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Orbital, precessional, and insolation quantities for the Earth from -20Myr to +10 Myr

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Abstract. The solution for the precession and obliquity of the Earth, issued from the orbital solution La90 (Laskar 1990) is presented. This solution provides the necessary data for the computation of insolation at the surface of the Earth from -20 Myr to +10Myr. When taking into account the tidal dissipation, this solution presents very good agreement with the 3Myr numerical integration of (Quinn et al.1991). The main source of uncertainty in the computation of precession and obliquity of the Earth is found to arise from the changes of dynamical ellipticity of the Earth which can occur during an ice age. This change is especially important because of the existence of a resonant effect with a secular term of frequency $s_6 - g_6 + g_5$ resulting from the perturbations of Jupiter and Saturn. In order to overcome this difficulty, the nominal solution is provided together with FORTRAN programs which would allow to fit the unknown parameters of the solution to geological records.

Key words: celestial mechanics: insolation, precession – Earth – solar system: numerical methods, resonances

1. Introduction

The insolation parameters of the Earth depend on its orbital parameters and on its precession and obliquity. Until 1988, the solution usually adopted for paleoclimate computation consisted of the orbital elements of the Earth from (Bretagnon 1974), complemented by the computation of the precession and obliquity of the Earth of (Berger 1976). In 1988, Laskar issued a solution for the orbital elements of the Earth, which was obtained in a new manner, making use of vast analytical computations and numerical integration (Laskar 1988). In this solution, denoted La88, the precession and obliquity quantities necessary for paleoclimate computations were integrated at the same time, which insured good consistency of the solutions. Unfortunately, for various reasons, this latter solution for the precession and obliquity was not widely distributed (Berger et al. 1988). On the other hand, the quasiperiodic approximation of the orbital part of the solution La88 for the Earth was used in (Berger & Loutre 1991) to derive another solution for precession and obliquity, aimed at climate computations.

Later on, Laskar issued a new solution (La90) which includes some slight improvements over the previous one. As previously, this solution contains orbital, precessional and obliquity variables, but in the paper of (Laskar 1990), only the orbital solutions are discussed. The precession solution was also distributed several times in magnetic form, but it seems important to present this solution in more detail in order to make it more widely available. It also now appears that some astronomical parameters may be improved by taking geological records into account, and in the present paper, the solution is augmented by some routines which allows one to test various hypotheses of long term changes in the tidal forces due to the Moon, or in the changes of dynamical ellipticity of the Earth. All the files and related programs can be obtain by requesting them from the authors at laskar@cosme.polytechnique.fr.

2. The orbital solution La90

The orbital solution La90 is obtained by the numerical integration of an extended averaged system, which represents the mean evolution of the orbits of the planets. All the 8 main planets of the solar system are taken into account, as well as the main lunar and relativistic perturbations. The use of numerical integration for computing the solution of the secular system is one of the reasons for the good quality of this solution, which was checked by comparing with the available ephemeris over a short time scale (Laskar 1986, 1988). In (Laskar 1988), the solution La88 was represented in quasi-periodic form over 10 Myr, but these representations are slowly convergent, which prevents good accuracy of the solution.

Later on, the reason for this slow convergence was understood to be due to the presence of multiple resonances in the secular system of the inner solar system (Laskar 1990). Because of these resonances, the motion of the solar system is chaotic, and not quasi-periodic, as was first demonstrated by the computation of its Lyapunov exponents which reaches 1/(5

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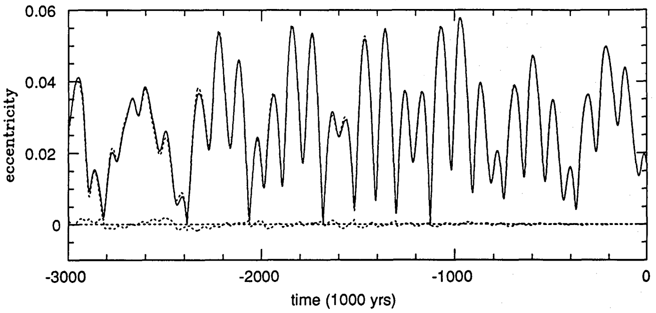

Fig. 1. Comparison of the solutions La90 (solid line) and the solution QTD (dotted line) (Quinn et al.1991) for the eccentricity of the Earth over the past 3Myr from J2000. The difference between the two solutions is also in dotted line

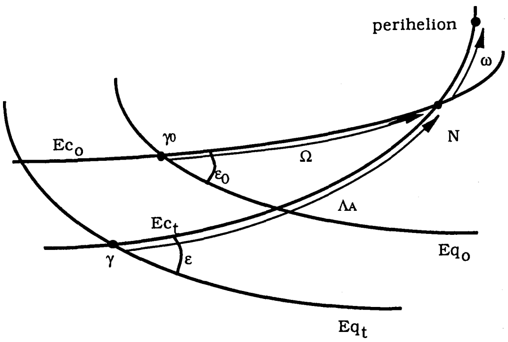

Fig. 2. Fundamental planes for the definition of the precession. Eq_t and Ec_t are the mean equator and ecliptic of the date, while Eq_0 and Ec_0 are the J2000 fixed mean equator and ecliptic. The general precession in longitude, p_A is defined as $p_A = \Lambda_A - \Omega$; $\varpi = \Omega + \omega$ is the longitude of perihelion from the equinox of reference γ_0; $\omega^{\bullet} = \varpi + p_A$ is the longitude of perihelion from the equinox of the date γ_t; ϵ denotes the obliquity

Myr) (Laskar 1989). This implies that it is not possible to give any precise solution for the motion of the Earth over more than about 100 Myr, and most probably, ephemerides can only be given with good precision for about 10 Myr to 20 Myr.

Since then, a direct numerical integration over 3Myr was published by (Quinn et al.1991), including also solutions for precession and obliquity, which will be denoted QTD in the sequel. The orbital solutions QTD have been compared with La90. The two solutions present very small discrepancies over 3Myr (Fig. 1), and the existence of the secular resonances in the inner solar system was confirmed by this new solution (Laskar et al.1992b). The very close agreement of these two orbital solutions over 3Myr shows that both solutions are of good quality, and confirms that the Earth parameters are now very well known over this time span. This also insures that the orbital solution La90 can be used with confidence over 10 Myr to 20 Myr for paleoclimate studies.

3. Precession and obliquity in the La90 solution

The precession quantities are completely determined by the two motions of the equatorial and ecliptic pole (Fig 2). In our computations, the motion of the ecliptic is given by the secular theory La90 while the precession quantities are integrated using the equations of the rigid-Earth theory of Kinoshita (Kinoshita 1977; Laskar 1986). The equations for the general precession in longitude p_A, and for the obliquity of the date ε are then

$$\begin{aligned}\frac{dp_A}{dt} &= R(\varepsilon) - \cot\varepsilon\left[\mathbf{A}(\mathbf{p},\mathbf{q})\sin p_A + \mathbf{B}(\mathbf{p},\mathbf{q})\cos p_A\right] \\ &\qquad -2\,\mathbf{C}(\mathbf{p},\mathbf{q}) - p_g \\ \frac{d\varepsilon}{dt} &= \quad -\mathbf{B}(\mathbf{p},\mathbf{q})\sin p_A + \mathbf{A}(\mathbf{p},\mathbf{q})\cos p_A\end{aligned} \tag{1}$$

with :

$$\begin{aligned}\mathbf{A}(\mathbf{p},\mathbf{q}) &= \frac{2}{\sqrt{1-\mathbf{p}^2-\mathbf{q}^2}}\;(\dot{\mathbf{q}} + \mathbf{p}(\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\dot{\mathbf{q}})) \\ \mathbf{B}(\mathbf{p},\mathbf{q}) &= \frac{2}{\sqrt{1-\mathbf{p}^2-\mathbf{q}^2}}\;(\dot{\mathbf{p}} - \mathbf{q}(\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\dot{\mathbf{q}})) \\ \mathbf{C}(\mathbf{p},\mathbf{q}) &= (\mathbf{q}\dot{\mathbf{p}} - \mathbf{p}\dot{\mathbf{q}})\end{aligned} \tag{2}$$

and :

$$\begin{aligned}R(\varepsilon) &= \frac{3k^2 m_M}{a_M^3\nu}\frac{C-A}{C} \\ &\times\left[(\mathrm{M}_0 - \mathrm{M}_2/2)\cos\varepsilon + \mathrm{M}_1\frac{\cos 2\varepsilon}{\sin\varepsilon}\right. \\ &\left.-\mathrm{M}_3\frac{m_M}{m_E+m_M}\frac{n_M^2}{\nu n_\Omega}\frac{C-A}{C}(6\cos^2\varepsilon - 1)\right] \\ &+\frac{3k^2 m_\odot}{a_\odot^3\nu}\frac{C-A}{C}\left[\mathrm{S}_0\cos\varepsilon\right],\end{aligned} \tag{3}$$

where $\mathbf{p} = \sin(i/2)\sin(\Omega)$, $\mathbf{q} = \sin(i/2)\cos(\Omega)$, ($i$ is the inclination of the Earth with respect to a fixed ecliptic, and Ω the longitude of the node). $R(\varepsilon)$ is the secular term due to the direct lunisolar perturbations. The quantities $\mathrm{M}_0, \mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$, and S_0 depend only on the orbital elements of the Moon and the Sun. The principal moments of inertia of the Earth are A, A, and C, and the angular velocity of the Earth is ν . The masses of the Sun, the Earth, and the Moon are denoted by $m_\odot$, m_E, and m_M; the sideral mean motion of the Sun and of the Moon by $n_\odot$ and n_M ; and the mean velocity of the node of the Moon by n_Ω. The other terms present in Eq. (3) represent the effects of the secular variation of the ecliptic, caused by the secular planetary perturbations. The numerical values of $\mathrm{M}_0, \mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$ are given in (Kinoshita 1977):

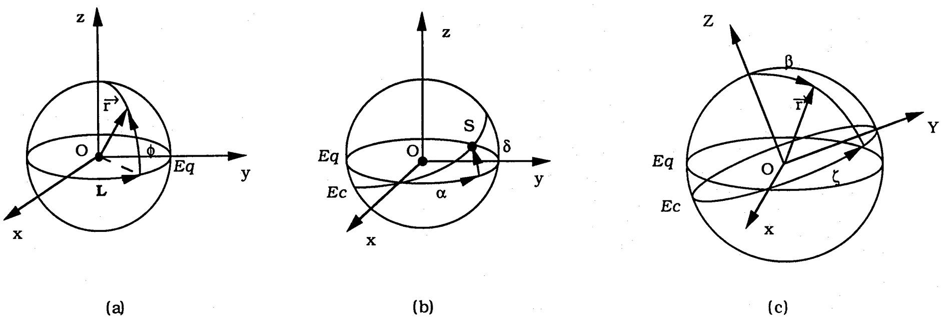

Fig. 3. (a) Equatorial coordinates for a point at the surface of the Earth. (b) Equatorial coordinates for the Sun; (c) Ecliptic coordinates

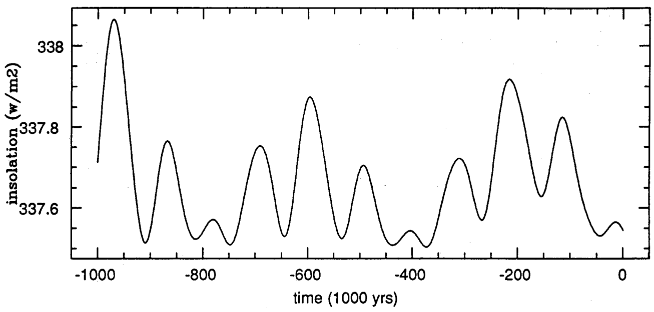

Fig. 4. Mean annual insolation of the Earth, for 1 Myr in the past

$$
\begin{aligned}
M_0 &= 496303.3 \times 10^{-6} \\
M_1 &= -20.7 \times 10^{-6} \\
M_2 &= -0.1 \times 10^{-6} \\
M_3 &= 3020.2 \times 10^{-6}
\end{aligned} \tag{4}
$$

and from (Laskar 1986),

$$S_0 = \frac{1}{2}(1-e^2)^{-3/2} - 0.522 \times 10^{-6}, \tag{5}$$

where e is the eccentricity of the Earth. It should be noted that the S_0 given here is equal to the quantity $S_0 - S_2/2$ in (Laskar 1986). The following numerical values were also used (see Laskar 1986, for complete references) :

$$
\begin{aligned}
\nu &= 474\,659\,981.597\,57\ ''/\text{yr} \\
n_M &= 17\,325\,593.4318\ ''/\text{yr} \\
n_\Omega &= -69\,679.193\,6222\ ''/\text{yr} \\
a_M &= 384\,747\,980.645\ \text{m} \\
k &= 0.017\,202\,098\,95 \\
m_\odot/(m_E+m_M) &= 328\,900.5 \\
m_\odot/m_E &= 332\,946.0
\end{aligned} \tag{6}
$$

The quantity p_g is the geodesic precession due to the general relativity,

$$p_g = 0.019188''/\text{yr} \tag{7}$$

The value of the dynamical ellipticity $E_D = (C-A)/C$ is obtained by adjustment at the origin J2000 to the values of the speed of precession and obliquity given by the IAU (Grenoble 1976) :

$$
\begin{aligned}
p &= 50.290966\ ''/\text{yr} \\
\varepsilon_0 &= 23^\circ 26' 21''.448\ .
\end{aligned} \tag{8}
$$

For $t = 0$, we have $p_A = 0$, $\varepsilon = \varepsilon_0$, $i = \Omega = 0$, and thus :

$$\left.\frac{dp_A}{dt}\right|_{t=0} = R(\varepsilon_0) - 2\ \dot{\mathbf{p}}|_{t=0} \cot\varepsilon_0 - p_g\ , \tag{9}$$

which gives the numerical value $E_D = 0.00328005$. This value may be slightly different from values obtained with other models, but it should be stressed that the value of the dynamical ellipticity depends on the model of precession used, and is adjusted in order to fit the observed initial conditions for the speed of precession and obliquity at the origin (8).

4. Computation of the insolation

The computation of the insolation quantities is now classical, (Sharaf & Boudnikova 1967; Ward 1974; Berger 1978), but we decided, for completeness and clarity, to briefly present them here.

4.1. Notations

The energy crossing one unit of terrestrial surface area, normally to the direction of the Sun, per unit time, at 1 AU (Astronomical Unit), although not constant on billion-year time scales, can be considered as fixed over a few millions of years and is called usually the "solar constant," S. In the numerical applications of the present work, we used the value $S = 1350$ W.m^{-2}. This quantity is a simple parameter in our programs and can be easily changed for different possible values of the solar constant. We also neglected the secular variations of the semi major axis a which are not present in the secular equations up to second order with respect to the masses. Under the assumption that the atmosphere is perfectly transparent, or in an equivalent way, if the measure is made at the top of the atmosphere, the insolation at the surface of the Earth will be

$$
\begin{aligned}
W &= S\frac{(\vec{r}\,.\vec{r_s})}{\rho^2} && \text{if} \quad \vec{r}\,.\vec{r_s} > 0 \\
W &= 0 && \text{if} \quad \vec{r}\,.\vec{r_s} \le 0
\end{aligned} \tag{10}
$$

where $\vec{r}$ is the unit vector defining the position of the surface element on the Earth, and $\vec{r_s}$ the unit vector directed towards the Sun. We will use the following notations:

The origin O is taken to be at the center of the Earth. The equatorial reference frame is (O,x,y,z) and the ecliptic reference frame (O,x,Y,Z). Ox is thus the direction of the vernal equinox, Oz the normal to the equator, and OZ the normal to the ecliptic. In the equatorial reference frame, (L,ϕ) denotes the right ascension and the declination of a point on the planet surface (Fig. 3a), (ϕ is thus also the latitude), while (α,δ) represent the same quantities for the Sun (Fig. 3b). In the ecliptic reference frame, (ζ,β) are the ecliptic longitude and colatitude of the point on the Earth (Fig. 3c).

The different elliptical elements for the orbit of the Earth are the semi major axis a, the eccentricity e, the obliquity ε, the true anomaly v, the argument of perihelion ω, the longitude of the ascending node Ω, the longitude of perihelion from a fixed origin $\varpi = \Omega + \omega$, and the longitude of perihelion from the moving equinox $\omega^\bullet = p_A + \varpi$.

4.2. Mean annual insolation

The daily mean insolation at the surface of the Earth is easily obtained by dividing the intercepted solar flux by the surface area of the Earth of radius $\mathscr{R}$, that is

$$W_{dm} = \frac{S}{\rho^2}\frac{\pi\mathscr{R}^2}{4\pi\mathscr{R}^2} = \frac{S}{4\rho^2} \,. \tag{11}$$

The mean annual insolation is then obtained by integration over a full orbit of the Earth

$$W_{am} = \frac{1}{2\pi}\int_0^{2\pi}\frac{S}{4\rho^2}dM \tag{12}$$

where M denotes the mean anomaly. This is easily computed using the area law $\frac{dv}{dM} = \frac{a^2}{r^2}\sqrt{1-e^2}$ (v is the true anomaly) and gives

$$W_{am} = \frac{S}{4}(1-e^2)^{-1/2} \,. \tag{13}$$

The mean annual insolation depends thus only on the eccentricity of the Earth, and its variations over 1 Myr are given in Fig. 4. It should be noted that these variations are very small, as they depend on the square of the eccentricity. In fact, this is not the main paleoclimate quantity, and it was recognized by Milankovitch (see Imbrie 1982) that the summer insolation at high latitudes had a larger influence on the climate of the past. If the insolation in summer is not high enough, the ice does not melt, and the ice caps can extend. This is why it is also important to compute the daily insolation at a given point on the Earth.

4.3. Daily insolation as a function of the latitude

The daily insolation at a given latitude on the Earth is the mean insolation during a full rotation of the Earth, for a location of given latitude ϕ. The orbital distance in units of the semimajor axis is

$$\rho = \frac{r}{a} = \frac{1-e^2}{1+e\cos v} \,. \tag{14}$$

In the equatorial frame, one obtains easily (Fig. 3) :

$$\begin{aligned} \vec{r} &= (\cos\phi\cos L,\ \cos\phi\sin L,\ \sin\phi) \\ \vec{r_s} &= (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta) \end{aligned} \tag{15}$$

thus

$$\vec{r}.\vec{r_s} = \sin\phi\sin\delta + \cos\phi\cos\delta\cos(L-\alpha) \,. \tag{16}$$

The insolation is null when the Sun is on the opposite side of the Earth, that is, when the scalar product $\vec{r}.\vec{r_s}$ is negative. We have

$$\vec{r}.\vec{r_s} \geq 0 \iff \cos(L-\alpha) \geq -\tan\phi\tan\delta \tag{17}$$

where the equalities holds for sunrise and sunset. Different cases can occur:

a) $1 < -\tan\phi\tan\delta$: in this case, we have always $\vec{r}.\vec{r_s} < 0$: and the Sun never rises.

b) $-1 > -\tan\phi\tan\delta$: in this case, we have always $\vec{r}.\vec{r_s} > 0$ and the Sun never sets.

c) $-1 \leq -\tan\phi\tan\delta \leq 1$: In this case, there exist sunrises and sunsets and it is convenient to introduce the horary angle of sunset and sunrise $H_0 \in [0,\pi]$ defined by $\cos(H_0) = -\tan\phi\tan\delta$. We thus have

$$\vec{r}.\vec{r_s} > 0 \iff -H_0+\alpha < L < H_0+\alpha \,. \tag{18}$$

To each of the three previous cases corresponds the following expression for the daily insolation at a point on the Earth with a given latitude ϕ, obtained by averaging over the longitude:

$$\begin{aligned} W_{j\phi}^{(a)} &= 0 \\ W_{j\phi}^{(b)} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{S}{\rho^2}(\vec{r}.\vec{r_s})dL \\ W_{j\phi}^{(c)} &= \frac{1}{2\pi}\int_{-H_0+\alpha}^{H_0+\alpha}\frac{S}{\rho^2}(\vec{r}.\vec{r_s})dL \,, \end{aligned} \tag{19}$$

that is

$$\begin{aligned} W_{j\phi}^{(a)} &= 0 \\ W_{j\phi}^{(b)} &= \frac{S}{\rho^2}\sin\phi\sin\delta \\ W_{j\phi}^{(c)} &= \frac{S}{\pi\rho^2}(H_0\sin\phi\sin\delta + \cos\phi\cos\delta\sin(H_0)) \,. \end{aligned} \tag{20}$$

The declination of the Sun δ is obtained through the true longitude of the date w_d (measured from the moving equinox) as $\sin\delta = \sin w_d \sin\varepsilon$. In particular, one can compute the daily insolation for the summer solstice ($w_d = \pi/2$) and for the winter solstice ($w_d = 3\pi/2$).

Some other insolation quantities appear in the paleoclimate literature, as monthly or annual averages, but they can all be easily deduced in a numerical manner from the previous ones. A remark should be make at this point, as otherwise some confusion could occur. For a given date t, the parameters of the Earth are all determined, including its longitude. Thus, theoretically, it should be sufficient to give the date t, and the computation of the insolation could be carried out. But in fact, it is useless to keep track of the position of the Earth along its orbit. Indeed, during one year, the elements of the orbit as well as the obliquity and precession will barely change. It is more important to know the different values of the insolation for different positions of the Earth along its orbit with respect to the location of the vernal equinox, which regulates the seasons. This is why, practically, we use one time t for defining the slowly changing parameters like the elements of the orbit and orientation of the Earth, and introduce a second, local time λ_d, which indicates the position of the Earth along its orbit, that is, the date in this local year, which will be given as the mean longitude from the equinox at time t, and which will range from 0 degrees to 360 degrees. The mean longitude λ_d can be related to the true longitude of date $w_d = \omega^\bullet + v$ through the Kepler equation

$$E - e\sin E = \lambda_d - \omega^\bullet \,, \tag{21}$$

where E is the eccentric anomaly, and its relation to the true anomaly v is given by:

$$\cos v = \frac{\cos E - e}{1 - e\cos E} \,. \tag{22}$$

Monthly insolations are also used. In this case, a month will correspond to a length of 30 degrees in mean longitude λ_d (the mean longitude is proportional to the time), starting with the vernal equinox. Sometimes in the literature the vernal equinox ($\lambda_d = 0$) is given by the conventional date of March 21, and referred to as "mid-month", but our feeling is that these conventions, trying to define some arbitrary calendar dates, should be avoided as they generate confusion. We will thus stick to dates given by the mean longitude from the equinox of the date λ_d. In Fig. 5, the variations of the insolation at 65 degrees of northern latitude are given over 1 Myr in the past, for $\lambda_d = 120$ degrees (summer). It can be seen that these variations reach about 20% and are thus much more important than the variations of the annual mean insolation given in Fig. 4.

5. Integration of the precession equations

The Eqs. (1) give a solution for precession and obliquity in agreement with the requirements of highly accurate ephemerides over a few thousand years (Laskar 1986), but their form is not well suited for numerical integration over an extended time span,

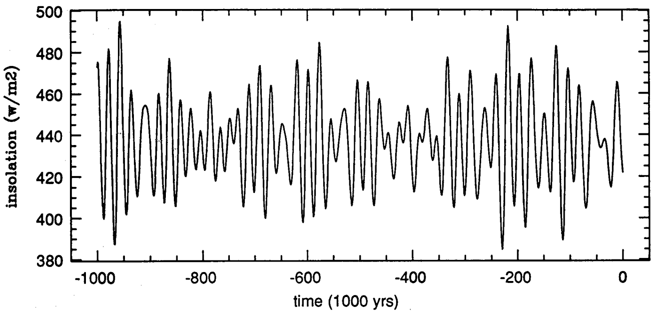

Fig. 5. Mean daily insolation of the Earth, at 65N, and $\lambda_d = 120$ degrees for 1 Myr in the past

as the precession angle, p_A will accumulate too much and could introduce some numerical errors. Another problem is the indeterminacy of p_A when the obliquity goes to zero (hopefully, this will not happen for the Earth !). This leads us to change the form of the equations slightly. Let

$$\begin{aligned} R'(\varepsilon) &= R(\varepsilon) - 2\,\mathbf{C}(\mathbf{p},\mathbf{q}) - p_g \\ \chi &= \sin\varepsilon e^{ip_A}\,; \end{aligned} \tag{23}$$

then Eq. (1) become

$$\frac{d\chi}{dt} = iR'(\varepsilon)\chi + \cos\varepsilon(\mathbf{A}(\mathbf{p},\mathbf{q}) - i\mathbf{B}(\mathbf{p},\mathbf{q}))\,. \tag{24}$$

With this formulation, which is only useful when $\varepsilon < \pi/2$, it appears more clearly that the system is an oscillator with proper frequency $R'(\varepsilon)$, excited by the secular change of orbital plane given by $(\mathbf{A}(\mathbf{p},\mathbf{q}) - i\mathbf{B}(\mathbf{p},\mathbf{q}))$. We have

$$R(\varepsilon) = C_1\cos\varepsilon + C_2\frac{\cos 2\varepsilon}{\sin\varepsilon} + C_3(6\cos^2\varepsilon - 1) + C_4 S_0\cos\varepsilon\,, \tag{25}$$

with the numerical values obtained with the constants (6)

$$\begin{aligned} C_1 &= 37.526603\ ''/\mathrm{yr} \\ C_2 &= -0.001565\ ''/\mathrm{yr} \\ C_3 &= 0.000083\ ''/\mathrm{yr} \\ C_4 &= 34.818618\ ''/\mathrm{yr}\,. \end{aligned} \tag{26}$$

A usual way to integrate these precession equations is to use some quasiperiodic expression for the eccentricity of the Earth and for the inclination, and then to integrate the equations by perturbation methods. As was stated before, this method presents severe drawbacks. The first one is that in the analytical resolution, one has to truncate the solutions, but the most important problem is the fact that quasiperiodic expressions do not approximate the orbital solutions for the motion of the Earth very well, as this motion is chaotic. Even over 5Myr, the presence of the strong secular resonances between the Earth and Mars prevents a good quasiperiodic approximation of the orbital solution of the Earth. This can be easily understood by looking

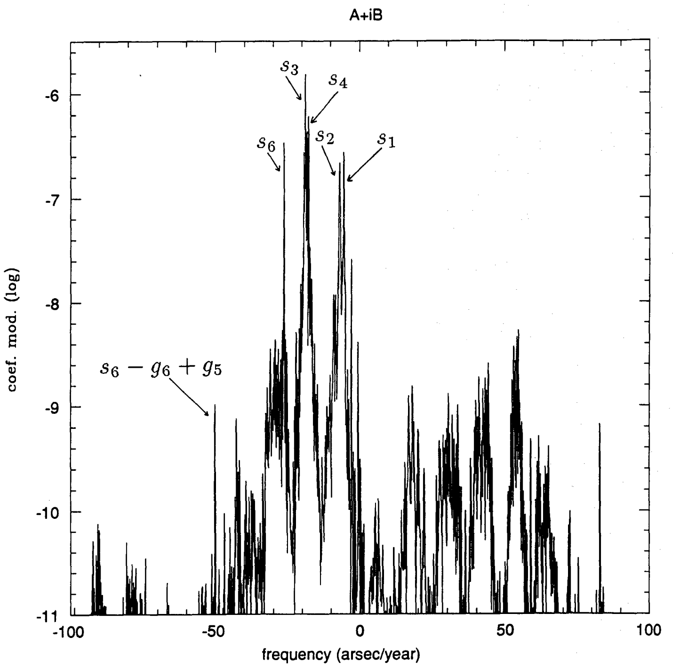

Fig. 6. Fourier spectrum of $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$ over 17 Myr obtained with a Hanning filter (Eq. 31). Only the main secular frequencies of the solar system can be identified, as well as the small isolated term $s_6 - g_6 + g_5$

at the Fourier spectrum of the $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$ function, obtained from the solution La90 over 17 Myrs (Fig. 6). In this plot, the logarithm of the spectrum amplitude is given against the frequency, expressed in arcseconds per year. Several peaks are well identified, as they correspond to the main secular frequencies g_i and s_i of the solar system, but very quickly, when one wants to go further than this crude approximation, one encounters lots of unidentified spectral lines which arise because of the presence of the secular resonances and because of the chaotic behavior of the solution. Indeed, the spectrum looks nearly continuous in several places. Nevertheless, a quasiperiodic approximation of $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$ will still be very useful for the qualitative understanding of the solutions.

The equations for precession were first integrated numerically at the same time as the orbital elements La90, but it appears that the orbital elements were in fact much more reliable than the precession quantities. In fact, several factors are not very well known which can induce sensible changes in the precession and obliquity solutions. We thus decided to make available to the community of researchers in this field, not only a solution for orbital and precessional motion, but also a complete package which enables anyone to change slightly the model of precession and the related constants in order to test the solution against some possible precise observations arising from geological records.

The computation of the precession and obliquity is a much simpler task than the computation of the Earth orbital elements, and we found that it could be very useful to provide the means for anyone to undertake these computations on an average computer.

The method to be used is to provide a file for the orbital parameters of the Earth, $\mathbf{k},\mathbf{h},\mathbf{q},\mathbf{p}$ defined as

$$\begin{aligned}\mathbf{k} &= e\cos(\varpi)\\ \mathbf{h} &= e\sin(\varpi)\\ \mathbf{q} &= \sin(i/2)\cos(\Omega)\\ \mathbf{p} &= \sin(i/2)\sin(\Omega)\,,\end{aligned} \tag{27}$$

where e is the eccentricity of the Earth, i denotes its inclination, ϖ the longitude of perihelion, and Ω the longitude of the node with respect to the fixed ecliptic and equinox J2000. These parameters are given every 1000 years, counting from J2000. We decided to provide the solution for 20 Myr in the past, and 10 Myr in the future. The precision should decrease after 10 Myr, but it seems that it could be used in the -10 Myr to -20 Myr range for qualitative studies.

The equations to be integrated numerically are the precession Eqs. (24). As the main frequency of the precession is about 50 $''$/yr, which corresponds to a period of about 20000 years, the stepsize to be taken in the Adams multistep method which we used is on the order of 200 yr. Using a multistep method, one also needs a starting table which will be constructed using a Runge Kutta scheme of order 8 (RK8(7)), as given by (Hairer et al.1987). This implies computing the orbital elements for all values of time, and they were obtained from the files of the positions every 1000 years by using interpolation polynomials of order 7. We also need the computation of the derivatives of $\mathbf{p}$ and $\mathbf{q}$, which are obtained in a very accurate way using the central difference method of (Mineur 1952).

Once this preliminary work is complete, the numerical integration is rapid; this led us to carry out several experiments on the possible changes in the model of precession, in order to test its reliability (Sect. 7).

6. Frequency analysis

The frequency analysis is a numerical method developed by J. Laskar for the analysis of the stability of the solutions of a conservative dynamical system, based on a refined numerical search for a quasiperiodic approximation of its solutions over a finite time span (Laskar 1990, 1992; Laskar et al.1992a). If $f(t)$ is a function with values in the complex domain, obtained numerically over a finite time span $[-T, T]$ the frequency analysis algorithm will consist in the search for a quasiperiodic approximation for $f(t)$ with a finite number of periodic terms of the form

$$\tilde{f}(t) = \sum_{k=1}^{N} a_n e^{i\sigma_k t}\,. \tag{28}$$

The frequencies σ_k and complex amplitudes a_k are found with an iterative scheme. The quoted papers fully describe this method, which we will only outline here. To determine the first frequency σ_1, one searches for the maximum of the amplitude of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t}\rangle \tag{29}$$

where the scalar product $\langle f(t), g(t)\rangle$ is defined by

$$\langle f(t), g(t)\rangle = \frac{1}{2T}\int_{-T}^{T} f(t)\bar{g}(t)\chi(t)dt\ , \tag{30}$$

and where $\chi(t)$ is a weight function, that is, a positive function with $1/2T\int_{-T}^{T}\chi(t)dt = 1$. In all our computations, we used the Hanning window filter, that is

$$\chi(t) = 1 + \cos(\pi t/T)\ , \tag{31}$$

although some other weight functions could be used. Once the first periodic term $e^{i\sigma_1 t}$ is found, its complex amplitude a_1 is obtained by orthogonal projection, and the process is started again on the remaining part of the function $f_1(t) = f(t) - a_1 e^{i\sigma_1 t}$. As all the different functions $e^{i\sigma_k t}$ are not orthogonal, it is also necessary to orthogonalize the set of functions $(e^{i\sigma_k t})_k$, when projecting f iteratively on these $e^{i\sigma_k t}$.

If $f(t)$ is a quasiperiodic function, like the regular solution of a Hamiltonian system, $f(t)$ can be written in the form

$$f(t) = \sum_{k=1}^{\infty}\alpha_k e^{i\nu_k t}\ , \tag{32}$$

where the α_k are of decreasing amplitude. The expression for the approximation $\tilde{f}(t)$ obtained by frequency analysis is then very close to the original function. This means that the a_k and σ_k are very close to the α_k and ν_k, which would not be the case with a simple FFT. For regular solutions, and when the coefficients a_k decrease rapidly, the algorithm allows a very accurate determination of the frequencies of largest amplitude, several orders of magnitude better than with a simple FFT. This refined method is very powerful in the analysis of the dynamics of conservative dynamical systems of many degrees of freedom (Laskar et al.1992a; Laskar 1992).

The analysis of the spectrum of the orbital solution of the Earth is important, as it could reveal the existence of some small terms which might be in resonance with the main precession frequency, which is about 50"$/yr$. Indeed, one can see a well isolated peak in the FFT spectrum of Fig. 6, which by frequency analysis has a frequency of -50.3021"$/yr$, and can be identified with no doubt as $s_6 - g_6 + g_5$, where g_5 and g_6 are the secular frequencies usually related to the perihelion of Jupiter and Saturn, while s_6 is related to the node of Saturne. The full application of the frequency analysis algorithm gives a quasiperiodic approximation of $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$, and is given in Table 1. In this table, only the 12 largest terms are given, and the 13th term is the $s_6 - g_6 + g_5$ term which was added because of its dynamical importance, which will be analyzed in more depth in Sect. 7. In fact, the algorithm stops automatically, when the periodic terms are no longer considered relevant.

Table 1. Quasiperiodic approximation of $\mathbf{A} + i\mathbf{B}$ obtained by frequency analysis over 18 Myr. The 12 Major terms are listed as well as a smaller, well isolated term, due to the perturbation of Jupiter and Saturn. $\mathbf{A} + i\mathbf{B} \approx \sum_{k=1}^{13} A_k e^{i(\sigma_k t + \phi_k)}$ (the A_k are expressed in yr^{-1}).

k		σ_k ("/yr)	$A_k \times 10^6$	ϕ_k (°)
1	s_3	−18.8504	1.616070	151.724
2	s_4	−17.7544	0.691588	199.002
3		−18.3016	0.478868	176.641
4	s_6	−26.3302	0.340738	37.294
5	s_1	−5.6128	0.274325	270.479
6		−19.3997	0.286930	305.514
7	s_2	−7.0772	0.237068	9.899
8		−19.1251	0.165838	46.398
9		−6.9564	0.132989	199.316
10		−7.2037	0.112089	176.470
11		−6.8283	0.108391	233.037
12		−5.4892	0.080168	289.422
13	$s_6 - g_6 + g_5$	−50.3021	0.001043	120.161

The approximation made with this quasiperiodic expression is not very precise, as can be deduced from the bad convergence of the amplitude of its terms, but it gives the most important terms, and going much further may not be dynamically very significant. Indeed, among these leading terms, there are already very complicated dependencies of the periodic terms which result from the secular resonances in the inner solar system (Laskar 1990). The only terms which are simply identified are the periodic terms corresponding to the linear part of the secular solution, that is, the terms of frequencies s_3, s_4, s_6, s_1, s_2. The other terms will be denoted by their generic frequency σ_i. The extra frequency $s_6 - g_6 + g_5$, resulting from the Jupiter and Saturn perturbations will be denoted by f. It should be stressed that in the La88 solution, which was given uniquely as a quasiperiodic approximation, this last term was not present due to its small amplitude. This would be of no importance unless, as presently, it appears to be nearly in resonance with the main frequency of precession.

In this work, we preferred to use numerical integration of the precession equations rather than perturbation methods with trigonometric series. The essential reason for this was already stated, and is the fact that the solutions for the orbital elements of the Earth are not quasiperiodic, and moreover are not conveniently approximated by quasiperiodic expressions over the long span of time of a few million years which we consider in this work. Moreover, very small periodic terms, like $s_6 - g_6 + g_5$, which have been neglected in quasiperiodic approximations may have some importance due to the resonance effect. All this favors numerical integrations, especially when these numerical integrations can be carried out on averaged equations and are thus very easy to undertake. Nevertheless, the orbital or precessional solutions are still not very far from quasiperiodic solutions over a limited time, and it is useful for the dynam-

Table 2. Quasiperiodic approximation of $\chi = \sin\varepsilon\, e^{ip_A}$ obtained by frequency analysis over 18 Myr.

$$\chi(t) \approx \sum_{k=1}^{34} B_k e^{i(\nu_k t + \psi_k)} ,$$

where the B_k coefficients are dimensionless. In column two, the terms wich are well identified are given, expressed in terms of the secular frequencies g_i and s_i of the solar system, of the fundamental precession frequency p, and of the resonant frequency $f = s_6 - g_6 + g_5$. The other terms can also be expressed in terms of the frequencies σ_k of $\mathbf{A} + i\mathbf{B}$ (Table 1) which might result from more complicated interactions.

k		$\nu_k('' /\mathrm{yr})$	B_k	$\psi_k(\mathrm{deg})$
1	p	50.4712	0.392066	7.718
2	f	50.3017	0.033291	329.456
3	$2p - f$	50.6408	0.032118	226.240
4	$-s_3$	18.8507	0.011062	298.608
5	$-s_4$	17.7533	0.004573	248.612
6	$-s_6$	26.3302	0.003191	52.704
7	$-\sigma_3$	18.3011	0.003258	272.424
8	$p + g_3 - g_4$	49.9245	0.002234	359.588
9	$p + g_4 - g_3$	51.0199	0.002173	200.934
10	$-\sigma_6$	19.4005	0.002030	146.363
11		50.8342	0.001956	139.149
12	$2p + s_3$	82.0908	0.001388	254.778
13	$-s_1$	5.6120	0.001284	176.832
14		50.1600	0.001285	2.051
15	$-\sigma_8$	19.1255	0.001120	43.820
16	$-s_2$	7.0773	0.001136	82.426
17	$p + g_2 - g_5$	53.6768	0.000902	173.456
18	$p - g_2 + g_5$	47.2655	0.000899	21.803
19	$p + g_5 - g_1$	49.1217	0.000679	98.474
20	$p + g_1 - g_5$	51.8203	0.000674	96.179
21	$-\sigma_9$	6.9492	0.000635	231.817
22	$-\sigma_{10}$	7.1961	0.000536	258.246
23	$2p + s_4$	83.1895	0.000551	307.846
24	$-\sigma_{11}$	6.8259	0.000519	208.423
25	$2p + s_6$	74.6121	0.000501	142.458
26		18.9599	0.000445	261.476
27	$2p + \sigma_3$	82.6417	0.000387	284.322
28	$-\sigma_{12}$	5.4865	0.000369	153.860
29		49.7240	0.000353	238.712
30		18.5455	0.000319	131.224
31		48.6168	0.000315	116.498
32		52.3235	0.000312	74.288
33		82.2652	0.000266	122.071
34		81.9390	0.000256	252.485

ical understanding of the solutions to also obtain the leading terms of a quasiperiodic approximation of the solutions of the precession equations.

The results of the frequency analysis of the precession solution, or more precisely of $\chi(t) = \sin\varepsilon(t)e^{ip_A(t)}$, are given in Table 2. This time, the frequency analysis provides us many more terms, and most of these can be easily identified as combinations of the periodic terms from Table 1, of the main precession frequency $p = 50.4712\ ''/\mathrm{yr}$, and also of some of the periodic terms coming from the eccentricity solution of the Earth, and involving the secular frequencies of the planets perihelions g_i.

In fact, the quasiperiodic approximation given in Table 2 converges quite well, and its use gives a fair approximation of the precession and obliquity over a few million years, although for precise use, we would better recommend the direct use of the numerical solutions. The differences between this quasiperiodic approximation and the numerical solutions are plotted in Fig. 7 over 18 Myr in the past, which corresponds to the time interval on which the frequency analysis is performed. It should be noted that the precision of this approximation decreases at the edges of the time interval.

In the solution given in Table 2, it should be stressed that apart from the first term, which represents the linear part of the precession solution, the next two terms are the terms related to the frequencies f and $2p - f$ which come from the very small term given as $\sigma_{13} = f$ in Table 1. We can see very well here that the resonance with the main frequency of precession p has very much increased the effect of this small exciting term due to the perturbations of Jupiter and Saturn. The next terms are due mostly to perturbations from Mars, and the 7th term is related to σ_3 and depends on the secular resonance between Mars and the Earth.

7. Testing the model of precession

In this section, the solution of the precession is studied against several possible changes in the model. In particular, the tidal terms and passage through resonance are analyzed.

7.1. Effect of the tidal dissipation

In the first computation of the precession quantities, in the La88 or La90 solutions, no tidal terms for the Moon were included in the equations. When we made the first comparisons with the numerical integration of Quinn et al.(1991), it appeared that the orbital elements of all the planets, and in particular of the Earth, were in very good agreement (Laskar et al.1992b), while precession and obliquity solutions presented a small shift in phase (Fig. 8a). These discrepancies are small, but we needed to understand their origin, and we suspected that they arose from the introduction of the tidal dissipation term in the solution QTD. With our new package for computing the precession, such a test can be easily done, and we can introduce the tidal dissipation terms which were used by (Quinn et al.1991) in our solution.

The tidal dissipation resulting from the action of the Sun and the Moon on the Earth induces slow changes in the speed of rotation of the Earth (ν), and in the mean motion of the Moon (n_M). If the rate of this slow change is assumed to be constant over the period considered, and denoting by ν_0 and ν_{M0} the values of ν and ν_M at the origin of date (J2000), we obtain

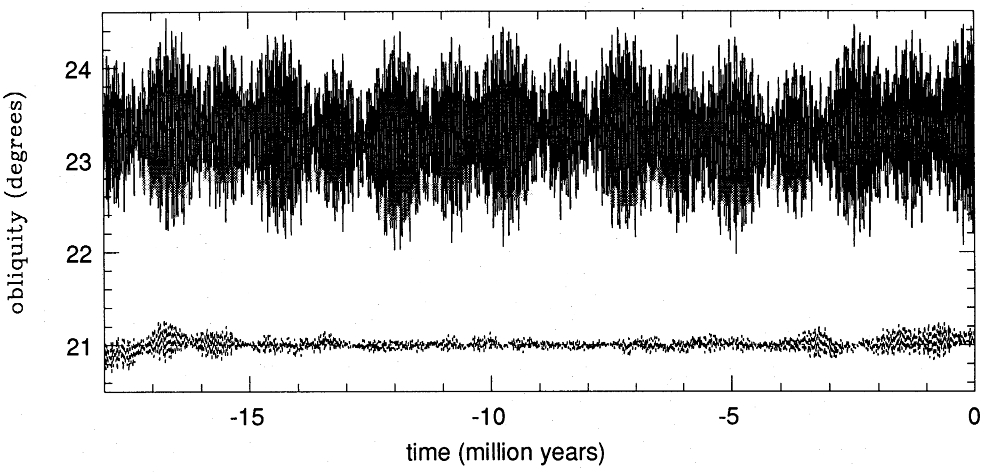

Fig. 7. Solution of the obliquity of the Earth (in degrees) from -18 Myr to 0Myr and differences (+21 degrees) between the solution obtained by numerical integration of Eq. 24, and its quasiperiodic approximation given in Table 2

$$\nu = \nu_0 \quad (1 + \frac{\dot{\nu}_0}{\nu_0} t)$$
$$n_M = n_{M0} \, (1 + \frac{\dot{n}_{M0}}{n_{M0}} t) \, . \qquad (33)$$

In our comparisons, we used the same values of $\dot{\nu}_0$ and $\dot{n}_{M0}$ which were used by (Quinn et al.1991), that is $\dot{\nu}_0/\nu_0 = -4.6\,10^{-18}\mathrm{s}^{-1}$, as given by Dyckey & Williams (1982), and the relation $\dot{\nu}_0 = 51\dot{n}_{M0}$ issued in Lambeck (1980). The different coefficients C_i of Eqs. (3) and (25) are thus changed in the following way: C_1 and C_2 are proportional to $E_D n_M^2/\nu$, C_3 is proportional to $(E_D n_M^2/\nu)^2$, and the solar term C_4 is proportional to E_D/ν. But the dynamical ellipticity E_D itself is proportional to ν^2, which gives the following dependence of the coefficients C_i:

$$C_1 = C_{10}(1 + \frac{\dot{\nu}_0}{\nu_0} t)(1 + 2\frac{\dot{n}_{M0}}{n_{M0}} t)$$
$$C_2 = C_{20}(1 + \frac{\dot{\nu}_0}{\nu_0} t)(1 + 2\frac{\dot{n}_{M0}}{n_{M0}} t)$$
$$C_3 = C_{30}(1 + \frac{\dot{\nu}_0}{\nu_0} t)^2(1 + 2\frac{\dot{n}_{M0}}{n_{M0}} t)^2 \qquad (34)$$
$$C_4 = C_{40}(1 + \frac{\dot{\nu}_0}{\nu_0} t) \, .$$

With these modifications, we obtain the solution plotted in Fig. 8b, where it can be seen that the discrepancies with QTD over 3Myr have nearly disappeared (the solutions are only plotted on the -2Myr to -3Myr interval where the differences are most important).

Looking at Fig. 8b, one may be tempted to conclude that the solution for the obliquity and precession of the Earth is now very well determined over time span extending to several millions years. Unfortunately, the very good agreement of Fig. 8b reflects more our ability to integrate properly the same physical model, using very different methods, over such long time scales. Indeed, when entering an ice age, the tidal dissipation is supposed to be much smaller than the present value. We should thus expect

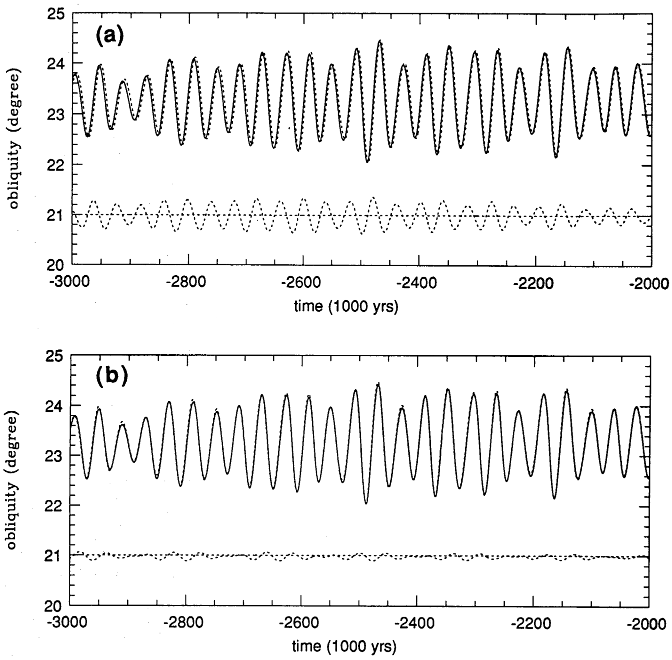

Fig. 8. Comparison of the solution of the obliquity of the Earth of the present paper (solid line), with the solution QTD. The differences of the two solutions are plotted on the line of ordinate 21 degrees: (a) without any the tidal dissipation term; (b) including the tidal contribution given in Eq. 34

the exact solution to be somewhat in between the two solutions given in Figs. 8a and 8b. Moreover, some other uncertainty can also result from the passage into an ice age, which result from change in the dynamical ellipticity.

7.2. Changes in dynamical ellipticity during an ice age

In the frequency decomposition of the $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$ planetary forcing term, we found the periodic term of frequency $f = s_6 - g_6 + g_5 = -50.3021\,''$/yr, which could enter into resonance with the precession frequency which was found to be $p = 50.4712\,''$/yr, corresponding to a period of 25678yr. The amplitude of this term in $\mathbf{A}(\mathbf{p},\mathbf{q}) + i\mathbf{B}(\mathbf{p},\mathbf{q})$ is not very large,

(0.104×10^{-8}) compared with the leading term related to s_3 of amplitude 0.1616×10^{-5}, but this term is very close to resonance, and it appears in the solution in χ as the second periodic term (Table 2).

We thus wanted to see what could be the effect of an exact resonance with this small periodic term (some modeling of the effect of the passage into resonance was already made in Ward 1982). In order to do this, we have sligthly changed the value of the dynamical ellipticity of the Earth, keeping fixed the angular momentum. This is what can happen for example during an ice age, where the redistribution of the ice changes a little the dynamical ellipticity of the earth. According to Thomson (1990), the redistribution of the ice at the surface of the Earth during an ice age induces a change in the inertial momentum of the Earth $\delta C/C \approx -15.105\,10^{-6}$, while $\delta(C-A) = 3\delta C/2$. On the other hand, the angular momentum νC of the Earth remains constant. Let $\gamma = E_D/\nu$; we obtain finally, for the changes during an ice age

$$\frac{\delta\gamma}{\gamma} = \frac{3}{2}\frac{\delta C}{C}\frac{1}{E_D}\,, \tag{35}$$

which gives $\dfrac{\delta\gamma}{\gamma} = -6.9\,10^{-3}$. This change induces a change in the precession frequency of about $-0.35\ ''$/yr, which is much more than the necessary change to drive the precession into resonance with the $s_6 - g_6 + g_5$ perturbation term. We have integrated over 18 Myr the precession equations for various values of the parameter $\gamma = E_D/\nu$ around the nominal value γ_0, obtained with the present values of E_D and ν, ranging from 0.9940 γ_0 to 1.0015 γ_0. For each of these integrations, a frequency analysis gives the precession frequency with an estimate of the precision of the determination of this frequency (Fig. 9a). The frequency changes regularly outside of the resonance, which effect is clearly visible. The minimum, mean, and maximum value of the obliquity obtained over 18 Myr of integration with these different values of γ are plotted in Fig. 9b. The passage through resonance is also clearly visible and lead to an increase of the maximum obliquity of the Earth of about 0.5 degrees. The small perturbation term $s_6 - g_6 + g_5$ can thus be of great importance in the computation of the past insolation of the Earth, as very small changes can make the solution enter into this resonance.

On Fig. 10 is plotted the behaviour of the obliquity from -3 Myr to +1 Myr for the actual adopted value of γ_0, and also for the possible value $\gamma = 0.9977\gamma_0$ which can be reached when entering an ice age (dotted line). The two solutions appear to be very different, presenting a shift in phase, but also a sensible change of amplitude after 2Myr. These differences are much larger than the effect of the tidal dissipation presented in Fig. 8.

The uncertainty on the possible changes of dynamical ellipticity resulting from entering an ice age thus appears to be the main obstruction for the computation of a precise solution for the precession and obliquity of the Earth over several million years. One possible way to overcome this difficulty would be to modelize more completely the passage through an ice age, and to try to fit this complete model to the geological data.

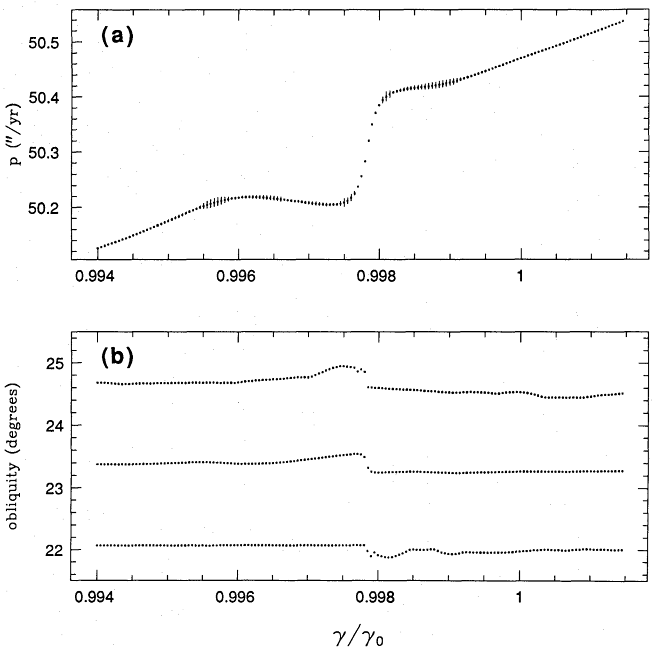

Fig. 9. Passage into the $s_6 - g_6 + g_5$ resonance. Several integrations are made over 18 Myr with γ ranging from $0.9940\gamma_0$ to $1.0015\gamma_0$.(a) Frequency of precession p. The vertical bars are the estimates of the precision of the measure given by the frequency analysis.(b) Maximum, mean, and minimum value of the obliquity over 18 Myr

7.3. Other possible change

With our tool which allows so easy integrations of the equations of precession, we also carried out a more dramatic experiment. Without changing any of the parameters of the Earth, we suddenly suppressed the Moon ! As was already forecasted by Ward (1982), the change in obliquity increases very much, ranging from 15 degrees to more than 30 degrees (Fig. 11a), with a corresponding dramatic change of the insolation in summer at 65 degrees of northern latitude (Fig. 11b). This is due to the fact that when the Moon is not here, we have $C_1 = C_2 = C_3 = 0$ in (6), and the fundamental frequency of precession decreases very much to $15.6\ ''$/yr, which is now very close to the leading frequencies of the inclination periodic exitation. In this crude experiment, the climate of the Earth will be changed in a large extend, if one admits that the changes of insolations in the past (negative time of Fig. 11b) are responsible for the existence of the ice ages. A more detailed analysis of this problem is under study.

8. Conclusions

In the present work, we provide a simple way to obtain practically the solution for precession and obliquity from a numerical solution of the secular evolution of the orbit of the Earth sampled every 1000 years. The orbital solution La90 used here was computed over 200 Myr, but due to the chaotic behavior of the solution, it should give a reliable solution for the Earth only

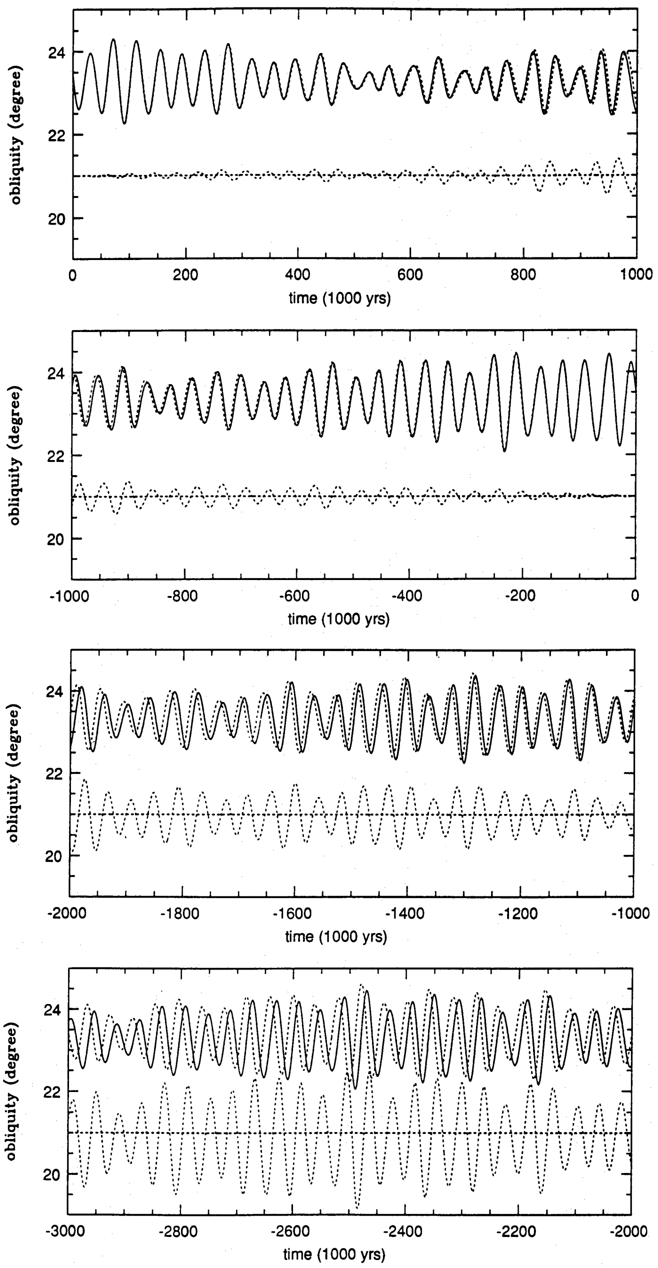

Fig. 10. Comparison from -3Myr to +1Myr of the obliquity solution of the Earth obtained with the current value γ_0 of $\gamma = E_D/\nu$, and with the value $\gamma = 0.9977\gamma_0$ which can be reached during an ice age (dotted line). The difference (+21 degrees) is also plotted in dotted line

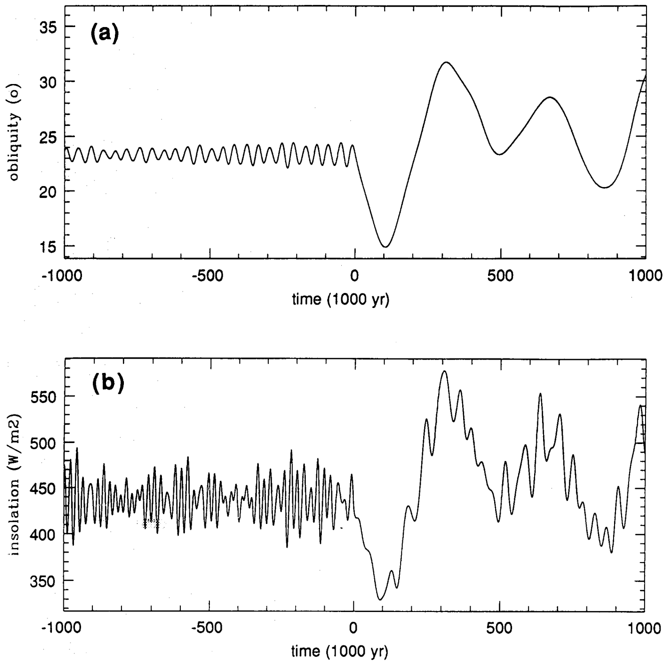

Fig. 11. Changes in obliquity (a) and insolation at 65N (λ_d = 120 deg) (b) resulting from the suppression at t = 0 of the Moon. The Moon is present from -1Myr to 0, and absent from 0 to +1Myr

over 10 to 20 Myr which should be enough for most paleoclimate computations. Over much longer time span, reaching 100 Myr, there is no hope to obtain such an ephemeris, due to the chaotic behaviour of the solar system. This orbital solution La90 is also in very good agreement with the recent numerical integrations over 3 Myr of (Quinn et al.1991).

The status of the solution for obliquity and precession is somewhat different. As the orbital solution is not quasiperiodic, perturbations series should be avoided for the computation of precise solutions for the precession and obliquity, and the solution presented here was obtained with a numerical integration of the precession equations, using the orbital solution La90 for the Earth. This allows to take into account very small periodic terms which may enter into resonance with the main precession frequency. For the understanding of the dynamics of the solution, a frequency analysis provides a quasiperidic approximation of the final solution, revealing the effect of the resonant terms.

The comparisons over 3 Myr with the solution QTD from (Quinn et al.1991) present some small discrepancies which are removed when taking into account the same tidal dissipative term which was used by (Quinn et al.1991), but these terms corresponds to the present value of the tidal dissipation, while the actual dissipation over 3 Myr is supposed to be smaller during the ice ages. Moreover, during the ice ages, there could be some changes in the dynamical ellipticity of the Earth which induces much larger changes in the precession and obliquity (Fig. 9). The recent passage trough ice ages during the previous few millions years can also induce changes resulting from the resonance of the main precession frequency with a small periodic term due to the perturbation of Jupiter and Saturn, which can cause changes in obliquity as large as 0.5 degree.

These reasons limit deeply the possibility for obtaining a solution for the precession and obliquity of the Earth over a time span of several million years, contrary to the solution fo the orbital motion of the Earth which can be considered as reliable over 10 to 20 Myr. For this reason, we decided to provide a nominal solution for the precession and obliquity from -20 Myr to +10 Myr, with no tidal term, and with the present value of the determined dynamical ellipticity, but we also provide to the interested community a complete set of FORTRAN routines, which would allow anyone to change some of the main parameters for the computation of the obliquity, precession, and insolation quantities. These routines can be obtained by request to the first author at `laskar@cosme.polytechnique.fr`. It may be the opportunity for trying to fit the unknown parameters to some precise geological data.

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