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THE EINSTEINIAN GRAVITATIONAL FIELD OF THE RIGIDLY ROTATING DISK OF DUST

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ABSTRACT

This paper presents the gravitational field of a uniformly rotating stationary and axisymmetric disk consisting of dust particles as a rigorous global solution to the Einstein equations. The problem is formulated as a boundary value problem of the Ernst equation and solved by means of inverse methods. The solution is given in terms of linear integral equations and depends on two parameters: the angular velocity Ω and the relative redshift z from the center of the disk. The Newtonian limit $z \ll 1$ represents the MacLaurin solution of a rotating fluid in the disk limit. For $z \to \infty$ the "exterior" solution is given by the extreme Kerr solution. This proves a conjecture of Bardeen & Wagoner (1969, 1971).

Subject headings: black hole physics — galaxies: general — gravitation — quasars: general — relativity

1. THE PHYSICAL PROBLEM

We consider incoherent matter (dust) with the energy-momentum tensor

$$T^{ik} = \epsilon u^i u^k \tag{1}$$

and assume axisymmetry as well as stationarity. The motion shall be a rigid rotation around the symmetry axis. As in Newtonian theory we expect the only finite mass configuration compatible with these assumptions to be an infinitesimally thin disk where the centrifugal forces balance the gravitational attraction.

There are two arguments in favor of a general-relativistic treatment of self-gravitating rotating disks of dust:

1. So far no exact global solution of Einstein's equations describing a rotating star or any other rotating isolated matter distribution is known. A uniformly rotating axisymmetric disk of dust is the simplest model of an isolated rotating perfect fluid body. Like the classical MacLaurin disk it represents a "universal" limit for any rigidly rotating fluid ball whose equation of state has a zero-pressure limit (vanishing ratio of pressure to energy density).

2. Disk models play an important role in astrophysics (galaxies, accretion disks). Hence a general-relativistic treatment is desirable. Especially in view of exotic objects as, e.g., quasars, general-relativistic disk models could become important.

It is known from Newton's theory that solutions with a disklike source can be constructed from any stationary gravitational potential which is singularity-free in a half-space by joining that potential to its reflection. The mass distribution in the disk plane and the (nonconstant) angular velocity can be calculated a posteriori. In the same way one can find general relativistic "disk solutions" and try to interpret the resulting energy-momentum tensor. It will differ, in general, from the dust tensor (eq. [1]).

Unlike this we prescribe a simple but astrophysically relevant matter model from the very beginning and consider gravitationally interacting dust particles moving on circular geodesic orbits of their own field. The classical analogs of these disk solutions are known to be unstable (Toomre 1964). Therefore some pressure (or velocity dispersion in the stellar dynamical case) will be necessary to stabilize them. For a discussion of the stability of the classical MacLaurin disks we refer to Binney & Tremaine (1987, chap. 5). Nevertheless, as a first step, we investigate the relativistic generalization of the zero-pressure MacLaurin disk. Future work has to clarify how the inclusion of pressure will affect our results. Note that we are not interested here in "counterrotating" disks, cf. Morgan & Morgan (1969).

Bardeen & Wagoner (1969, 1971) developed a powerful series expansion technique for an approximate solution of the above described problem. In this paper we want to present the exact solution. It was gained with the aid of the inverse (scattering) method which was first utilized for the axisymmetric stationary vacuum Einstein equations by Maison (1978), Belinski & Zakharov (1978), Harrison (1978), Neugebauer (1979, 1980), Hauser & Ernst (1979, 1980), Hoenselaers, Kinnersley, & Xanthopoulos (1979), and Aleksejev (1980). Many of these authors followed the line of Geroch (1972), Kinnersley (1977), Kinnersley & Chitre (1977, 1978), and Herlt (1978).

It should be stressed that the inverse method is restricted to the vacuum equations. Therefore, infinitesimally thin disks are extremely favored objects for an application of that method. Since the interior problem may be reduced to boundary conditions it is possible to solve the global problem in this way. Generalizations of our result without changing the solution techniques might be possible in the following directions:

1. Models with pressure (acting only in the plane of the disk), i.e., generalizations of the classical MacLaurin disks with pressure.
2. Models with differential rotation.
3. Models including electromagnetic fields.

2. THE BOUNDARY VALUE PROBLEM OF THE ERNST EQUATION

In Weyl-Lewis-Papapetrou-coordinates,[1]

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a\, d\varphi)^2 , \tag{2}$$

[1] Throughout the paper (except Fig. 2) we use units where Newton's gravitational constant G as well as the velocity of light c are equal to 1.

the vacuum Einstein equations are equivalent to the Ernst equation

$$(\Re f)\left(\frac{1}{\rho}f_{,\rho}+f_{,\rho\rho}+f_{,\zeta\zeta}\right)=f_{,\rho}^{\,2}+f_{,\zeta}^{\,2} \qquad (3)$$

for the complex function

$$f(\rho,\zeta)=e^{2U}+ib \qquad (4)$$

with

$$b_{,\rho}=-\frac{e^{4U}}{\rho}a_{,\zeta}\,;\quad b_{,\zeta}=\frac{e^{4U}}{\rho}a_{,\rho}\,. \qquad (5)$$

(The metric function k may be calculated from the Ernst potential f by a path integral.) The reflectional symmetry of our problem with respect to the plane $\zeta=0$ allows us to assume

$$f(\rho,-\zeta)=\bar{f}(\rho,\zeta) \qquad (6)$$

and to formulate a boundary value problem for the Ernst equation in the upper half space $\zeta>0$ (a bar denotes complex conjugation). The boundary conditions may be taken from Figure 1.

The quantity Ω is the angular velocity of the disk (as measured by an observer at infinity) and ρ_0 is the coordinate radius of the disk. The "surface potential" V_0 is related to the relative redshift z from the center of the disk:

$$z=e^{-V_0}-1\,. \qquad (7)$$

Note that $z/(1+z)$ is just the expansion parameter used in the work of Bardeen & Wagoner (1969, 1971).

The surface density $\sigma(\rho)$ corresponding to the baryonic mass M_0,

$$M_0=\int u^4\epsilon\sqrt{-g}\,d^3x=2\pi\int u^4\sigma(\rho)\rho\,d\rho\,, \qquad (8)$$

cannot be prescribed beforehand but has to be calculated from the solution:

$$\sigma(\rho)=\frac{1}{2\pi}\,U'_{,\zeta}\bigg|_{\zeta=0^+}\,. \qquad (9)$$

In the next sections we will present the result of our analysis in two steps:

1. We calculate the Ernst potential on the symmetry axis, $f(0,\zeta)$, from a linear integral equation (which we call the "small" integral equation).
2. From the solution of the small integral equation we obtain the kernel of a second linear integral equation, the "big" integral equation, whose solution provides us with the Ernst potential $f(\rho,\zeta)$ everywhere.

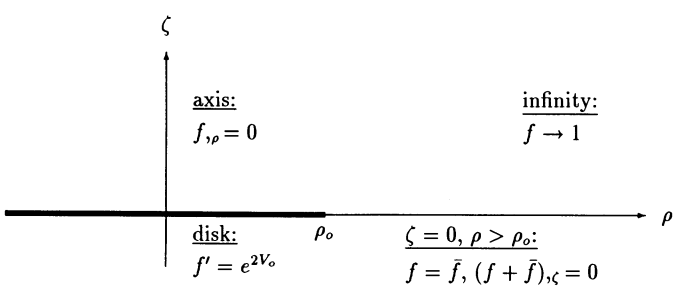

FIG. 1.—Boundary conditions, f' is the Ernst potential in the corotating frame of reference defined by $\rho'=\rho$, $\zeta'=\zeta$, $\varphi'=\varphi-\Omega t$, $t'=t(u^{i'}=e^{-V_0}\delta_4^{i'})$.

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

3.1. *The Axis Potential*

On the symmetry axis $\rho=0$ the Ernst potential is given by

$$f(\rho=0,\zeta>0)$$
$$=\left[2\pi+\int_{-1}^{1}\frac{\beta(x)dx}{ix-\zeta/\rho_0}\right]\bigg/\left[2\pi+\int_{-1}^{1}\frac{\alpha(x)dx}{ix-\zeta/\rho_0}\right], \qquad (10)$$

where $\alpha(x)$ algebraically depends on $\beta(x)$ and x:

$$\alpha(x)=\frac{(1-w)\beta+i\sqrt{4w^2e^{-4V_0}(b_0^2+4\Omega^2\rho_0^2x^2)-(e^{4V_0}+w^2)\beta^2}}{ib_0-2\Omega\rho_0x} \qquad (11)$$

with

$$w=2\Omega^2\rho_0^2(1-x^2)\,,\quad b_0\equiv b(\rho=0,\zeta=0^+)\,, \qquad (12)$$

and $\hat{\beta}=\beta/(8\Omega^3\rho_0^3e^{-4V_0})$ (the normalized β) satisfies the "small" integral equation

$$\hat{\beta}=x(1-x^2)+\mu^2\mathbb{D}\hat{\beta}\,, \qquad (13)$$

with

$$\mathbb{D}[f(x)]=-(1-x^2)^2f(x)-(1-x^2)\mathbb{C}[(1-x^2)\mathbb{C}f(x)]\,, \qquad (14)$$

and

$$\mathbb{C}f(x)=\frac{1}{\pi}\oint_{-1}^{1}\frac{f(x')dx'}{x'-x}\,, \qquad (15)$$

where $\oint$ denotes Cauchy's principal value. The parameter μ is defined by

$$\mu=2\Omega^2\rho_0^2e^{-2V_0}\,. \qquad (16)$$

The solution of the small integral equation can be written in the form of a von Neumann series,

$$\hat{\beta}=(1-\mu^2\mathbb{D})^{-1}[x(1-x^2)]=\sum_{n=0}^{\infty}\mu^{2n}\mathbb{D}^n[x(1-x^2)]\,. \qquad (17)$$

It converges for

$$0<\mu<\mu_0\,, \qquad (18)$$

where μ_0 is related to the first nontrivial eigenvalue of the homogeneous equation

$$\beta_0=\mu_0^2\,\mathbb{D}\beta_0\,. \qquad (19)$$

This value is given numerically by

$$\mu_0=4.62966184347\ldots \qquad (20)$$

In each order $(n=0,1,2,\ldots)$ $\mathbb{D}^n[x(1-x^2)]$ may be expressed in terms of elementary functions (polynomials and logarithms).

Each of the parameters V_0, $\Omega\rho_0$, and b_0 is a function of the parameter μ alone,

$$\sinh 2V_0=-\mu\left\{1+\frac{\mu^2}{\pi^2}\left[\int_{-1}^{1}\frac{\hat{\beta}(x,\mu)}{x}\,dx\right]^2\right\}, \qquad (21)$$

$$\Omega^2\rho_0^2=\tfrac{1}{2}\mu e^{2V_0}\,, \qquad (22)$$

and

$$b_0^2=1-2\mu e^{2V_0}-e^{4V_0}\,. \qquad (23)$$

Note that $b_0<0$ for $\Omega>0$.

The Newtonian limit can be characterized as $\mu\ll 1$ and the "ultrarelativistic limit" is given by $\mu\to\mu_0$.

An interesting rigorous result is the relation

$$2[1 - 2\Omega^2(\rho_0^2 + \zeta^2)]b(0, \zeta) + (2\Omega\zeta - b_0)[e^{4U(0,\zeta)} + b^2(0, \zeta)] = 2\Omega\zeta + b_0\,, \quad \zeta > 0\,, \quad (24)$$

which connects the real part and the imaginary part of the Ernst potential on the axis algebraically.

3.2. *The Ernst Potential Everywhere*

For arbitrary ρ, ζ, the Ernst potential is given by

$$f(\rho, \zeta) = \chi(\lambda = 1, \rho, \zeta)\,, \quad (25)$$

where $\chi(\lambda, \rho, \zeta)$ is a function of the complex variable λ which is normalized by $\chi(\lambda = -1, \rho, \zeta) = 1$. For fixed (but arbitrary) values of ρ and ζ the function $\chi(\lambda)$ is regular everywhere in the λ-plane except the curve Γ,

$$\Gamma:\quad \lambda = \sqrt{\frac{i\zeta + \rho_0 x - \rho}{i\zeta + \rho_0 x + \rho}}\,, \quad -1 \le x \le 1 \quad (\Re\lambda > 0 \text{ as } \zeta > 0)\,. \quad (26)$$

On Γ the function $\chi(\lambda)$ jumps in a well-defined way:

$$[\chi(\lambda)]_- = A(x)[\chi(\lambda)]_+ - B(x)\chi(-\lambda)\,. \quad (27)$$

The jump coefficients $A(x)$ and $B(x)$ (the "scattering data") are algebraic functions of $\beta(x)$ and x. Equation (27) belongs to a matrix Riemann-Hilbert problem and can be reformulated as a linear integral equation (our "big" integral equation). For parameter values $\mu > \mu^* = 1.0313067\ldots$ additional linear constraints on χ must be taken into consideration. This leads to a system consisting of a linear integral equation and two linear algebraic equations. The details will be published in a subsequent paper.

4. DISCUSSION

Here we confine ourselves to two subjects: (1) the ultrarelativistic limit of the space-time of the rotating disk and (2) the interrelationship between physical parameters as, e.g., the dependence of the angular velocity Ω on the total gravitational mass M and the angular momentum J.

1. For $\mu \to \mu_0$ the solution of the small integral equation diverges. As a consequence $V_0 \to -\infty$, $\rho_0 \to 0$ and $b_0 \to -1$. With the aid of the big integral equation it can be shown in a rigorous way that, for $\rho^2 = \zeta^2 \neq 0$, the solution approaches the extreme ($J = M^2$) Kerr solution. On the other hand, the metric in the vicinity of the disk will be obtained if the limit $\rho_0 \to 0$ is performed for finite $(\rho^2 + \zeta^2)/\rho_0^2$. We refer to the interesting discussion in Bardeen & Wagoner (1971). In our formalism this "interior" solution can be expressed in terms of the eigenfunction β_0 in equation (19). The relative binding energy $(M_0 - M)/M_0$ approaches the value of 37.328358... percent in the limit $\mu \to \mu_0$, cf. the remarkably excellent Padé approximation value of 37.323% given by Bardeen & Wagoner (1971).

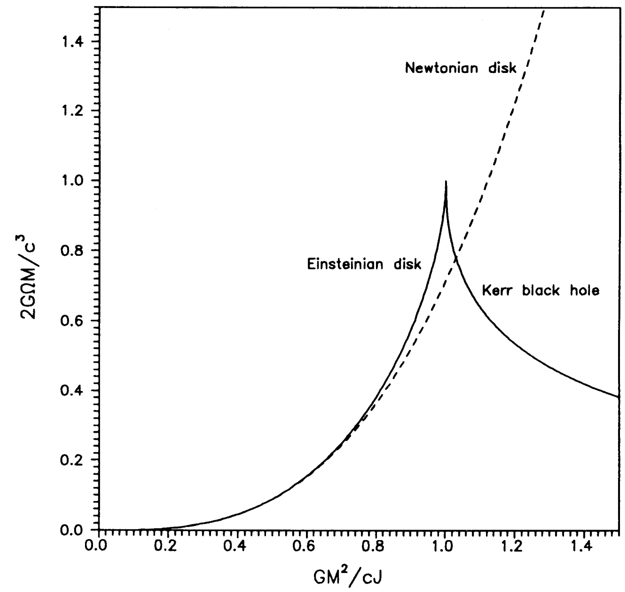

FIG. 2.—Rigidly rotating axisymmetric dust in general relativity (*solid line*) compared with the Newtonian case (*dashed line*). The latter is a good approximation for $GM^2/cJ \ll 1$ only. For $GM^2/cJ > 1$ the only possible state is the Kerr black hole where Ω denotes the "angular velocity of the horizon." The limit $GM^2/cJ \to \infty$ leads to the Schwarzschild solution ($\Omega = 0$).

2. The relations between the source parameters V_0 and Ω and the far field quantities M and J can be treated systematically in the framework of a "parameter thermodynamics," cf. Neugebauer & Herold (1992). The transition to the Kerr solution may provide new insights into black hole thermodynamics. As an example we consider the equation of state $\Omega = \Omega(M, J)$. It turns out that ΩM depends on M^2/J alone, see Figure 2. The rigidly rotating disk solution exists only for $M^2/J \le 1$ or, equivalently, $M_0^2/J \le 2.5459968\ldots$ and reaches its ultrarelativistic limit at $M^2/J = 1$. (Note that the surface mass density eq. [9] is indeed positive as $0 < \mu \le \mu_0$.) On the other hand, the Kerr solution is characterized by $M^2/J \ge 1$. The transition at $M^2/J = 1$ could be interpreted as a phase transition from normal matter to the black hole state. This transition is continuous in the "exterior" field and in all global (far field) parameters but discontinuous in the interior. The dashed line of Figure 2 shows $2\Omega M$ of the zero velocity dispersion MacLaurin disk which is given by

$$2\Omega M = \frac{9\pi^2}{125}\left(\frac{M^2}{J}\right)^3\,, \quad (28)$$

(cf. Binney & Tremaine 1987).

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