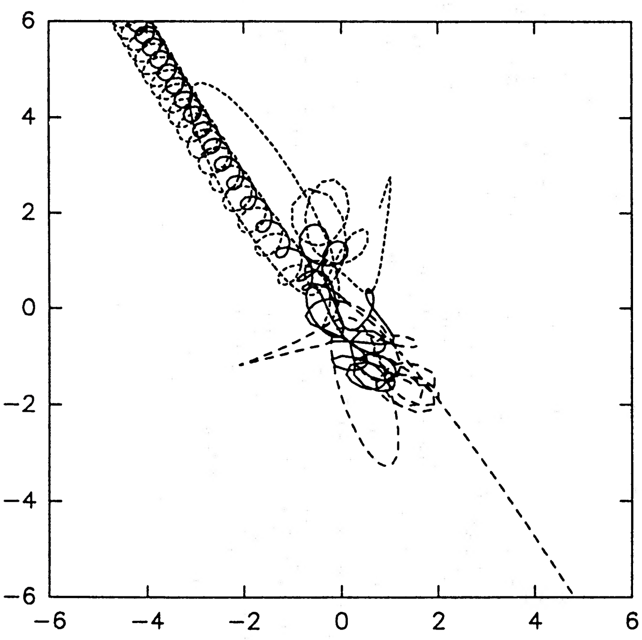

Figure 3. Burrau's problem with mass unit 10^5 M$_\odot$. The orbits are shown after $t = 29.8$ units. The ejection of a single body takes place at $t = 55.4$ units. The single object escapes with speed 4 km s^{-1}. Distance unit = 1 pc.

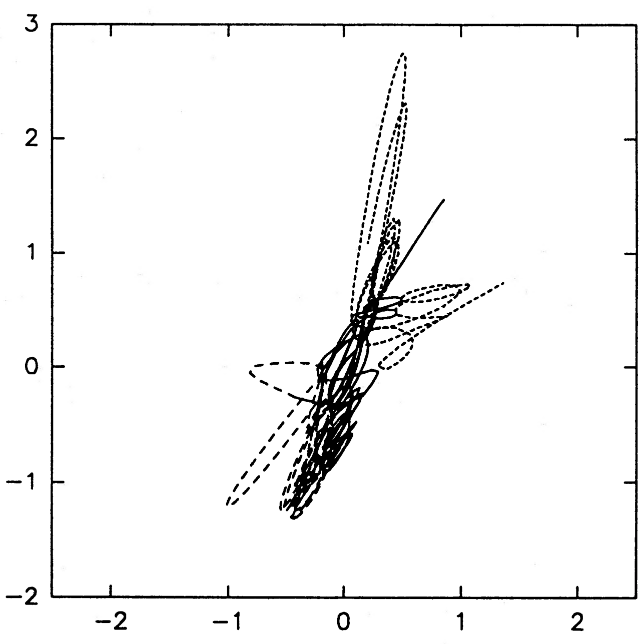

Figure 5. Burrau's problem with mass unit 10^4 M$_\odot$. The orbit is shown from $t = 16.5$ to 33.2. Distance unit = 1 pc.

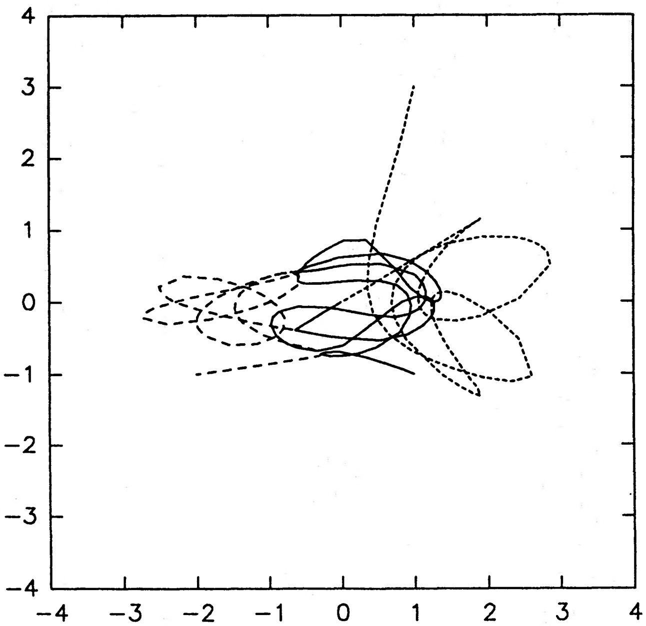

Figure 6. Burrau's problem with mass unit 10^8 $M_\odot$. The orbit is shown from $t=0$ until a single black hole forms at $t=17.1$. Distance unit = 1 pc.

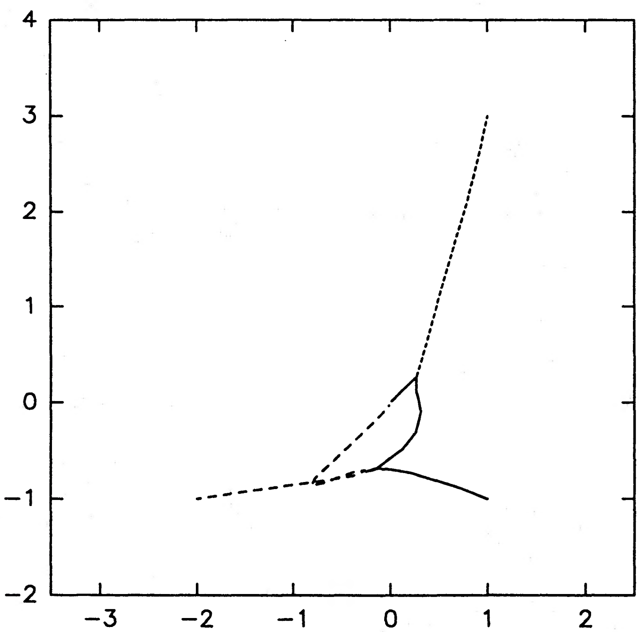

Figure 7. Burrau's problem with mass unit 10^9 $M_\odot$. The orbit is shown from $t=0$ until a single black hole forms at $t=3.0$. Distance unit = 1 pc.

black holes may have been accumulated in successive mergers of galaxies.

We have solved Burrau's problem with the mass unit being randomly selected in the range 6×10^6 to 2×10^7 $M_\odot$. We find that the ejections still dominate at 6×10^6 $M_\odot$, but a transition to collision-dominated dynamics takes place at about 8×10^6 $M_\odot$ when the total mass of the system is 10^8 $M_\odot$. At 10^7 $M_\odot$, the end result is generally a collision.

The largest values of the escape speed occur just beyond this limit; they are about 1500 km s^{-1} for single bodies and 500 km s^{-1} for binaries. Note that the limiting mass is directly proportional to the initial size of the system, while the maximum escape speeds are independent of scaling.

We note that the typical escape speed of a single body of about 400 km s^{-1} near the transition mass-scale of 10^8 $M_\odot$ is of the same order of magnitude as the escape speed from the centre of a small galaxy, but is no higher than the typical velocity dispersion in centres of giant elliptical galaxies. Thus ejections of black holes from galaxies are not a common process, if the nuclei of galaxies contain three-black-hole systems with zero angular momentum. Ejection probabilities are much greater, and ejection speeds higher, for black hole systems of relatively large angular momentum (Valtonen et al. 1994).

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