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SELF-CONSISTENCY CONSTRAINTS ON THE DYNAMO MECHANISM

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ABSTRACT

It is shown that the requirements of self-consistency constrain the functional form of the turbulent dynamo field in an incompressible plasma. These requirements involve the back-reaction of the magnetic field through the Lorentz force in the momentum equation and the conservation laws of magnetohydrodynamic turbulence. The dynamo field is calculated in the weak-field limit when the turbulence is isotropic, as well as in the strong-field limit when the turbulence is anisotropic. For magnetic fields of a nontrivial topology, it is shown that the results of kinematic dynamo theory are strongly modified by the production of hyperresistivity (in the mean-field induction equation) which is left as a remnant after a near-cancellation between the alpha and beta effects. An interpolation formula for alpha quenching, encompassing weak-field and strong-field regimes, is proposed.

Subject headings: MHD — plasmas — turbulence

1. INTRODUCTION

The dynamo effect has been invoked as a mechanism for the generation and sustainment of astrophysical magnetic fields. The most well-developed branch of dynamo theory is the kinematic dynamo theory, which has been comprehensively discussed in several existing monographs (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin, & Sokoloff 1983). Kinematic dynamo theory is essentially concerned with the question of growth (or decay) of a magnetic field $\boldsymbol{B}$, given a velocity field $\boldsymbol{v}$. The magnetic field is assumed to obey the induction equation of resistive magnetohydrodynamics (MHD),

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} , \tag{1}$$

where η is the resistivity of the conducting field. In turbulent systems, we can separate the variables $\boldsymbol{B}$ and $\boldsymbol{v}$ into an averaged part and a fluctuation, i.e.,

$$\boldsymbol{B} = \langle \boldsymbol{B} \rangle + \delta \boldsymbol{B} \equiv \boldsymbol{B}_0 + \delta \boldsymbol{B} , \tag{2a}$$

$$\boldsymbol{v} = \langle \boldsymbol{v} \rangle + \delta \boldsymbol{v} \equiv \boldsymbol{v}_0 + \delta \boldsymbol{v} , \tag{2b}$$

where $\langle \delta \boldsymbol{B} \rangle = \langle \delta \boldsymbol{v} \rangle = 0$. The angle brackets represent either an ensemble average or an average over the small-space and fast-time scales of the fluctuations. Averaging equation (1), we obtain

$$\frac{\partial \boldsymbol{B}_0}{\partial t} = \nabla \times (\boldsymbol{v}_0 \times \boldsymbol{B}_0) + \nabla \times \mathscr{E} + \eta \nabla^2 \boldsymbol{B}_0 , \tag{3}$$

where

$$\mathscr{E} \equiv \langle \delta \boldsymbol{v} \times \delta \boldsymbol{B} \rangle \tag{4}$$

is the turbulent dynamo field. Subtracting equation (3) from equation (1), we obtain an exact equation for the magnetic field fluctuation:

$$\frac{\partial}{\partial t} \delta \boldsymbol{B} = \nabla \times (\boldsymbol{v}_0 \times \delta \boldsymbol{B} + \delta \boldsymbol{v} \times \boldsymbol{B}_0) + \eta \nabla^2 \delta \boldsymbol{B} + [\nabla \times (\delta \boldsymbol{v} \times \delta \boldsymbol{B}) - \langle \delta \boldsymbol{v} \times \delta \boldsymbol{B} \rangle] . \tag{5}$$

In the "first-order smoothing" or "quasi-linear" approximation, the term in brackets on the right-hand side of equation (5) is neglected. The advantage of making this approximation is that it yields a linear equation for the fluctuation $\delta \boldsymbol{B}$, i.e.,

$$\frac{\partial}{\partial t} \delta \boldsymbol{B} = \nabla \times (\boldsymbol{v}_0 \times \delta \boldsymbol{B} + \delta \boldsymbol{v} \times \boldsymbol{B}_0) + \eta \nabla^2 \delta \boldsymbol{B} . \tag{6}$$

Equation (6) can be easily inverted to give $\delta \boldsymbol{B}$. If $\boldsymbol{v}_0$ is a constant, then it can be eliminated by a Galilean transformation. For turbulence that is isotropic but not reflectionally symmetric, one then obtains from equation (4) the well-known result (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983)

$$\mathscr{E} = \alpha_0 \boldsymbol{B}_0 - \beta_0 \boldsymbol{J}_0 , \tag{7}$$

where

$$\alpha_0 = -\frac{\tau}{3} \langle \delta \boldsymbol{v} \cdot \delta \boldsymbol{\omega} \rangle , \tag{8}$$

$$\beta_0 = \frac{\tau}{3} \langle | \delta \boldsymbol{v} |^2 \rangle , \tag{9}$$

and τ is an approximate eddy correlation time. Whereas the α_0-effect can amplify a seed magnetic field, the β_0-effect enhances the diffusion rate, typically to values much larger than that due to the classical resistivity η. The violation of mirror symmetry, a requirement for the α_0-effect, can occur due to the effects of natural cyclonic and nonuniform rotation on both small and large scales. If $\boldsymbol{v}_0$ is not spatially uniform, it cannot be eliminated by a Galilean transformation and leads to the so-called Ω-term in the mean-field induction equation (Moffatt 1978; Parker 1979; Zeldovich et al. 1983). If we write the mean magnetic field as the sum of a poloidal and a toroidal component, the Ω-term can generate a toroidal component from the poloidal component. The α_0-term, on the other hand,

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regenerates the poloidal field from the toroidal component due to the effect of small-scale turbulence. In this paper, we focus on the α_0-effect, which is the heart of the mean-field amplification process.

From the inception of turbulent dynamo theory, it has been widely realized that the kinematic dynamo models are incomplete. If a weak seed magnetic field is amplified exponentially in time, the back-reaction of the magnetic field on the turbulent flow that generates it must eventually be taken into account and may alter qualitatively the predictions of kinematic dynamo theory. This realization gives rise to two important questions: first, what is the approximate magnitude of the mean magnetic field for which the kinematic theory ceases to be valid, and second, what happens to α_0 and β_0 when the kinematic theory no longer applies?

An answer to the first question might be that kinematic dynamo models will be invalid roughly when the energy of the mean magnetic field and the kinetic energy of the characteristic tubulent flow reach equipartition. It turns out that this answer is not supported by theory at the present time. This is because of the "Alfvén effect," a process by which the *small-scale* magnetic fluctuation energy reaches equipartition with the energy of the turbulent flow long before the large-scale magnetic field has picked up enough energy to reach equipartition with the turbulence (Pouquet, Frisch, & Leorat 1976). If one uses the simple estimate (Zeldovich 1957) (for which there is no rigorous justification in three dimensions),

$$\langle \delta B^2 \rangle^{1/2} \sim R_m^{1/2} B_0 , \tag{10}$$

where R_m is the magnetic Reynolds number, the Alfvén effect constrains the large-scale field B_0 by the inequality (Cattaneo & Vainshtein 1991; Vainshtein & Cattaneo 1992),

$$B_0 \lesssim \frac{\langle \delta U^2 \rangle^{1/2} \sqrt{4\pi\rho}}{R_m^{1/2}} . \tag{11}$$

Since R_m typically varies from 10^9 in stellar plasmas to 10^{14} in galactic plasmas, Cattaneo & Vainshtein claim that the inequality (11) severely restricts the magnitude of B_0 for which the predictions of kinematic dynamo models hold.

A different point of view, but one that also casts doubt on the validity of kinematic dynamo theory, is developed by Kulsrud & Anderson (1992). They note that in order for mean-field dynamo theory to be successful, it is important that the small-scale fluctuations be subdominant to the growing mean field. Unfortunately, they find the opposite to be the case in their calculations: there is much more energy on small scales than large, and the mean field generated by the kinematic dynamo effect is completely overwhelmed by the faster growing magnetic fluctuations. Thus, the "dynamo field quickly becomes unobservable under such conditions and the kinematic approximation fails before the mean field grows significantly" (Anderson & Kulsrud 1993, p. 1).

If we accept inequality (11) as the regime of validity of the kinematic theory, the next question is what happens to α_0 and β_0 when the kinematic approximation breaks down. In this paper, we attempt to answer this question by going beyond the kinetic approximation and imposing the constraints of self-consistency. We consider magnetic topologies that are more general than currently available models of α-quenching. (See, e.g., Rüdiger & Kichatinov 1993 and references therein, which assume that the mean magnetic field is uniform in space.) The standard result (7) is contained as a "weak-field" limit of our results. In the "strong-field" limit, we show that

$$\mathscr{E}_{\parallel} = \frac{\boldsymbol{B}_0}{B_0^2} \nabla \cdot \left(\kappa^2 \nabla \frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2} \right) , \tag{12}$$

where $\boldsymbol{J}_0 = \nabla \times \boldsymbol{B}_0$ and κ^2 is a positive definite functional. The form of equation (12) is popularly known as "hyperresistivity;" it conserves magnetic helicity and dissipates magnetic energy (Boozer 1986; Bhattacharjee & Hameiri 1986; Strauss 1986). We demonstrate here that for magnetic fields of nontrivial topology, hyperresistivity is left as a remnant after a remarkable near-cancellation between the α- and β-effects of kinematic dynamo theory.

The following is the plan of the paper. In § 2, we discuss the "weak-field" corrections to kinematic dynamo theory due to self-consistent dynamics. In § 3, we consider the "strong-field" limit in which the turbulence is strongly anisotropic. Though the results obtained in §§ 2 and 3 hold in different asymptotic regimes, we suggest that it is possible to interpolate between those regimes to obtain a form for $\mathscr{E}_{\parallel}$ that contains both asymptotic limits. In § 4, we give a derivation of the expression (12), using quasi-linear theory. We conclude in § 5 with a summary and a brief discussion of the implications of our results.

2. SELF-CONSISTENCY: THE WEAK-FIELD LIMIT

From the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\eta \boldsymbol{J} - \boldsymbol{v} \times \boldsymbol{B}) = 0 , \tag{13}$$

we can obtain the equation for the vector potential $\boldsymbol{A}(\boldsymbol{B} = \nabla \times \boldsymbol{A})$:

$$\frac{\partial \boldsymbol{A}}{\partial t} + \eta \boldsymbol{J} - \boldsymbol{v} \times \boldsymbol{B} = -\nabla \phi . \tag{14}$$

Here ϕ is a scalar function. Taking the scalar product of equation (13) with $\boldsymbol{A}$ and equation (14) with $\boldsymbol{B}$, and adding the results, we obtain

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) + 2\eta \boldsymbol{J} \cdot \boldsymbol{B} + \nabla \cdot \left(2\boldsymbol{E} \times \boldsymbol{A} + \frac{\partial \boldsymbol{A}}{\partial t} \times \boldsymbol{A} \right) = 0 , \tag{15}$$

where $\boldsymbol{E} = \eta \boldsymbol{J} - \boldsymbol{v} \times \boldsymbol{B}$. Averaging equation (15), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}(\boldsymbol{A}_0 \cdot \boldsymbol{B}_0) + \frac{\partial}{\partial t} \langle \delta \boldsymbol{A} \cdot \delta \boldsymbol{B} \rangle + 2\eta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0 \\ &\quad + 2\eta \langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B} \rangle + \nabla \cdot \Bigg(2\boldsymbol{E}_0 \times \boldsymbol{A}_0 + 2\langle \delta \boldsymbol{E} \times \delta \boldsymbol{A} \rangle \\ &\quad + \frac{\partial \boldsymbol{A}_0}{\partial t} \times \boldsymbol{A}_0 + \left\langle \frac{\partial \delta \boldsymbol{A}}{\partial t} \times \delta \boldsymbol{A} \right\rangle \Bigg) = 0 . \end{aligned} \tag{16}$$

An alternative form for $\partial/\partial t(\boldsymbol{A}_0 \cdot \boldsymbol{B}_0)$ can be obtained by using the averaged form for equation (14), i.e.,

$$\frac{\partial \boldsymbol{A}_0}{\partial t} + \eta \boldsymbol{J}_0 - \boldsymbol{v}_0 \times \boldsymbol{B}_0 - \mathscr{E} = -\nabla \phi_0 , \tag{17}$$

and the averaged form for equation (13), i.e.,

$$\frac{\partial \boldsymbol{B}_0}{\partial t} + \nabla \times (\eta \boldsymbol{J}_0 - \boldsymbol{v}_0 \times \boldsymbol{B}_0 - \mathscr{E}) = 0 . \tag{18}$$

We take the scalar product of $\boldsymbol{B}_0$ with equation (17) and $\boldsymbol{A}_0$ with equation (18) and add the two equations. The result is

$$\frac{\partial}{\partial t}(\boldsymbol{A}_0 \cdot \boldsymbol{B}_0) + 2\eta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0 - 2\mathscr{E}_0 \cdot \boldsymbol{B}_0 + \nabla \cdot \left(2\boldsymbol{E}_0 \times \boldsymbol{A}_0 + \frac{\partial \boldsymbol{A}_0}{\partial t} \times \boldsymbol{A}_0\right) = 0\,, \quad (19)$$

where $\boldsymbol{E}_0 = \eta \boldsymbol{J}_0 - \boldsymbol{v}_0 \times \boldsymbol{B}_0 - \mathscr{E}$. Subtracting equation (19) from equation (16), we obtain

$$\mathscr{E} \cdot \boldsymbol{B}_0 = -\eta\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle - \frac{1}{2}\frac{\partial}{\partial t}\langle \delta \boldsymbol{A} \cdot \delta \boldsymbol{B}\rangle - \nabla \cdot \left(\langle \delta \boldsymbol{E} \times \delta \boldsymbol{A}\rangle + \frac{1}{2}\left\langle \frac{\partial \delta \boldsymbol{A}}{\partial t} \times \delta \boldsymbol{A}\right\rangle\right), \quad (20)$$

which reduces to

$$\mathscr{E} \cdot \boldsymbol{B}_0 = -\eta\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle + \langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\,. \quad (21)$$

Equation (20) was derived by Bhattacharjee & Hameiri (Bhattacharjee & Hameiri 1986; Hameiri & Bhattacharjee 1987) in their study of the dynamo effect in laboratory plasmas. Equations (20) and (21) are both exact relations and impose constraints on the allowable functional forms for $\mathscr{E}$. (In deriving eq. [21], we have assumed, for simplicity, that the operations of spacetime differentiation and ensemble averaging commute.)

The back-reaction of the magnetic field on the turbulence that generates it modifies the kinematic relations (8) and (9). In the weak-field regime, when the turbulence can be regarded as isotropic, it has been shown (Pouquet et al. 1976; Vainshtein 1980; Zeldovich et al. 1983; Gruzinov & Diamond 1994) that the relation (7) changes to

$$\mathscr{E} \simeq \alpha \boldsymbol{B}_0 - \beta \boldsymbol{J}_0\,, \quad (22)$$

where

$$\alpha = -\frac{\tau}{3}\left(\langle \delta \boldsymbol{v} \cdot \delta \boldsymbol{\omega}\rangle - \frac{1}{\rho}\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle\right). \quad (23)$$

Vainshtein reports that $\beta = 2\beta_0$ for incompressible isotropic turbulence. However, if the effect of the perturbed pressure is taken into account, then it can be shown that

$$\beta = \beta_0 \quad (24)$$

(Avinash 1991; Gruzinov & Diamond 1994).

From equations (21) and (22), we obtain

$$\eta\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle = -\alpha B_0^2 + \beta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0 + \langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\,. \quad (25)$$

Equation (25) should be compared with equation (9) of Gruzinov & Diamond (after a typographical error in eq. [9] of Gruzinov & Diamond is corrected). Gruzinov & Diamond maintain that their relation $\eta\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle = -\alpha B_0^2 + \beta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0$ is an exact expression, but this is not so because it omits the last term in equation (25). It is this last term that yields hyperresistivity in the strong-field, anisotropic regime. Eliminating $\langle \delta \boldsymbol{J} \cdot \delta \boldsymbol{B}\rangle$ between equations (23) and (25), we obtain

$$\alpha = \frac{\alpha_0 + (\tau/3\rho\eta)(\beta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0 + \langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle)}{1 + (\tau/3\eta\rho)B_0^2}\,. \quad (26)$$

In the limit $(\tau/\eta\rho)B_0^2 \ll 1$, equation (26) gives $\alpha \simeq \alpha_0$, the kinematic result. (The second and third terms in the numerator of eq. [26] are much smaller than α_0 in this limit.)

We have not shown that equation (26) continues to hold in the strong-field regime, which is the subject of § 3. However, in anticipation of the results derived in §§ 3 and 4, we heuristically take the limit $(\tau/\eta\rho)B_0^2 \gg 1$ of equation (26). We obtain

$$\alpha \simeq \beta \frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2} + \frac{1}{B_0^2}\langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\,. \quad (27)$$

Substituting equation (27) in equation (22), we obtain

$$\mathscr{E} = \left(\beta \frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2} + \frac{1}{B_0^2}\langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\right)\boldsymbol{B}_0 - \beta \boldsymbol{J}_0\,, \quad (28)$$

$$= -\beta \boldsymbol{J}_{0\perp} + \frac{\boldsymbol{B}_0}{B_0^2}\langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\,. \quad (29)$$

The first term on the right-hand side of equation (29) is perpendicular to $\boldsymbol{B}_0$ and does not contribute to $\mathscr{E}_\parallel$. Thus a significant cancellation has occurred between the β-effect (third term in eq. [28]) and (a part of) the α-effect (first term in eq. [28]), to yield

$$\mathscr{E}_\parallel = \frac{\boldsymbol{B}_0}{B_0^2}\langle \delta \boldsymbol{E} \cdot \delta \boldsymbol{B}\rangle\,. \quad (30)$$

In §§ 3 and 4, we will demonstrate that equation (30) has the functional form of equation (12) in the strong-field limit.

3. SELF-CONSISTENCY: THE STRONG-FIELD LIMIT

As a large-scale magnetic field $\boldsymbol{B}_0$ grows in strength, the turbulent velocity field adjusts itself to the growing anisotropy induced by $\boldsymbol{B}_0$. To motivate the results that follow, we begin with a simple discussion of the main physical consequences of this growing anisotropy. For this purpose, neglecting collisional dissipation, we write the linearized equation for the fluctuating vector potential

$$\frac{\partial}{\partial t}\delta \boldsymbol{A} = \delta \boldsymbol{v} \times \boldsymbol{B}_0 - \nabla \delta\phi\,, \quad (31)$$

which can be resolved into parallel and perpendicular components:

$$\frac{\partial}{\partial t}\delta A_\parallel = -\nabla_\parallel \delta\phi\,, \quad (32)$$

$$\frac{\partial}{\partial t}\delta \boldsymbol{A}_\perp = \delta \boldsymbol{v} \times \boldsymbol{B}_0 - \nabla_\perp \delta\phi\,. \quad (33)$$

Hence, we have

$$\mathscr{E} = \left\langle \delta \boldsymbol{v} \times \nabla \times \int dt\, \frac{\partial \delta \boldsymbol{A}}{\partial t}\right\rangle, = \left\langle \delta \boldsymbol{v} \times \nabla \times \int dt(\delta \boldsymbol{v} \times \boldsymbol{B}_0 - \nabla_\perp \delta\phi)\right\rangle + \langle \delta \boldsymbol{v} \times \nabla \times (\delta A_\parallel \hat{\boldsymbol{b}})\rangle\,, \quad (34)$$

where $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}_0/B_0$. The plasma beta, defined by $\beta_p \equiv 2p/B^2$, is a convenient parameter with which we can track the growth of the magnetic field (for fixed p). As discussed later in this section, for very large $\boldsymbol{B}_0$, the perturbed quantities $\delta \boldsymbol{A}_\perp$, $\delta v_\parallel$, and $\delta \boldsymbol{J}_\perp$ are very small. Then equation (33) reduces to

$$\delta \boldsymbol{v} \times \boldsymbol{B}_0 \simeq \nabla_\perp \delta\phi\,. \quad (35)$$

When equation (35) holds, the first angle bracket on the right-hand side of equation (34) is nearly zero. Then the second term in equation (34) can be written, for incompressible plasmas, as

$$\mathscr{E}_{\parallel} = -\frac{\boldsymbol{B}_0}{B_0^2}\nabla \cdot \langle B_0 \delta A_{\parallel} \delta \boldsymbol{v}_{\perp} \rangle . \quad (36)$$

Equation (36) is a *local* relation, and the averaged terms under the divergence operator cannot be omitted in general by appealing to boundary conditions. We shall demonstrate that relation (36) (which is equivalent to the relation [30]) eventually leads to the functional form of equation (12).

Equation (36) can be derived in a more formally rigorous way by introducing a β_p-ordering on the self-consistent resistive MHD equations. For fusion plasmas, this procedure was first developed by Rosenbluth et al. (1976) and Strauss (1976), but a pedagogically more satisfactory derivation for astrophysical plasmas is given by Zank & Matthaeus (1993), who delineate three regimes $\beta_p \gg 1$, $\beta_p \sim 1$, and $\beta_p \ll 1$. We will use these regimes to classify the three cases of weak, moderate, and strong magnetic field, respectively. The case $\beta_p \gg 1$ is described well by fully three-dimensional, incompressible resistive MHD equations, and the turbulence is nearly isotropic. The results of § 2 pertain to this case. In the cases $\beta_p \sim 1$ and $\beta_p \ll 1$, the growth of $\boldsymbol{B}_0$ leads to the development of anisotropy. Assuming, for simplicity, that $\boldsymbol{B} \simeq \boldsymbol{B}_0 =$ constant to leading order, Zank & Matthaeus (1993) demonstrate formally that the effect of this large-scale field is "to introduce a preferred direction into the system that manifests itself by reducing the 'dimensionality' of the underlying incompressible description." Using a small parameter ϵ which is equal to the Alfvén Mach number $M_A = v_0/V_A \equiv v_0(\rho_0)^{1/2}/B_0$, Zank & Matthaeus derive, for $\beta_p \sim 1$, the system of equations

$$\nabla_{\perp} \cdot \boldsymbol{v}_{\perp} = 0 , \quad \nabla_{\perp} \cdot \boldsymbol{B}_{\perp} = 0 , \quad (37)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\perp} \cdot \nabla_{\perp}\right)\boldsymbol{v}_{\perp} = -\nabla_{\perp}\left(p + \frac{B_{\perp}^2}{2}\right) + (\boldsymbol{B}_{\perp} \cdot \nabla_{\perp})\boldsymbol{B}_{\perp} , \quad (38)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\perp} \cdot \nabla_{\perp}\right)\boldsymbol{B}_{\perp} = (\boldsymbol{B}_{\perp} \cdot \nabla_{\perp})\boldsymbol{v}_{\perp} + \eta \nabla^2 \boldsymbol{B}_{\perp} , \quad (39)$$

where $\boldsymbol{B} = \boldsymbol{B}_{\perp} + \boldsymbol{B}_0 \hat{z}$ and $\boldsymbol{v} = \boldsymbol{v}_{\perp}$ to leading order. In this ordering, $\boldsymbol{A}_{\perp}$ and $v_{\parallel}$ are zero to leading order, as asserted earlier. It is then possible to write $\boldsymbol{v} = \nabla_{\perp}\hat{\phi}(x, y, t) \times \hat{z}$, where $\hat{\phi} = -\phi/B_0$ and $\boldsymbol{B}_{\perp} = \nabla_{\perp} A_{\parallel}(x, y, t) \times \hat{\boldsymbol{b}}$. Hence, we have

$$\mathscr{E}_{\parallel} = \langle \delta \boldsymbol{v} \times \delta \boldsymbol{B} \rangle_{\parallel} = \frac{\boldsymbol{B}_0}{B_0^2}\nabla \cdot \langle -B_0 \delta \boldsymbol{v}_{\perp} \delta A_{\parallel} \rangle$$

$$= \frac{\boldsymbol{B}_0}{B_0^2}\nabla \cdot \langle B_0 \delta\hat{\phi}\, \delta \boldsymbol{B}_{\perp} \rangle , \quad (40)$$

which is identical to equation (36).

In the next section, we will use quasi-linear theory to show that equation (40) has the form of equation (12), with κ^2 related to the spectrum of fluctuations.

4. DERIVATION OF HYPERRESISTIVITY FROM QUASI-LINEAR THEORY

We introduce Elsasser variables $\boldsymbol{Z}^+ = \boldsymbol{B}_{\perp}/\rho_0^{1/2} + \boldsymbol{v}_{\perp}$ and $\boldsymbol{Z}^- = \boldsymbol{B}_{\perp}/\rho_0^{1/2} - \boldsymbol{v}_{\perp}$. Then, neglecting collisional dissipation equations (37)–(39) can be combined to give

$$\nabla_{\perp} \cdot \boldsymbol{Z}^+ = 0 , \quad \nabla_{\perp} \cdot \boldsymbol{Z}^- = 0 , \quad (41)$$

$$\frac{\partial}{\partial t}\boldsymbol{Z}^+ - \boldsymbol{Z}^- \cdot \nabla_{\perp}\boldsymbol{Z}^+ = -\nabla_{\perp} P_* , \quad (42)$$

and

$$\frac{\partial}{\partial t}\boldsymbol{Z}^- + \boldsymbol{Z}^+ \cdot \nabla_{\perp}\boldsymbol{Z}^- = \nabla_{\perp} P_* , \quad (43)$$

where $P_* \equiv p/\rho_0 + (\boldsymbol{Z}^+ + \boldsymbol{Z}^-)^2/8$. Hereafter, we set $\rho_0 = 1$. We assume that $\langle \boldsymbol{B} \rangle = B_y(x)\hat{y} + B_0\hat{z}$ and that the inhomogeneity depends only on x. Since $\delta \boldsymbol{B}_{\perp} = \nabla \delta\psi \times \hat{\boldsymbol{b}}$ (where $A_{\parallel} \equiv \psi$) and $\delta \boldsymbol{v}_{\perp} = \nabla \delta\hat{\phi} \times \hat{\boldsymbol{b}}$, we have

$$\delta \boldsymbol{Z}^+ = \nabla \xi_+ \times \hat{\boldsymbol{b}} , \quad \delta \boldsymbol{Z}^- = \nabla \xi_- \times \hat{\boldsymbol{b}} , \quad (44)$$

where $\xi_+ = \delta\psi + \delta\hat{\phi}$ and $\xi_- = \delta\psi - \delta\hat{\phi}$. Then the turbulent electromotive force $\mathscr{E}$ can be written as

$$\mathscr{E} = \langle \delta \boldsymbol{v}_{\perp} \times \delta \boldsymbol{B}_{\perp} \rangle = \frac{\hat{\boldsymbol{b}}}{4}\nabla \cdot \langle \xi_+ \delta \boldsymbol{Z}^- - \xi_- \delta \boldsymbol{Z}^+ \rangle . \quad (45)$$

We Fourier-analyze along y- and z-directions, and write

$$\xi_+ \delta \boldsymbol{Z}^- = \frac{1}{(2\pi)^2}\sum_{k_y, k_{y'}, k_z, k_{z'}} \xi_{+k_y k_z}(x, t)\delta \boldsymbol{Z}^-_{k_{y'} k_{z'}}(x, t)$$

$$\times \exp\left[i(k_y + k_y')y - i(k_z + k_z')z\right]$$

$$\simeq \frac{1}{(2\pi)^2}\sum_{\boldsymbol{k}} \xi_{+k_y k_z}(x, t)\delta \boldsymbol{Z}^{-*}_{k_y k_z}(x, t) , \quad (46)$$

where * denotes complex conjugation. Similarly, we have

$$\xi_- \delta \boldsymbol{Z}^+ \simeq \frac{1}{(2\pi)^2}\sum_{\boldsymbol{k}} \xi_{-k_y k_z}(x, t)\delta \boldsymbol{Z}^{+*}_{k_y k_z}(x, t) , \quad (47)$$

where $\boldsymbol{k} = (k_y, k_z)$. From equation (44), we obtain

$$\nabla_{\perp}^2 \xi_+ = \hat{\boldsymbol{b}} \cdot \nabla \times \delta \boldsymbol{Z}^+ , \quad (48)$$

which gives

$$\xi_{+\boldsymbol{k}} = k_{\perp}^{-2}\hat{\boldsymbol{b}} \cdot [i\boldsymbol{k} \times \delta \boldsymbol{Z}^+_{\boldsymbol{k}}(x, t) + \nabla \times \delta \boldsymbol{Z}^+_{\boldsymbol{k}}(x, t)] . \quad (49)$$

Similarly, from

$$\nabla_{\perp}^2 \xi_- = \hat{\boldsymbol{b}} \cdot \nabla \times \delta \boldsymbol{Z}^- , \quad (50)$$

we obtain

$$\xi_{-\boldsymbol{k}} = k_{\perp}^{-2}\hat{\boldsymbol{b}} \cdot [i\boldsymbol{k} \times \delta \boldsymbol{Z}^-_{\boldsymbol{k}}(x, t) + \nabla \times \delta \boldsymbol{Z}^-_{\boldsymbol{k}}(x, t)] . \quad (51)$$

It is clear by inspection of equations (45), (48), and (51) that in order to compute $\mathscr{E}$, we need a renormalized turbulence theory to deal with the nonlinear terms $\boldsymbol{Z}^- \cdot \nabla \boldsymbol{Z}^+$, $\boldsymbol{Z}^+ \cdot \nabla \boldsymbol{Z}^-$, and $(\boldsymbol{Z}^+ + \boldsymbol{Z}^-)^2$ in equations (42)–(43). For simplicity, in order to keep the underlying physics as transparent as possible, we choose to use the quasi-linear approximation, neglecting mode-coupling effects. The linear propagators in our derivation can be shown to be broadened by nonlinear effects (Strauss 1986; Tetreault 1989). In the quasi-linear approximation, equations (42) and (43) yield

$$\left(\frac{\partial}{\partial t} - \langle \boldsymbol{Z}^- \rangle \cdot \nabla\right)\delta \boldsymbol{Z}^+ = \delta \boldsymbol{Z}^- \cdot \nabla_{\perp}\langle \boldsymbol{Z}^+ \rangle , \quad (52)$$

$$\left(\frac{\partial}{\partial t} + \langle \boldsymbol{Z}^+ \rangle \cdot \nabla\right)\delta \boldsymbol{Z}^- = -\delta \boldsymbol{Z}^+ \cdot \nabla_{\perp}\langle \boldsymbol{Z}^- \rangle , \quad (53)$$

where we have used the condition $\delta p + \langle \boldsymbol{B} \rangle \cdot \delta \boldsymbol{B} = 0$, valid for a wide class of resistive modes (see, e.g., Hameiri & Bhattacharjee 1987). In Fourier space, equations (52) and (53) give

$$\left(\frac{\partial}{\partial t} - ik_y\langle \boldsymbol{Z}^- \rangle_y\right)\delta \boldsymbol{Z}_{\boldsymbol{k}}^+ = \delta \boldsymbol{Z}_{\boldsymbol{k}}^- \cdot \nabla_\perp \langle \boldsymbol{Z}^+ \rangle \tag{54}$$

and

$$\left(\frac{\partial}{\partial t} + ik_y\langle \boldsymbol{Z}^+ \rangle_y\right)\delta \boldsymbol{Z}_{\boldsymbol{k}}^- = -\delta \boldsymbol{Z}_{\boldsymbol{k}}^+ \cdot \nabla_\perp \langle \boldsymbol{Z}^- \rangle\,, \tag{55}$$

respectively. Equations (54) and (55) can then be easily inverted. Formally, we write

$$\delta \boldsymbol{Z}_{\boldsymbol{k}}^+ = G_{\boldsymbol{k}}^- \delta \boldsymbol{Z}_{\boldsymbol{k}}^- \cdot \nabla \langle \boldsymbol{Z}^+ \rangle \tag{56}$$

and

$$\delta \boldsymbol{Z}_{\boldsymbol{k}}^- = -G_{\boldsymbol{k}}^+ \delta \boldsymbol{Z}_{\boldsymbol{k}}^+ \cdot \nabla \langle \boldsymbol{Z}^- \rangle\,, \tag{57}$$

where $G_{\boldsymbol{k}}^{\pm}$ are the Green's functions, given by

$$G_{\boldsymbol{k}}^{\pm} = (\gamma_{\boldsymbol{k}} \pm ik_y\langle Z^{\pm}\rangle_y)^{-1}\,. \tag{58}$$

Using equations (56) and (57) in equations (49) and (51), respectively, we obtain

$$\xi_{+\boldsymbol{k}} \simeq k_\perp^{-2} G_{\boldsymbol{k}}^- \delta \boldsymbol{Z}_{\boldsymbol{k}}^- \cdot \nabla \langle J_\parallel \rangle \tag{59}$$

and

$$\xi_{-\boldsymbol{k}} \simeq -k_\perp^{-2} G_{\boldsymbol{k}}^+ \delta \boldsymbol{Z}_{\boldsymbol{k}}^+ \cdot \nabla \langle J_\parallel \rangle\,. \tag{60}$$

Substituting equations (59) and (60) in equation (45), we obtain

$$\mathscr{E} \simeq \frac{\boldsymbol{B}_0}{B_0^2}\,\nabla \cdot \mathbf{K} \cdot \nabla\left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right), \tag{61}$$

where

$$\mathbf{K} = \frac{B_0^2}{(4\pi)^2} \sum_{\boldsymbol{k}} k_\perp^{-2}(G_{\boldsymbol{k}}^- \delta \boldsymbol{Z}_{\boldsymbol{k}}^{-*} \delta \boldsymbol{Z}_{\boldsymbol{k}}^- + G_{\boldsymbol{k}}^+ \delta \boldsymbol{Z}_{\boldsymbol{k}}^{+*} \delta \boldsymbol{Z}_{\boldsymbol{k}}^+)\,. \tag{62}$$

An equation similar to equation (62) has been derived earlier by Tetreault (1989) in the context of MHD clump turbulence in toroidal plasmas, confined by a strong toroidal field $\boldsymbol{B}_0$. If the fluctuations are dominated by one component orthogonal to $\boldsymbol{B}_0$ (say, the x-component) which is a case of considerable physical interest, then equation (61) can be approximated by the functional form of equation (12). We note that equation (12) obeys the integral relations,

$$\int_V \mathscr{E} \cdot \boldsymbol{B}_0\, d\boldsymbol{r} = \int_S \kappa^2 \nabla\left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right) \cdot d\boldsymbol{a}\,, \tag{63}$$

and

$$\int_V \mathscr{E} \cdot \boldsymbol{J}_0\, d\boldsymbol{r} = -\int_V \kappa^2 \left[\nabla\left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right)\right]^2 d\boldsymbol{r} + \int_S \kappa^2 \left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right) \nabla\left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right) d\boldsymbol{r}\,, \tag{64}$$

where S denotes the surface of V. The physical significance of equations (63) and (64) is clear: the turbulent dynamo field $\mathscr{E}$ neither creates nor destroys helicity in any volume V but dissipates energy (Boozer 1986, 1993; Bhattacharjee & Hameiri 1986) within the volume. (If the dynamo field would have created or destroyed helicity, there would have been a volume term in eq. [63]. There is a volume term in eq. [64], and it is negative definite, representing dissipation.)

We discuss the physical implications of the calculation given above in the context of astrophysical plasmas. In the presence of a large-scale background magnetic field $\boldsymbol{B}_0$, the plasma turbulence is envisioned to be a bath of Alfvénic fluctuations. Indeed, if we set the right-hand side of equations (52) and (53) to zero, we obtain uncoupled Alfvén wave fluctuations which obey the relation $\delta \boldsymbol{v} = \pm \delta \boldsymbol{B}$. Observations of incompressible MHD turbulence in the solar wind (see, for instance, Belcher & Davis 1971; Burlaga & Turner 1979) indicate a tendency of alignment or anti-alignment between the fluctuations $\delta \boldsymbol{v}$ and $\delta \boldsymbol{B}$. Dobrowolny, Mangeney, & Veltri (1980) have shown from considerations of the inertial range of the turbulence (ignoring source or dissipation terms) that this tendency is a general consequence of the dynamical relaxation of self-consistent MHD turbulence if the initial excitation favors one type of Alfvén fluctuation (+ or −). It should be emphasized that alignment (or anti-alignment) is merely a tendency and not realized in practice. (If this asymptotic state were realizable, then according to eq. [23], α would be exactly zero.) However, the tendency in itself is indicative of the fact that $\mathscr{E}$ calculated by the kinematic theory will be strongly reduced by the Alfvén effect. In the neighborhood of the asymptotic state, one expects nonlinear mode-coupling effects to be weak (Dobrowolny et al. 1980). There are many Fourier modes in the turbulent bath, and quasi-linear theory, which sums over the modes but neglects nonlinear mode-coupling effects, is a reasonable first approximation. In the context of this physical picture, our calculation shows that when the two types of propagating Alfvén wave fluctuations are coupled by the terms on the right-hand side of equations (52) and (53), we obtain the result (61) in which hyperresistivity is left as a remnant after a near-exact cancellation between the alpha and beta effects of the kinematic theory.

Before we conclude this section, we draw the attention of the reader to an instructive discussion of the Alfvén effect in Biskamp's recent monograph (Biskamp 1993). Biskamp gives a qualitative discussion of the importance of hyperresistivity in the context of MHD turbulence and the inverse cascade phenomenon that underlies the conservation of magnetic helicity.

5. CONCLUSIONS

In this paper, we have examined the constraints imposed by self-consistency on the turbulent dynamo in the weak-field as well as strong-field regimes. Synthesizing the results of §§ 2–4, we propose the interpolation formula

$$\alpha = \left\{\alpha_0 + \frac{\tau}{3\rho\eta}\left[\beta \boldsymbol{J}_0 \cdot \boldsymbol{B}_0 + \nabla \cdot \kappa^2 \nabla\left(\frac{\boldsymbol{J}_0 \cdot \boldsymbol{B}_0}{B_0^2}\right)\right]\right\} \times \left[1 + \frac{\tau}{3\rho\eta} B_0^2\right]^{-1}. \tag{65}$$

As demonstrated in §§ 2 and 3, when equation (65) is substituted in equation (22), there is a near-cancellation between the α- and β-effects, and in the strong-field limit, we are left with the functional form of equation (12) for $\mathscr{E}_\parallel$, known as hyperresistivity. Even though our derivation of hyperresistivity is based on the quasi-linear approximation, we believe that the functional dependencies of this term on mean physical variables is robust because it is consistent with well-known properties of three-dimensional MHD turbulence (Taylor 1974; Pouquet et al. 1976; Matthaeus & Montgomery 1980; Boozer 1986; Bhattacharjee & Hameiri 1986). Hyperresistivity does not amplify either magnetic flux or energy. It can, for example, convert toroidal flux to poloidal flux as long as the conversion is consistent with helicity conservation and dissipates magnetic energy in the process.

While these conclusions pose critical challenges for traditional turbulent MHD dynamo models, they do not negate the relevance of the traditional theory for all astrophysical magnetic fields. The mechanism and affectiveness of the saturation mechanism discussed in this paper may not apply in all circumstances, particularly if the mean magnetic field is very weak. Thus, the galactic dynamo problem may require considerations rather different from those relevant to the solar or planetary dynamo. Field (1994) has recently given a useful summary of the issues raised by recent criticisms of the dynamo theory for galactic magnetic fields, and it is clear that a final resolution of the problem of origin of galactic magnetic fields depends on reliable calculations of alpha and beta over relevant timescales, after freeing "the classical theory from having to assume the first-order smoothing approximation." The present self-consistent calculation is a step in that direction.

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