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ONE-DIMENSIONAL MERGING OF MAGNETIC FIELDS WITH COOLING

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ABSTRACT

A correct understanding of the process of magnetic reconnection is very important for the proper understanding of many astrophysical situations in which magnetic fields are involved. The main element in magnetic reconnection is the existence of a very thin layer over which the magnetic field reverses. In order to understand how such a layer forms spontaneously, we consider an initial one dimensional equilibrium with magnetic field going linearly through zero. Then we assume that plasma starts to cool at a rate proportional to pressure. This throws the situation out of equilibrium, and adiabatic compression commences in order to keep the situation in equilibrium. This leads quickly to a narrow layer across which the magnetic field reverses, and in which the resistivity destroys the magnetic field. We are able to analyze the resistive evolution analytically by a boundary layer analysis. It turns out that the layer forms in a cooling time multiplied by the logarithm of the magnetic Reynolds number. After the layer forms all the magnetic flux is destroyed in a few more cooling times.

Subject headings: MHD — plasmas

1. INTRODUCTION

An important element in magnetic reconnection is the existence of a very thin layer over which the magnetic field reverses. For a highly conducting plasma, this layer must be thin enough for resistive effects to be important.

One can easily understand how this layer can form in the case of Earth's magnetosphere boundary with the solar wind. Here two distinct plasmas carrying distinct magnetic fields are brought together by the flow of the solar wind. On the other hand, it is not simple to see how such a layer would form in the middle of a single plasma if the boundary conditions on the magnetohydrodynamic fluid are smooth, although on occasion it can happen.

If it does happen that fields of opposite signs begin to be brought together, then Parker (1963) and Sweet (1958) would predict that these field lines will reconnect at a certain rate, once the layer becomes thin enough. However, reconnection removes the magnetic lines as they are brought in and will tend to keep the layer from becoming too thin. This will tend to limit the rate of reconnection.

From these remarks it is clear that for the understanding of reconnection, it is as important to determine the physics of layer formation as it is to understand the magnetohydrodynamic of the plasma in the layer. Besides, it is often the case that more energy is released during the layer formation than during the actual reconnection itself.

There are two possible ways that a layer might form: either (1) it forms spontaneously due to some physical process or (2) it forms by external forcing due to changing boundary conditions. In this paper we consider an example of the first process. The process we invoke is a hypothetical sort of cooling of the plasma. An example of the second kind is illustrated by the solar corona. As Parker (1963) has emphasized, motion of footpoints of magnetic field lines in the solar surface may lead to current layer formation through a change in the topology of the magnetic field lines.

For simplicity, we restrict ourselves to a purely one-dimensional situation. We take the magnetic field to be in the y-direction and varying with x and time. Let the initial dependence of the magnetic field on x be linear and vanish at $x = 0$. Let the system be bounded by rigid walls which by a choice of the unit of length are at $x = \pm 1$.

We take the initial plasma pressure to be uniform as in Figure 1*a*. Then clearly the forces are out of balance, the plasma is compressed, and the magnetic field gradient is modified as in Figure 1*b*. We assume that the plasma resistivity is small enough that the flux is frozen if the scale size of variation is of order unity. We ignore any transient motions between Figures 1*a* and 1*b* assuming they are somehow damped out.

(We could have started with a parabolic pressure profile whose gradient would balance the magnetic force, that is, so that $p + B^2/2$ is constant initially. However, the choice of an initially constant pressure leads to greater analytic simplicity. Since we are really only interested in the behavior near $x = 0$, we can make this simpler choice. Then adjustment to Figure 1*b* takes us to a slightly different initial state.)

Now, take Figure 1*b* as the initial equilibrium. It is a magnetohydrodynamic equilibrium and will stay constant for a long time, of order the resistive timescale $\sim 1/\eta$. Let us assume that by some process energy is removed from the plasma, say by radiation, so that the plasma cools slowly compared to the dynamic timescale. As the pressure drops, the system will go out of equilibrium, and further compression occurs. Clearly the plasma pressure near the origin must be maintained near $B_0^2/2$, where $2B_0$ is the initial value of B near the wall. As time proceeds, more and more cooling occurs, and more and more compression is needed to keep the pressure finite at the origin. This compression of matter toward the origin drags the magnetic field lines inward as long as the plasma resistivity is negligible. After a while the profiles are as in Figure 1*c*.

The pressure is very small away from the origin, and B is nearly constant. Close to the origin, the pressure is finite and drops quickly to zero, and the magnetic field changes rapidly in space near the origin. For zero resistivity, it is clear that the layer of finite pressure over which the magnetic field changes will continue to grow thinner unless cooling stops. No matter

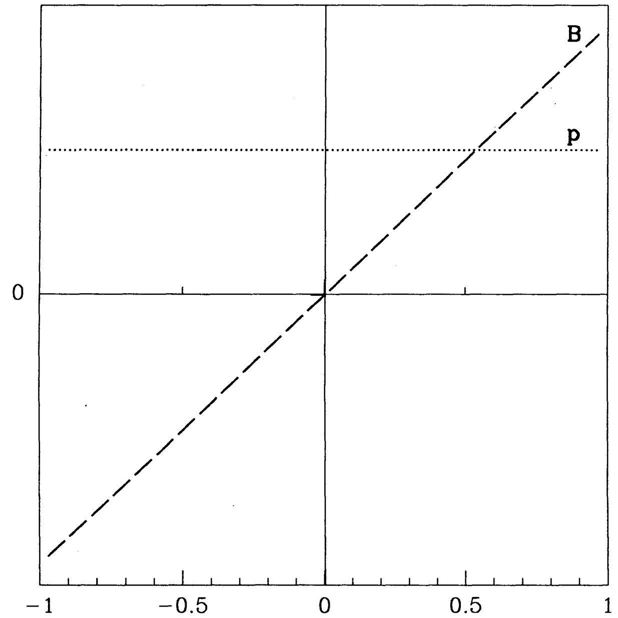

FIG. 1*a*

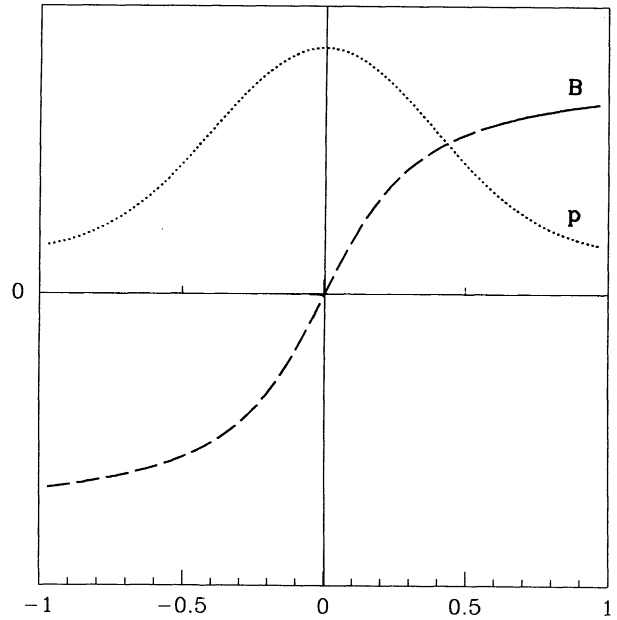

FIG. 1*b*

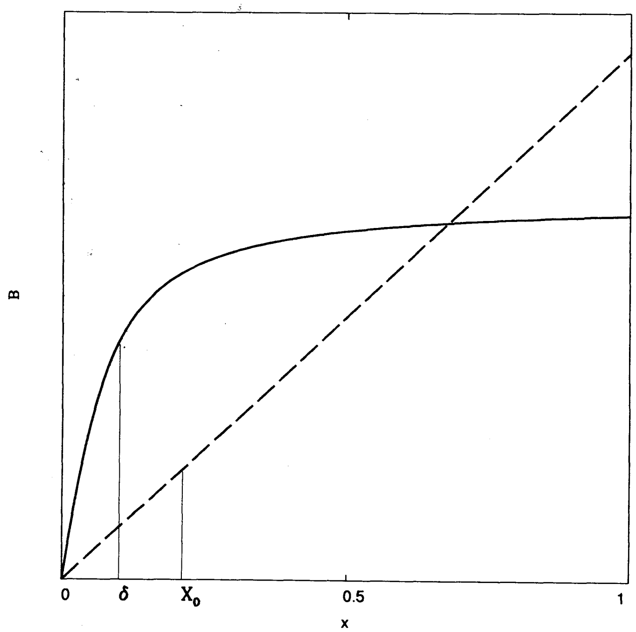

FIG. 1*c*

FIG. 1.—The time evolution of $B(x, t)$ (*dashed line*) and $p(x, t)$ (*dotted line*). (*a*) The initial nonequilibrium state. (*b*) The state after the first equilibrium is achieved. (*c*) The magnetic field after the layer of thickness δ has formed (*solid line*). Also the initial region X_0 that has moved into the layer is indicated. The dashed line is the initial magnetic field of (*a*).

how small the resistivity, the layer will become so thin that resistive terms are important and the magnetic field will be resistively destroyed; the two different signs of field will merge.

After resistivity becomes important, the plasma will continue to compress and flow toward the origin, and the field will start to be destroyed. Eventually all the field will be gone as well as the pressure.

The question we wish to discuss in this paper is: how long is it before the entire field is destroyed? It is clear that since the density at the origin grows without bound, and the pressure is fixed the temperature will drop to nearly zero. In general, this will reduce the rate of cooling. In addition, the amount of Joule heating from the resistive layer will tend to keep the plasma hot. Both of these effects will tend to stabilize the rate of collapse of the layer, and to slow down the reconnection rate. But the physical behavior in the boundary layer will be quite complex. Thus, instead of addressing this more physical problem, we choose to treat the rate of change of the plasma by a cooling that is proportional to the pressure. That is, we assume the rate of change of the entropy is given by

$$\frac{d}{dt} c_v \ln \frac{p}{\rho^\gamma} = -kc_v \,, \tag{1}$$

where k has the dimensions of a frequency, γ is the ratio of specific heats, and c_v is the specific heat at constant volume per unit mass. The entropy decrease due to heat loss minus ohmic heating, is a constant. Even this simplified problem has interesting ramifications which we now discuss.

It turns out that the merging rate of the oppositely directed fields are relatively insensitive to the value of γ. Therefore, for analytic simplicity we choose $\gamma = 2$, throughout the paper. We consider more general values of γ in the Appendix A.

We make the additional assumption that the motions are slow enough that the inertial terms are negligible, so after the first rapid transition from the nonequilibrium state with constant pressure to the equilibrium state discussed above, the plasma continues in an equilibrium state. That is, $p + B^2/2$ is constant in space.

In this introduction we discuss what should happen in a qualitative way, which will serve as a guide to the different boundary layer regions that are treated quantitatively in this paper. We restrict ourselves in this qualitative discussion to times during which only a small fraction of the magnetic flux has been destroyed.

Let δ be the thickness of the layer over which B changes from 0 to near its final value. The plasma in this region has a pressure of order the total pressure. This total pressure is nearly constant, at least until substantial amount of magnetic flux is destroyed. It is near $B_0^2/2$ where B_0 is the initial mean value of the magnetic field. The field quickly settles down to a constant value throughout the entire region outside $\delta \ll 1$.

Now, we know $p/\rho^2 = (p_0^2/\rho_0^2) \exp(-kt)$ by equation (1). Since p/ρ^2 decreases exponentially, this means that the density near the origin where $p \approx p_0$ is of order $\rho_0 \exp kt/2$, and the mass in this region which is

$$M \approx \delta \rho_0 e^{kt/2} \tag{2}$$

must originally have come from a region extending from 0 to X_0 where

$$M = \rho_0 X_0 . \tag{3}$$

Thus, the plasma which has entered the boundary layer comes from the region 0 to X_0 with

$$X_0 = \delta e^{kt/2} . \tag{4}$$

For early times when resistivity is negligible we expect the flux to be conserved. The flux in the region δ is of order

$$\Phi = \delta B_0 , \tag{5}$$

while the initial flux in the region out to X_0 is

$$\Phi = B_0 X_0^2 , \tag{6}$$

since B is initially linear. Thus,

$$\delta = X_0^2 \tag{7}$$

if flux is conserved. Combining equation (4) with equation (7) we get

$$\delta = e^{-kt} , \tag{8}$$

$$X_0 = e^{-kt/2} . \tag{9}$$

Now, B satisfies the one-dimensional equation

$$\frac{\partial B}{\partial t} = -\frac{\partial(vB)}{\partial x} + \eta \frac{\partial^2 B}{\partial x^2} . \tag{10}$$

The rate of change of B in the region δ is of order

$$\frac{\partial B}{\partial t} \approx -kB . \tag{11}$$

This is because δ changes exponentially at the rate k, and therefore, B changes from a small value to a value of order B_0 in the time it takes δ to change by a factor 2.

The resistive term is clearly of order

$$\eta \frac{\partial^2 B}{\partial x^2} = \left(\frac{\eta}{\delta^2}\right) B_0 \tag{12}$$

so resistivity is negligible and the above equations are correct for times such that

$$\eta/\delta^2 \ll k . \tag{13}$$

From equation (8) this gives

$$\delta \approx e^{-kt} \gg (\eta/k)^{1/2} \tag{14}$$

or $t \ll t_r \equiv \ln(\eta/k)/2k$.

For later times than this, resistivity becomes important and flux is no longer conserved so equation (8) is no longer valid.

Is it possible that δ stops decreasing and takes on the value $(\eta/k)^{1/2}$ of equation (14)? The answer is no. This can be seen as follows: if δ were constant, the amount of mass in the δ region increases. Since mass is conserved equation (4) is still valid, and X_0 grows exponentially. Since the amount of flux in the δ region is finite, an amount of flux,

$$\Phi = B_0 X_0^2 \gg B_0 \delta , \tag{15}$$

must either be destroyed or appear in the δ region. Since X_0 continues to grow, this flux is much larger than that in the δ region for times later then the time when δ becomes constant.

On the other hand, the rate of change of the total flux is

$$\frac{\partial}{\partial t} \Phi = \eta \int_0^1 \frac{\partial^2 B}{\partial x^2} dx \approx \frac{-\eta B_0}{\delta} . \tag{16}$$

From equation (4)

$$\frac{\partial}{\partial t} B_0 X_0^2 \approx k X_0^2 B_0 . \tag{17}$$

These two expressions are equal when $t = t_r$. But since X_0 increases exponentially, they cannot stay equal.

In other words, if δ is constant, more flux flows into the boundary layer than can be resistively destroyed. Thus, δ must continue to decrease. It is easy to obtain the time behavior of δ and X_0 when δ is smaller than $(\eta/k)^{1/2}$. We know that essentially all the initial flux up to X_0 must be resistively destroyed. Thus, the rate of resistive destruction given by equation (16) must be equal to the rate at which flux is disappearing equation (17).

Thus,

$$k X_0^2 \approx \eta/\delta . \tag{18}$$

We also know that since mass is conserved eq. (4) is still valid. Multiplying these two equations we have

$$k X_0^3 \approx \eta e^{kt/2} \tag{19}$$

or

$$X_0 \approx (\eta/k)^{1/3} e^{kt/6} . \tag{20}$$

Also, from equation (4)

$$\delta \approx (\eta/k)^{1/3} e^{-kt/3} . \tag{21}$$

From experience with boundary layer theory, it is tempting to scale the equations for B and p to δ and solve them together with the equilibrium condition. The density ρ satisfies a continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 , \tag{22}$$

and p is given by

$$p = \frac{p_0}{\rho^2} \rho^2 e^{-kt} . \tag{23}$$

Unfortunately, this will not work for the following reason:

From equation (22) and the known exponential behavior of ρ, we see that in the δ region v is of order $k\delta$. But from equation (10) we see that $\partial B/\partial t \approx kB$, $\partial(vB)/\partial x \approx kB$, and $\eta \partial^2 B/\partial x^2 \approx \eta/\delta^2$. Since, $\delta \ll (\eta/k)^{1/2}$, we see that these terms can only balance if $\partial^2 B/\partial x^2 = 0$, i.e., the last term vanishes to lowest order. Thus, from our physical condition we see that resistivity actually dominates over most of the δ region. This forces B to increase linearly up to nearly the maximum $\sim B_0$.

There is a second boundary layer region that occurs in the transition region passing from the resistivity dominated region to the ideal region. In this region the mass flow ρv is constant. Because ρ decreases greatly from its value at the center since $p \approx B_0^2 - B^2$ is becoming small, v becomes large enough for it to balance the resistive diffusion velocity in equation (10). In this region B, v, and p can be scaled as in the standard boundary layer theory.

The solution in this transition region matches the ideal solution at larger x and the linear solution at smaller x. The matching conditions are just those required to determine the rate of destruction of magnetic flux by merging. The mathematical details completely bear out the correctness of our qualitative discussion.

To summarize, there are three time periods of behavior. For early times the plasma is ideal. During this time the δ region shrinks exponentially until resistivity becomes important. The second period is a time of transition during which all terms balance and it is difficult to derive analytic results. During this time $\delta \approx (\eta/k)^{1/2}$. For later times, during the third period, δ again shrinks exponentially but at a slower rate than during the first period. During this third period three different spatial regions emerge in each of which the physical behavior is different. In the inner region resistivity dominates and B is linear in x. In the outer region the plasma is ideal and flux is frozen. Finally in the intermediate transition region plasma flow balances resistive diffusion. For clarity, we denote the inner region as region 1, the transition region as region 2, and the outer ideal region as region 3. We distinguish the early and late times as the ideal and resistivity periods, respectively.

The behavior in the resistive layer which we sketch above and which we derive in detail in the body of the paper is quite different from that usually assumed in the standard theories of the reconnection layer. That there might be such an inner region as region 1, where resistivity dominates is not generally appreciated. The implications of this are not clear to us.

In § 2, we formulate the problem precisely. In § 3, we treat the first time period during which the plasma is everywhere ideal and there is only one region to consider. In § 4 we treat the third resistive period. In § 4.1, the region 3 where the plasma is still ideal is analyzed. In § 4.2, the innermost region where resistivity dominates and in § 4.3 the transition region are analyzed. In § 5 the solution for these three regions are matched. This allows us to determine the time evolution of the flux until it all disappears. The implications of our solution are discussed in § 6. In Appendix A the case of general γ not necessarily equal to 2, is treated. Finally, in Appendix B certain assumptions made in § 4.3 in the transition region, are justified.

2. FORMULATION OF THE PROBLEM

To begin, we write the resistive MHD equations with cooling. We assume cooling is proportional to pressure. We consider a one-dimensional model. All quantities are independent of y and z, and the system possesses mirror symmetry about $x = 0$. The magnetic field $B(x, t)$ is a function of x and t, B is in the y direction, $p(x, t)$ is pressure, $v(x, t)$ is velocity, v is in x direction. The plasma resistivity which we take to be constant in space and time, is denoted by η, and k is a cooling coefficient. The equations (see eq. [1]) are

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(vB) = \eta \frac{\partial^2 B}{\partial x^2}, \quad (24)$$

$$\frac{\partial p}{\partial t} + \gamma p \frac{\partial v}{\partial x} + v \frac{\partial p}{\partial x} = -kp, \quad (25)$$

where γ is a ratio of specific heats. As determined in the introduction we assume $\gamma = 2$. Since the pressure cools uniformly we can write it (with a proper choice of units for p_0) in the form

$$p = e^{-kt}\rho^2, \quad (26)$$

where $\rho(x, t)$ is effectively the density. Substituting equation (26) into equation (25), we get the continuity equation for ρ

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0. \quad (27)$$

We assume that the initial density is sufficiently small that inertial terms are negligible at all times. Therefore, the equation of motion reduces to

$$\frac{\partial}{\partial x}\left(p + \frac{B^2}{2}\right) = 0 \quad (28)$$

or

$$p + B^2/2 = B_t^2/2, \quad (29)$$

where $B_t(t)$ is constant in space. It is determined once we know the amount of flux present. This is in turn determined by the amount of flux that has reconnected. The system is bounded by rigid walls, and we choose the units of length so that the walls are at $x = -1$ and $x = 1$ and $v = 0$ at $x = \pm 1$. Initially, we have mirror symmetry about $x = 0$, i.e., $B(-x) = -B(x)$, $p(-x) = p(x)$, and $v(-x) = -v(x)$. Since from equations (24)–(29) the system remains symmetric about $x = 0$, we can restrict ourselves to $x \geq 0$ and take the boundary conditions, as $v = 0$, $B = 0$ at $x = 0$. We take B initially to be linear in x

$$B = 2B_0 x, \quad (30)$$

where the constant B_0 is the initial mean value of B for $x \geq 0$ and is equal to the initial amount of magnetic flux. We take ρ to be initially constant and equal ρ_0, where

$$\rho_0^2 = 2B_0^2. \quad (31)$$

This choice for the initial situation clearly does not satisfy the equilibrium equation (28), but it makes many of the formula of the paper considerably simpler. What happens is that the plasma is accelerated inward. We imagine it to reach a new equilibrium by removing energy but preserving flux and p/ρ^2. This happens quickly and we consider this new equilibrium to be our initial one.

The problem is now completely determined and could be solved numerically. B, p, and ρ are given by equations (24), (25), and (26), and v is determined to maintain equation (28). This requirement with the boundary conditions $v = 0$ at $x = 0$, and $x = 1$ determines v. However, if η is small, it is possible to solve the problem by asymptotic analysis and matching. In this paper we carry out this analytical approach.

3. IDEAL TIME PERIOD

Because η is small, and B does not at first vary rapidly in x, the right-hand side in equation (24) is small and can be neglected. In the introduction the period of time when this is valid is termed the ideal time period. That is equation (24) reduces to the ideal MHD equation

$$\frac{\partial B}{\partial t} + v \frac{\partial B}{\partial x} + B \frac{\partial v}{\partial x} = 0. \quad (32)$$

However, after a certain time B varies rapidly with x in the region near $x = 0$. This leads to resistive reconnection, and the ideal equation is no longer applicable here. The time at which this happens is given at the end of the section.

Equations (27), (29), and (32) can be most easily solved using Lagrangian coordinates x_0, t, with $x = x(x_0, t)$, where x is a position of that fluid element at time t, that was initially at x_0. The velocity v is $(\partial x/\partial t)_{x_0}$. Since $v = 0$ at $x_0 = 0$ and $x_0 = 1$, our boundary conditions become $x(0, t) = 0$ and $x(1, t) = 1$. Making use of this relation and transforming equation (32) to x_0 and t we get

$$\frac{\partial}{\partial t}\left[B\left(\frac{\partial x}{\partial x_0}\right)_t\right]_{x_0} = 0 . \tag{33}$$

Integrating equation (33) we have

$$B(x, t)\left(\frac{\partial x}{\partial x_0}\right)_t = B(x_0, 0) . \tag{34}$$

Taking into account the initial conditions (30),

$$B(x, t) = \frac{B(x_0, 0)}{\partial x/\partial x_0} = \frac{2B_0 x_0}{\partial x/\partial x_0} . \tag{35}$$

Similarly, using equation (27), Lagrangian coordinates, and initial conditions (31) it is possible to show that

$$\rho = \frac{\rho(x_0, 0)}{\partial x/\partial x_0} = \frac{\rho_0}{\partial x/\partial x_0} . \tag{36}$$

Now using equations (26), (35), and (36) we can transform equation (29) to

$$2B_0^2 x_0^2\left(\frac{\partial x}{\partial x_0}\right)^{-2} + 2B_0^2\left(\frac{\partial x}{\partial x_0}\right)^{-2} e^{-kt} = \frac{A^2B_0^2}{2} , \tag{37}$$

where $A(t) \equiv B_t(t)/B_0$ and essentially measures the amount of unreconnected flux. Thus

$$\left(\frac{\partial x}{\partial x_0}\right)_t = \frac{2}{A(t)}\sqrt{x_0^2 + e^{-kt}} . \tag{38}$$

Now from equations (35)–(36) we have

$$B = \frac{A(t)B_0 x_0}{\sqrt{x_0^2 + \epsilon^2}} , \tag{39}$$

$$p = \frac{A^2(t)B_0^2 \epsilon^2}{2(x_0^2 + \epsilon^2)} , \tag{40}$$

$$\rho = \frac{A(t)\rho_0}{2\sqrt{x_0^2 + \epsilon^2}} , \tag{41}$$

where $\epsilon^2 \equiv e^{-kt}$.

Equations (39)–(41) give B, p, and ρ in terms of Lagrangian coordinates x_0, t. To express them in terms of x, t we must integrate equation (38). Making use of the boundary conditions $x(0, t) = 0$ we get

$$x = \frac{2}{A}\int_0^{x_0}\sqrt{x_0^2 + \epsilon^2}dx_0$$
$$= \frac{1}{A}\left[x_0\sqrt{x_0^2 + \epsilon^2} + \epsilon^2 \ln\left(\frac{x_0 + \sqrt{x_0^2 + \epsilon^2}}{\epsilon}\right)\right] . \tag{42}$$

Making use of equation (42) and the boundary condition at the wall $x(1, t) = 1$, we find A

$$A(t) = 2\int_0^1 \sqrt{x_0^2 + \epsilon^2}\, dx_0 = \sqrt{1 + \epsilon^2} + \epsilon^2 \ln\left(\frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon}\right) . \tag{43}$$

When $t = 0$ ($\epsilon^2 = 1$), $A = 2^{1/2} + \ln(1 + 2^{1/2})$ [$\epsilon = 1$ corresponds to the initial equilibrium after the readjustment from the constant density, as discussed in the introduction. At this time B has a simple dependence on x_0, $B = AB_0 x_0/(x_0^2 + 1)^{1/2}$, but x_0 is a complicated function of x given by eq. (42)].

From equation (43) it follows that

$$\dot{A} = -k\epsilon^2 \ln\left(\frac{1 + \sqrt{1 + \epsilon^2}}{\epsilon}\right) , \tag{44}$$

where dot means time derivative. The velocity of a fixed fluid element as a function of x_0 is

$$v = \frac{\partial x}{\partial t} = -\frac{2\dot{A}}{A^2}\int_0^{x_0}\sqrt{x_0^2 + \epsilon^2}\, dx_0 + \frac{1}{A}\int_0^{x_0}\frac{-k\epsilon^2}{\sqrt{x_0^2 + \epsilon^2}}\, dx_0$$
$$= \frac{-\dot{A}}{A}\, x + \frac{-k\epsilon^2}{A}\ln\left(\frac{x_0 + \sqrt{x_0^2 + \epsilon^2}}{\epsilon}\right) . \tag{45}$$

As ϵ^2 becomes very small at larger t the behavior of B, ρ and x with x_0 simplifies. From equation (43) we see that $A \approx 1$. From equation (39) we see that B becomes nearly constant and equal to B_0 (the initial total magnetic flux) for $x_0 \gg \epsilon$, or from equation (42), $x \gg \epsilon^2$. Also, $p \approx \epsilon^2 B_0^2/2x_0^2 \approx B_0^2/2x \ll B_0^2/2$, while $\rho \approx \rho_0/2x_0$ in the same region. For $x_0 \leq \epsilon$, or $x \leq \epsilon^2$, B changes rapidly from 0 to B_0 while p is of order $B_0^2/2$ and ρ is of order ρ_0/ϵ^2.

When $\epsilon^2 \ll 1$, equations (43), (44), and (45) become

$$A = 1 , \tag{46}$$

$$\dot{A} = -k\epsilon^2 \ln\left(\frac{2}{\epsilon}\right) , \tag{47}$$

$$v = k\epsilon^2 x \ln\left(\frac{2}{\epsilon}\right) - k\epsilon^2 \ln\left(\frac{x_0 + \sqrt{x_0^2 + \epsilon^2}}{\epsilon}\right) . \tag{48}$$

From equation (48) we verify that $v = 0$ at $x = x_0 = 1$ and at $x = x_0 = 0$.

In order to see graphically how B varies with x it is convenient to express our solution in scaled variables. Let $x' = xA/\epsilon^2$ and $x_0' = x_0/\epsilon$. In these variables equation (42) reduces to

$$x' = x_0'\sqrt{1 + x_0'^2} + \ln(x_0' + \sqrt{1 + x_0'^2}) \tag{49}$$

Let $B' = B/AB_0$, then from equation (39)

$$B' = \frac{x_0'}{\sqrt{1 + x_0'^2}} . \tag{50}$$

Thus, the scaled B is a fixed function of the scaled x and is independent of t. This scaled solution corresponds to the qualitative results given in § 1 (see eqs. [8] and [9]).

In Figure 2 we give the dependence of B' versus x', where x_0' ranges from 0 to 3. This behavior is correct as long as the term on the right-hand side of equation (24) is negligible; that is, the plasma is ideal. To find out how long this assumption is valid we transform this term to the scaled variables

$$\eta\frac{\partial^2 B}{\partial x^2} = \frac{\eta}{\epsilon^4} B_0 A^3 \frac{\partial^2 B'}{\partial x'^2} . \tag{51}$$

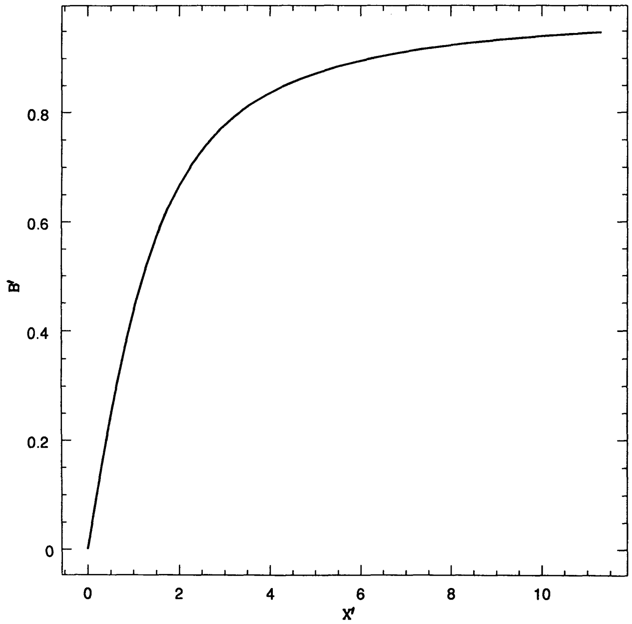

FIG. 2.—The scaled magnetic field vs. the scaled length during the ideal period.

Since B' is a finite function of x', we see that this is negligible if $\epsilon^4 \gg \eta/k$. After a time such that $\epsilon^4 \approx \eta/k$ [$t \approx \ln(k/\eta)/2k$] the ideal assumption breaks down near the reconnection region for finite x', that is $x \approx \epsilon^2$. This agrees with the estimate equation (14) of § 1.

4. THE RESISTIVE TIME PERIOD

From the results of the last section, we see that the resistivity becomes important near $x = 0$ when ϵ^4 is of the order η/k. The region near $x = 0$ is very complex at this time. However, when $\epsilon^4 \ll \eta/k$ this boundary layer region breaks into two regions where the behavior is simplified. In addition, for larger x, we still have the ideal behavior.

In § 1 we defined the time period during which $\epsilon^4 \approx \eta/k$ as the intermediate period, while the later times, $\epsilon^4 \ll \eta/k$, we defined as the resistive period. We skip the intermediate period and pass to the resistive period, during which the evolution can be followed by asymptotic matching.

We label the three regions 1, 2, and 3, where 1 is the region closest to the origin, 2 is the intermediate region, and 3 is the farthest away from the origin, where the field still varies so slowly that its behavior is ideal. By matching the solutions in each region in x, the time behavior of the reconnection will be determined as in the usual method of asymptotic matching. (Also the condition on the time that this matching works is given in Appendix B.) We solve for B and ρ in each region in turn. We begin with the ideal region.

4.1. *Region 3*

In our model the ideal MHD equations are valid in the entire region from 0 to 1 only when $\epsilon^4 \gg \eta/k$. During the later resistive period, when $\epsilon^4 \ll \eta/k$, ideal MHD equations apply only in the range away from $x_0 = 0$. This means, we no longer can use boundary condition $x(0, t) = 0$. Near zero we have to include resistive terms (regions 1 and 2). Our consideration of the ideal region will be similar to that in § 2. But now integrating equation (38) away from $x = 0$ we obtain in place of equation (42)

$$1 - x = \frac{2}{A}\int_{x_0}^{1} \sqrt{x_0^2 + \epsilon^2}\, dx_0 = \frac{1}{A}\left[\sqrt{1+\epsilon^2} - x_0\sqrt{x_0^2+\epsilon^2} + \epsilon^2 \ln\left(\frac{1+\sqrt{1+\epsilon^2}}{x_0 + \sqrt{x_0^2+\epsilon^2}}\right)\right], \tag{52}$$

where we still make use of boundary condition $x(1, t) = 1$, but because we no longer have $x(0, t) = 0$, A is not yet determined. Our boundary conditions on the ideal solution equation (52) are

$$x = 1\,, \quad x_0 = 1\,, \tag{53}$$

$$x = 0\,, \quad x_0 = X_0(t)\,, \tag{54}$$

where by $X_0(t)$ we denote the initial position of the fluid that is just moving into the boundary region at time t.

For the resistive period when $\epsilon^2 \ll \eta^{1/2} \ll 1$, it turns out, as we will show later, that the resistive region splits into two regions 1 and 2 in each of which the behavior is simple if $X_0 \gg \epsilon^2$. With this approximation for region 3, we can express $A(t)$ in terms of X_0. From equation (52) and the boundary condition (54) we get

$$A \approx 1 - X_0^2\,. \tag{55}$$

Similar to the derivation of equation (45), we find

$$v = \frac{\partial x}{\partial t} = -\frac{\partial(1-x)}{\partial t} = \frac{2\dot{A}}{A^2}\int_{x_0}^{1}\sqrt{x_0^2+\epsilon^2}\,dx_0 + \frac{k\epsilon^2}{A}\ln\left(\frac{2}{X_0+\sqrt{X_0^2+\epsilon^2}}\right), \tag{56}$$

where the dot indicates a time derivative.

$A(t)$ will be determined by matching these results to those of regions 1 and 2. We carry this out in § 5. There we will find that $\dot{A}/A$ is of order k so that the second term is much smaller than the first one and the expression for v simplifies to

$$v = \frac{\dot{A}}{A}(1 - x)\,. \tag{57}$$

In order to match to region 2 we need the mass flow m in the region 3

$$m \equiv -\rho v\,. \tag{58}$$

Using the expressions for ρ, equation (41), and v, equation (57), equation (58) becomes

$$m = \rho_0 \dot{X}_0(1 - x) = \sqrt{2}B_0\dot{X}_0(1 - x) \tag{59}$$

4.2. *Region 1*

We next consider region 1, the resistive region closest to the origin. As discussed in § 1 resistivity dominates in this region, so that the resistive term on the right-hand side of equation (24) is much bigger than the terms on the left-hand side. This means that to lowest order eq. (24) becomes

$$\eta\frac{\partial^2 B}{\partial x^2} = 0\,. \tag{60}$$

Therefore, B is a linear function of x

$$B = AB_0 \frac{x}{\delta} + B_1 , \tag{61}$$

where $\delta(t)$ is the extent of the region 1, and is a function of time. If $x \approx \delta$ then $B \approx B_t = AB_0$. We will show below that B_1 is small for $x < \delta$. In order to find the density ρ to the lowest order, we drop B_1 in equation (61) and substitute the result and the expression (26) into equation (29) obtaining

$$\rho = \sqrt{\epsilon\left(\frac{A^2B_0^2}{2} - \frac{A^2B_0^2x^2}{2\delta^2}\right)} = \frac{AB_0}{\sqrt{2}}\sqrt{\left(1 - \frac{x^2}{\delta^2}\right)}e^{kt/2} . \tag{62}$$

Differentiating equation (62) with respect to t we have

$$\frac{\partial\rho}{\partial t} = \left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\frac{x^2/\delta^2}{1 - x^2/\delta^2}\right)\rho . \tag{63}$$

In order to match the solutions of regions 1 and 2 we need the mass flow. Integrating equation (27) we get

$$m \equiv -\rho v = \int_0^x \frac{\partial\rho}{\partial t}\, dx . \tag{64}$$

We consider the mass flow near the boundary of the region 1, $x \to \delta$. Substituting equation (63) into equation (64) and integrating from 0 to δ we obtain

$$m(x \to \delta) = \frac{\pi}{4\sqrt{2}} AB_0\, \delta e^{kt/2}\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right) . \tag{65}$$

We can also find velocity v by dividing equation (65) by equation (62)

$$v = \frac{\pi}{4}\frac{\delta}{\sqrt{1 - x^2/\delta^2}}\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right) . \tag{66}$$

Employing equation (64) for x not near δ we may find v throughout region 1. We can then go to next order in equation (24) to find B_1

$$\frac{\partial}{\partial t}\left(AB_0 \frac{x}{\delta}\right) + \frac{\partial}{\partial x}\left(vAB_0 \frac{x}{\delta}\right) = \eta \frac{\partial^2 B_1}{\partial x^2} . \tag{67}$$

Thus for $\delta \ll (\eta/k)^{1/2}$, B_1 is small, compared to AB_0, as long as the square root in equation (66) is finite, i.e., x is not too close to δ. However, as $x \to \delta$ it is necessary to include B_1 in equation (61), and for this we must pass to region 2. The linear behavior in B breaks down near $x = \delta$. This can be seen by integrating equation (67) once to get

$$\frac{\partial B_1}{\partial x} = \frac{\pi}{4} AB_0 \frac{\delta}{\eta\sqrt{1 - x^2/\delta^2}}\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right) , \tag{68}$$

which becomes comparable with AB_0/δ when $x - \delta \approx \delta^3k^2/\eta^2$.

4.3. *Region 2*

Following the discussion in the end of the previous section we next consider the intermediate region 2. The magnetic field B in this region is close to B_t, and we set

$$b = B_t - B , \tag{69}$$

where $b(x, t)$ is a function of x, t, and $b \ll B_t$. In order to find the density ρ we substitute expressions (26) and (69) into equation (29):

$$\rho^2 e^{-kt} = \frac{B_t^2}{2} - \frac{B^2}{2} \approx B_t b . \tag{70}$$

From equation (70) it follows that

$$\rho = \sqrt{B_t b}e^{kt/2} . \tag{71}$$

The velocity v can be expressed in terms of mass flow $m \equiv -\rho v$, which is nearly constant in region 2 (Appendix B). From equation (71)

$$v = \left(-\frac{me^{-kt/2}}{\sqrt{B_t}}\right)\frac{1}{\sqrt{b}} . \tag{72}$$

We may now determine b from equation (24). It is the case that $\partial B/\partial t$ is small, as is justified in Appendix B. Thus,

$$\frac{\partial}{\partial x}(vB) = \eta \frac{\partial^2}{\partial x^2}(B_t - b) = -\eta \frac{\partial^2 b}{\partial x^2} . \tag{73}$$

Using expression (72) and integrating equation (73) once we get

$$Bv + \eta \frac{\partial b}{\partial x} = C . \tag{74}$$

The constant C can be evaluated by matching the expression for B to that of region 1 for x near δ ($B = B_t x/\delta + B_1$). Thus,

$$Bv + \eta \frac{\partial b}{\partial x} = Bv \frac{-\eta B_t}{\delta} - \eta \frac{\partial B_1}{\partial x} . \tag{75}$$

Taking into account that in the region 1, $Bv = \eta\, \partial B_1/\partial x$, we obtain $C = -\eta B_t/\delta$. Since b and v are small we can replace B by B_t in equation (74). Substituting equation (72) for v into equation (74), we get

$$-\sqrt{B_t} m\epsilon \frac{1}{\sqrt{b}} + \eta \frac{\partial b}{\partial x} = -\frac{\eta B_t}{\delta} . \tag{76}$$

This can be written as

$$\frac{\partial b}{\partial x} = \frac{\sqrt{B_t} m\epsilon}{\eta\sqrt{b}} - \frac{B_t}{\delta} = \frac{B_t}{\delta}\left(\sqrt{\frac{b_c}{b}} - 1\right) , \tag{77}$$

where

$$b_c \equiv \frac{m^2 e^{-kt}\delta^2}{\eta^2 B_t} . \tag{78}$$

It is convenient to introduce a new variable

$$y^2 \equiv b/b_c . \tag{79}$$

Then equation (77) becomes

$$\frac{2y^2\, dy}{y - 1} = -\frac{B_t}{b_c\, \delta}\, dx . \tag{80}$$

Integrating equation (80) yields

$$y^2 + 2y + 2\ln(y - 1) = \frac{B_t}{b_c\, \delta}(c - x) , \tag{81}$$

where c is a constant of integration. It is easy to show that, for small x, $y^2 \equiv b/b_c \approx B_t(c - x)/b_c\, \delta$. To match this solution to equation (61) of region 1 we must have $c = \delta$. As $x \to \infty$, $y \to 1$, which means $b \to b_c$. A plot of b/b_c versus $(x/\delta - 1)B_t/b_c$ is given in Figure 3.

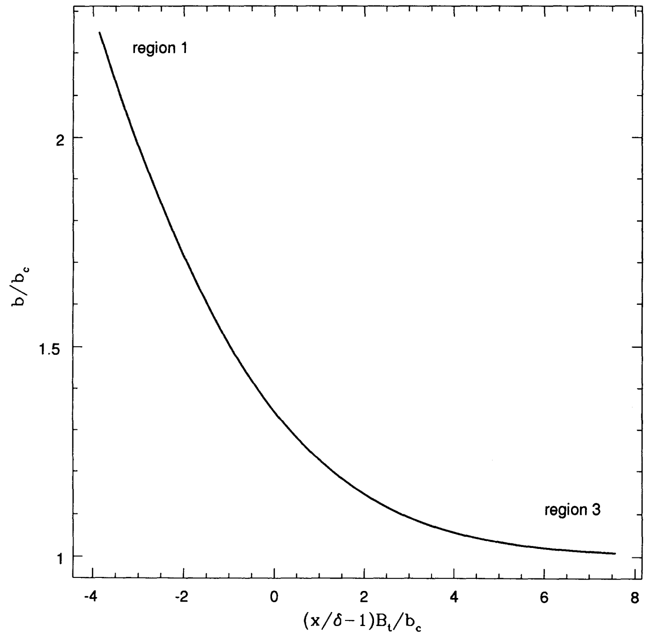

FIG. 3.—The variation of the magnetic field in the intermediate region 2. b/b_c is plotted vs. $B_t(x/\delta - 1)/b_c$. The matching to the regions 1 and 3 is indicated.

5. MATCHING CONDITIONS

The solution for the magnetic field B is given by equation (61) in the resistive region 1, equation (69) in the intermediate region 2, and equation (39) in the ideal region 3. The unknowns are m, δ, and X_0, as well as $\dot{\delta}$, $\dot{X}_0$. We match the magnetic field for large x in region 2 (in this limit $b \to b_c$, so $B \to B_t - b_c$) with B given in the region 3 in the limit $x_0 \to X_0$:

$$B = B_t - b_c = \frac{B_t x_0}{\sqrt{x_0^2 + e^{-kt}}} \approx B_t\left(1 - \frac{e^{-kt}}{2X_0^2}\right). \quad (82)$$

In equation (82) we take $e^{-kt} \ll X_0$, because we are interested in the later time period, see discussion in § 3. From equations (82) and (78) we get

$$b_c \equiv \frac{m^2 e^{-kt}\delta^2}{\eta^2 B_t} = B_t \frac{e^{-kt}}{2X_0^2}. \quad (83)$$

Since the mass flow m is constant throughout the region 2, we match the expression for m in region 1, equation (65) as $x \to \delta$, with m in region 3, equation (59) as $x_0 \to X_0$ and $x \to 0$.

$$m = \frac{\pi}{4\sqrt{2}} A B_0 \delta e^{kt/2}\left[\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right] = \sqrt{2} B_0 \dot{X}_0. \quad (84)$$

This relation involves the derivatives of X_0 and δ. Later in this section we treat the entire later time period, during which all the flux is reconnected, that is A goes from 1 to 0 and X_0 goes from 0 to 1. However, at the beginning of this resistive period, when X_0 is small and A is near 1, the behavior of δ and X_0 is exponential, as has been indicated in the introduction. To show this we set $\alpha = 2\dot{X}_0/X_0$ and $\beta = -\dot{\delta}/\delta$, and assume that α and β are constant. Then from equations (83) and (84),

$$\delta = \frac{A\eta}{\alpha X_0^2}, \quad (85)$$

$$\alpha X_0 = \frac{\pi}{4} \frac{A\eta}{\alpha X_0^2}\left[-\alpha X_0^2 + \left(\frac{k}{2} - \beta\right)A\right] e^{kt/2}. \quad (86)$$

Taking $X_0 \ll 1$, $A = 1$ we get

$$X_0 = e^{kt/6}\eta^{1/3}C_1^{1/3}, \quad (87)$$

$$\delta = e^{-kt/3}\eta^{1/3}\frac{1}{\alpha}C_1^{-2/3}, \quad (88)$$

where $C_1 = \pi(k/2 - \beta)/4\alpha^2$. Thus $\alpha = k/3$ and $\beta = k/3$ are constants as we assumed and $C_1 = 1.18/k$. (These results are essentially in agreement with eqs. [20] and [21] in § 1.]

Now we consider the general case when A is no longer near 1, but changes from 1 to 0. From equations (85) and (86) with $A = (1 - X_0^2)$ (eq. [55]) and from our definition of α we get

$$\alpha = \frac{2}{X_0}\frac{dX_0}{dt} = \frac{(1 - X_0^2)\eta}{\delta X_0^2}. \quad (89)$$

It is convenient to introduce new variables: $\tau = kt$, $\bar{\delta} = \delta k/\eta$, $\xi = e^{(\tau - \tau_0)/2}$, where $\tau_0 \equiv 2\ln k/\eta$. Transforming equations (89) and (86) to these new variables we get

$$\frac{1}{2}\frac{dX_0}{d\xi} = \frac{1}{\xi}\frac{(1 - X_0)^2}{2\bar{\delta}X_0}, \quad (90)$$

$$\frac{1}{2}\frac{d\bar{\delta}}{d\xi} = \frac{1}{\xi} - \frac{\bar{\delta}}{2\xi} + \frac{4}{\pi\xi^2 X_0 \bar{\delta}}. \quad (91)$$

These equations are dimensionless. They involve neither k nor η. For small ξ the early time behavior is given by equations (87)–(88), which in the new variables become

$$X_0 = \xi^{1/3}C_1, \quad (92)$$

$$\bar{\delta} = \xi^{-2/3}3C_1^{-2}. \quad (93)$$

Using these initial conditions we have numerically integrated equations (90) and (91). The solutions are displayed in Figure 4. Since from the initial situation the total flux inside X_0 is $\Phi = X_0^2 B_0$, and this is the amount of reconnected flux, we see that 50% of the flux is destroyed when $t = (2\ln k/\eta - 0.7)/k$.

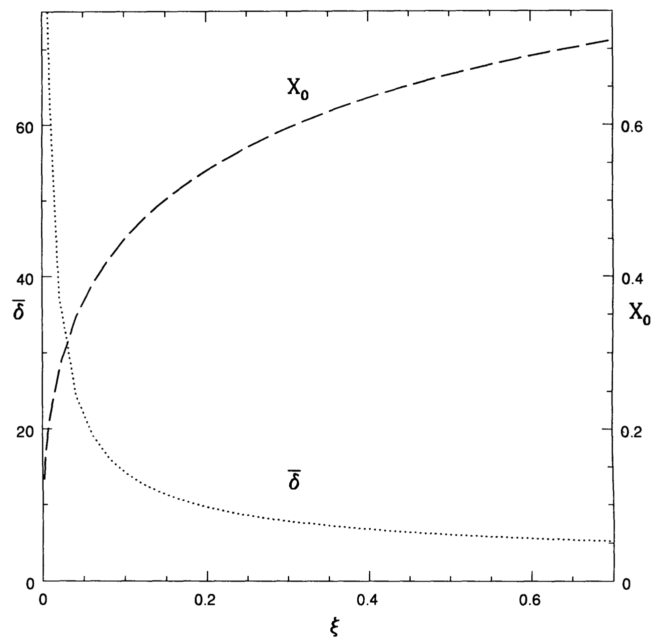

FIG. 4.—The time evolution of the $\bar{\delta} = \delta k/\eta$ vs. $\xi = e^{kt/2}(\eta/k)$ (*left scale and dotted line*) and X_0 vs. ξ (*right scale and dashed line*).

6. DISCUSSION

The principle interest of this paper is that it is an essentially complete analytic treatment of a reconnection problem. That is, we start with a finite system with no small layer. Then, by invoking a hypothetical cooling mechanism, we can trace the evolution to the situation to one in which a resistive layer forms. Finally, we can follow the evolution of this layer until all the magnetic flux has disappeared. This paper is parallel to a similar paper by Hahm & Kulsrud (1984) in which a resistive layer is formed in a natural way, and its evolution to a steady state is followed analytically. Because the treatment in both papers is analytical, the treatment is valid for any value of the magnetic Reynold's number, no matter how large. In this aspect, these analyses of magnetic reconnection are superior to numerical simulations (Biskamp 1986).

The main defect of the paper is the very hypothetical cooling mechanism that is invoked to produce the thin resistive layer and to lead to the rapid destruction of flux. If one examines the temperature T, one finds that $T/T_0 \approx R_M^{-1/4}$ when the resistive layer forms, and $T/T_0 \approx R_M^{-1}$ when the flux has entirely disappeared. T_0 is the initial temperature and $R_M \approx L^2/v_a \eta$ is the magnetic Reynold's number. From the enormous values of R_M which occur in astrophysics (10^{12} in the solar corona and 10^{19} in the interstellar medium) we see that the temperatures become fantastically low and any realistic cooling mechanism must fail. In fact, if one turns off the cooling when $T/T_0 \approx R_M^{-1/4}$ one only gets destruction of the flux which starts at the Sweet-Parker rate. As more flux is destroyed the layer becomes thicker and the rate gets even slower. If one turns off the cooling at higher temperatures, the resistive layer never forms.

We have assumed that the flows are one-dimensional. This leads to an accumulation of all the mass that was on the destroyed magnetic flux surfaces near the central, $x = 0$, point. In standard theories it is assumed that such mass will flow laterally out of the reconnection region. But if R_M is large, and the main driver in forming the reconnection layer is cooling, then by the time that the layer is thin enough for reconnection to occur, the speed of sound is too slow to drive such lateral flows. (In addition, it may be the case that there are barriers or back pressures which hinder these lateral flows, so that the one-dimensional approximation is appropriate from the beginning.) Thus, it is important to elucidate the actual mechanism which leads to the reconnection layer in the first place. The one we have chosen, i.e., exponential cooling, really does not produce rapid reconnection unless the exponential cooling persists to very low temperatures. Nevertheless, driven mainly by curiosity, we have pursued the analysis of the evolution of the layer to extremely low temperatures and assumed exponential cooling persists to these low temperatures.

Perhaps, the most interesting aspect of this paper is the existence of a purely resistive region through which the magnetic field changes from zero to a finite value in which the x velocity is much too slow to balance the resistive diffusion of the magnetic field. Such an additional layer is not normally considered to be present in the standard treatments of the resistive layer. (Sweet 1958; Parker 1963; Petschek 1963; Vasyliunas 1975; Priest & Forbes 1986.) It could conceivably be important, although, admittedly, its presence here is largely due to the cooling assumption.

A particularly interesting result comes from considering the rate at which the energy changes. Of course, at the end all the energy initially present in the form of thermal and magnetic energy has disappeared. But surprisingly, 80% of the energy is already gone by the time the layer has formed and before any flux has been destroyed. That is the bulk of the energy disappears during the ideal period. This is probably the case in more realistic scenarios and indicates that a change in topology is not always necessary for a release of energy.

One might ask is there a more realistic physical process that would be equivalent to our cooling process. Indeed if the plasma were removed at an exponential rate, say by recombination, and did not interact collisionally with neutral gas into which it combined, or alternatively if it were an electron-positron plasma that annihilated at an exponential rate, and the annihilation energy were removed without heating the plasma one would have essentially the same equations.

If the rate of cooling were less drastic, or if the recombination rate were algebraic, we would have a similar situation to the one we have treated in this paper. We believe the resulting more realistic reconnection process would also be susceptible to analysis. We do not attempt to carry out this analysis in this paper but leave it to a future paper.

The main point, as emphasized above, is that we have achieved an essentially complete solution of a physically consistent reconnection problem in which we can follow the creation of the reconnection layer from an equilibrium state in which the layer is not initially present. Also, we have treated the physics of the layer self consistently, keeping its interaction with the outside equilibrium. This solution, while astrophysically not of great practical value, should provide a good test of some aspects of more practical reconnection calculations.

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APPENDIX A

THE CASE $\gamma \neq 2$

We have considered the model with the ratio of specific heats $\gamma = 2$. In this appendix we want to show, that our approach will be valid and that the solution will not differ significantly in a more general case when γ is not equal to 2. To start with we have to write instead of equation (26) a new relationship between pressure and density:

$$p = e^{-kt} \rho^{\gamma} . \tag{A1}$$

However, the equations (24), (27), and (29) are not changed, and that means the expressions for B equation (35) and ρ equation (36) are not changed either. The initial condition for B is the same, but it is more convenient to choose new ρ_0

$$\rho_0^\gamma = 2B_0^2 . \tag{A2}$$

In order to find $\partial x/\partial x_0$, instead of equation (37), we now have

$$2B_0^2 x_0^2\left(\frac{\partial x}{\partial x_0}\right)^{-2} + 2B_0^2\left(\frac{\partial x}{\partial x_0}\right)^{-\gamma} e^{-kt} = \frac{A^2 B_0^2}{2} . \tag{A3}$$

Thus

$$\left(\frac{\partial x}{\partial x_0}\right)^2 = \frac{4}{A(t)^2}\left[x_0^2 + e^{-kt}\left(\frac{\partial x}{\partial x_0}\right)^{2-\gamma}\right] . \tag{A4}$$

Taking into account that during the resistive time period $X_0^2 \gg \epsilon^2$ and using equation (38) we can make an approximation:

$$\frac{\partial x}{\partial x_0} \approx \frac{2}{A} x_0 . \tag{A5}$$

With this approximation for region 3, we can get from equation (A4),

$$\left(\frac{\partial x}{\partial x_0}\right)^2 \approx \frac{2}{A} x_0\left[1 + \left(\frac{2}{A}\right)^{2-\gamma} \frac{e^{-kt}}{2x_0^\gamma}\right] . \tag{A6}$$

We are interested in finding the magnetic field, so substituting (A6) into equation (35) we obtain

$$B \approx \frac{AB_0}{1 + e^{-kt}/(2x_0^\gamma)} \approx AB_0\left[1 - \left(\frac{2}{A}\right)^{2-\gamma} \frac{\epsilon^2}{2x_0^\gamma}\right] . \tag{A7}$$

It is clear that now equations (52) and (56) are modified, but it is easy to show that in our approximation the dominant expressions for A and v can be written similarly to (55) and (57), notably $A \approx 1 - X_0^2$ and $v = (1 - x)\dot{A}/A$. For the matching conditions we need the mass flow, which is now

$$m = 2^{1/\gamma} B_0^{2/\gamma} \dot{X}_0 (1 - x) . \tag{A8}$$

We now consider the region 1. In the case of $\gamma \neq 2$, instead of expression (62) for ρ, we get

$$\rho = \left[\frac{A^2 B_0^2}{2}\left(1 - \frac{x^2}{\delta^2}\right) e^{kt}\right]^{1/\gamma} . \tag{A9}$$

Differentiating equation (A9) with respect to t

$$\frac{\partial \rho}{\partial t} = \frac{2}{\gamma}\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\frac{x^2/\delta^2}{1 - x^2/\delta^2}\right)\rho . \tag{A10}$$

In order to calculate the mass flow m we integrate equation (A10)

$$m = \int_0^x \frac{\partial \rho}{\partial t}\, dx = e^{kt/\gamma} \frac{2}{\gamma} \sqrt[\gamma]{\frac{A^2 B_0^2}{2}}\, \delta\left[\left(\frac{\dot{A}}{A} + \frac{k}{2}\right) \int_0^y (1 - y^2)^{1/\gamma} dy + \frac{\dot{\delta}}{\delta} \int_0^y (1 - y^2)^{1/\gamma - 1} y^2\, dy\right] , \tag{A11}$$

where $y \equiv x/\delta$. Making use of the beta function to evaluate these integrals at $x \to \delta$, we find

$$m(x \to \delta) = e^{kt/\gamma} \frac{\sqrt{\pi}\Gamma(1 + 1/\gamma)}{\gamma\Gamma(3/2 + 1/\gamma)} \sqrt[\gamma]{\frac{A^2 B_0^2}{2}}\, \delta\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right) . \tag{A12}$$

If $\gamma = 2$, equation (A11) reduces to equation (65).

In the intermediate region 2, expression (71) for the density is modified to

$$\rho = \sqrt[\gamma]{B_t b}\, e^{kt/\gamma} , \tag{A13}$$

and expression (72) for the velocity is now

$$v = \left(-\frac{me^{-kt/\gamma}}{\sqrt[\gamma]{B_t}}\right)\frac{1}{\sqrt[\gamma]{b}} \tag{A14}$$

Equations (73)–(75) are not changed, so the equation for b is (instead of eqs. [77] and [78])

$$\frac{\partial b}{\partial x} = \frac{B_t}{\delta}\left(\sqrt[\gamma]{\frac{b_c}{b}} - 1\right) , \tag{A15}$$

where

$$b_c \equiv \frac{m^\gamma \delta^\gamma e^{-kt}}{\eta^\gamma B_t} . \tag{A16}$$

Similarly to equation (79), we introduce a new variable $y^\gamma \equiv b/b_c$ and the equation (A15) becomes

$$\frac{\gamma y^\gamma dy}{y-1} = -\frac{B_t}{b_c \delta} dx . \tag{A17}$$

Analyzing this equation it is easy to show that as before, for large x, $y \rightarrow 1$, which means $b \rightarrow b_c$.

We have considered the differences, which occur in our three regions in the case of general γ. To make the model complete we discuss the new matching conditions. The first matching condition is for the magnetic field in regions 2 and 3. Using equations (A7) and (A16), we get instead of equation (83),

$$\frac{m^\gamma \delta^\gamma e^{-kt}}{\eta^\gamma A B_0} = A B_0 \left(\frac{2}{A}\right)^{2-\gamma} \frac{e^{-kt}}{2X_0^\gamma} . \tag{A18}$$

The second matching condition is for the mass flow in regions 1 and 3. Using equation (A8) (for $x_0 \rightarrow X_0$ and $x \ll 1$) and equation (A12), we get instead of eq. (84)

$$e^{kt/\gamma} \frac{\sqrt{\pi}\Gamma(1+1/\gamma)}{\gamma\Gamma(3/2+1/\gamma)} \sqrt[\gamma]{\frac{A^2 B_0^2}{2}} \delta\left(\frac{\dot{A}}{A} + \frac{k}{2} + \frac{\dot{\delta}}{\delta}\right) = \sqrt[\gamma]{2} B_0^{2/\gamma} \dot{X}_0 . \tag{A19}$$

Following the same discussion as in § 6, it is possible to show that we can get solutions for X_0 and δ. For example, in the approximation $X_0 \ll 1$ ($A = 1$) we have expressions similar to equations (87) and (89):

$$X_0 = e^{kt/3\gamma} \eta^{1/3} C_2^{1/3} , \tag{A20}$$

$$\delta = e^{-2kt/3\gamma} \eta^{1/3} \frac{1}{\alpha} C_2^{-2/3} , \tag{A21}$$

where C_2 is different from C_1.

It is clear, that for any γ the results are very similar to the case $\gamma = 2$, and the physical behavior of the reconnection is essentially the same, although the formulae are more complicated.

APPENDIX B

JUSTIFICATION OF THE ASSUMPTIONS OF THE TRANSITION REGION

For the analysis in the region 2 we assumed m was constant in equation (72) and that $\partial \boldsymbol{B}/\partial t$ was negligible in equation (73). Making use of this approximation we were able to match the solutions for ρ, v, and B in the three regions to find all the parameters and to determine these quantities explicitly. We will first assume that these solutions are valid and justify a posteriori our assumptions.

We first consider the assumption that m the mass flow is constant in x. The equation for the mass flow is equation (64)

$$\frac{\partial m}{\partial x} + \frac{\partial \rho}{\partial t} = 0 . \tag{B22}$$

Thus, the change in the mass flow across region 2 is

$$\Delta m = \int_0^x \frac{\partial \rho}{\partial t} dx < \left(\frac{\partial \rho}{\partial t}\right)_m \Delta x , \tag{B23}$$

where Δx is the extent of the region 2 δ and $(\partial \rho/\partial t)_m$ is the maximum value of $\partial \rho/\partial t$. Now ρ is given in terms of b and t by equation (71). Thus, the rate of change of ρ at a fixed position is given by the change of b and the change of the exponential,

$$\frac{\partial \rho}{\partial t} \approx \frac{k}{2} \rho + \frac{\rho}{2} \frac{1}{b} \frac{\partial b}{\partial t} ; \tag{B24}$$

Equation (81) gives b implicitly:

$$y^2 + 2y + 2\ln(y-1) = \frac{B_t}{b_c}\left(1 - \frac{x}{\delta}\right) , \tag{B25}$$

where $y^2 \equiv b/b_c$ and b_c is given through matching by equation (83). Now, y is of order unity and changes by a finite amount when the right-hand side changes by a finite amount. Thus, in order of magnitude

$$\frac{\partial y}{\partial t} \approx -\frac{\dot{\delta}}{\delta} - \frac{\dot{b}_c}{b_c} \approx -\frac{\dot{\delta}}{\delta} - k - \frac{\dot{X}_0^2}{X_0^2} . \tag{B26}$$

Hence,

$$\frac{1}{b}\frac{\partial b}{\partial t} = \frac{1}{2y}\frac{\partial y}{\partial t} + \frac{1}{b_c}\frac{\partial b_c}{\partial t} \approx \frac{\dot{\delta}}{2\delta} + \frac{3k}{2} + \frac{3\dot{X}_0^2}{2X_0^2} . \tag{B27}$$

In the early phases of the resistive period when δ and X_0 are exponential, this is a numerical factor times k. Later, in the resistive period, it is somewhat smaller. Combining this with eq. (B23) and noting $\Delta x \approx \delta$ we get

$$\Delta m \approx k\rho\,\delta \ll k\rho_0\,\delta e^{kt/2} . \tag{B28}$$

The latter inequality follows from equation (71) since $b \ll B_t$ in region 2. But from equation (84),

$$m \approx \rho_0 \dot{X}_0 \approx \rho_0 k X_0 . \tag{B29}$$

Since from equations (87) and (88), $\delta e^{kt/2} \approx X_0$, we see $\Delta m \ll m$.

In order to show that we can neglect $\partial B/\partial t$ in equation (73), we compare the change it would make in equation (74), with the constant term $C = -\eta B_t/\delta$. Thus,

$$\int \frac{\partial B}{\partial t}\,dx \approx \frac{\partial b}{\partial t}\,\Delta x \approx kb\,\delta , \tag{B30}$$

where we have used the estimate of equation (B27) and $\Delta x \approx \delta$. Now from $b \approx b_c$ and equation (83), we get that the condition to neglect $\partial B/\partial t$ reduces to

$$\frac{ke^{-kt}}{2x_0^2} B_t \delta \ll \frac{\eta B_t}{\delta} \tag{B31}$$

or

$$\delta \ll e^{kt/2} X_0 \sqrt{\eta/k} . \tag{B32}$$

Taking δ and X_0 from equations (87) and (88), this reduces to

$$e^{-kt} \equiv \epsilon^4 \ll \eta/k . \tag{B33}$$

Since this inequality is just the condition that we are in the resistive time period (see the beginning of the § 3), we are justified in neglecting $\partial B/\partial t$. However, in the intermediate time period this is not the case, and it makes the treatment of the intermediate period complex. The same remark applies to the assumption that m is constant.

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