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Solution of the N-body problem expanded into Taylor series of high orders. Applications to the solar system over large time range

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Abstract. A method for numerical integration of the N-body problem is carried out and described in this paper, the solution obtained being expanded into Taylor series of high orders with the aid of recurrent formulae. An easy to use Fortran program having been written, the accuracy of this method is then tested integrating some planetary problems with respect to time, in a direction and its reverse, such as:

(a) The nine major planets in translation around the Sun are integrated over intervals of 40 000 d with a near constant integration step-size of 4 d. The results are compared to the ephemeris DE200 of the JPL (Standish 1982a), to which the relativistic perturbations and those due to the Moon and minor planets were first subtracted. Differences of about 10^{-10} AU are obtained on the rectangular coordinates of all the planets.

(b) In the same way, the eight first major planets (Pluto is excluded) are integrated over intervals of 1000 yr and the results especially estimated on the mean longitudes. An accuracy of 0″0025 is reached on Mercury.

(c) The four outer planets (Jupiter, Saturn, Uranus, Neptune) are integrated over intervals of 6000 yr with a near constant integration step-size of 400 d. It is shown here that the results got by the numerical integrations of Schubart & Stumpff (1966) are improved by a factor of about 15.

Key words: celestial mechanics – numerical integration – solar system

1. Introduction

Apart from Gauss Jackson or Runge–Kutta methods, usually applied in order to solve numerically celestial problems, we can also seek solutions expanded in terms of power-series in time. Since Steffensen (1956), several authors have used such methods before as e.g. Broucke (1971), Black (1973), Roberts (1975) for the N-body problem, Roy et al. (1972), Moran (1973), Emslie & Walker (1979), Schwarz & Walker (1982) for the two-body and three-body problem and these works have been tested by Fox (1984).

In this paper we carry out and we test a similar numerical integration using Taylor series expansions as follows: We begin by presenting the differential equations of the N-body problem written with the aid of auxiliary variables; we explain the computation of the successive derivatives of the coordinates, i.e. as their direct calculation becomes quickly very difficult, we carry out recurrent formulae that allow to obtain easily developments into Taylor series of high orders, and then we give the final calculation of the solution.

Then, in order to test the accuracy of this method, we integrate different planetary problems with respect to time in a direction and its reverse, and we compare our results, either to the JPL DE200 ephemeris (Standish 1982a), or to those got by the numerical integration program of Schubart & Stumpff (1966). This is the main object of this paper.

2. The N-body problem

Let us consider, as point masses, N bodies $P_0, P_1, \ldots, P_{(N-1)}$ with their respective masses $m_0, m_1, \ldots, m_{(N-1)}$, and let P_0 be the origin of the rectangular coordinates system, we study the relative motions of the $(N-1)$ others bodies P_i around P_0, $i=[1,2,\ldots,(N-1)]$.

Within this frame, and with classical notations, $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are the position and velocity vectors of the body P_i,

$$\boldsymbol{r}_i=(x_i, y_i, z_i), \qquad \dot{\boldsymbol{r}}_i=(\dot{x}_i, \dot{y}_i, \dot{z}_i);$$

R_i is the distance between the body P_i and the origin body P_0,

$$R_i^2=(x_i^2+y_i^2+z_i^2),$$

and $\Delta_{j,i}$ the distance between the bodies P_j and P_i,

$$\Delta_{j,i}^2=[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2] \qquad j=[1,2,\ldots,(N-1)], \quad j\neq i.$$

2.1. Differential equations

The differential equations of motion of P_i are

$$\ddot{\boldsymbol{r}}_i = -K(m_0+m_i)\frac{\boldsymbol{r}_i}{R_i^3} + \sum_{\substack{j=1\\ j\neq i}}^{N-1} Km_j\left[\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)}{\Delta_{j,i}^3}-\frac{\boldsymbol{r}_j}{R_j^3}\right], \tag{1}$$

where K is the square of the Gaussian constant.

Let us now define $\frac{3}{2}(N-1)(N-2)$ auxiliary variables $\boldsymbol{r}_l$ by

$$(\boldsymbol{r}_q-\boldsymbol{r}_p)=\boldsymbol{r}_l \qquad \text{with } q>p, \quad l,p,q>0,$$

$p=[1,\ldots,(N-2)]$, and p being fixed, q varies from $(p+1)$ up to and including $(N-1)$, $q=[2,\ldots,(N-1)]$.

$$l=(N-p)+\cdots+(N-1)+(q-p)$$

and varies within the range $[N,N+1,\ldots,N']$ with $N'=\frac{1}{2}N(N-1)$.

Therefore the quantities $(\boldsymbol{r}_j-\boldsymbol{r}_i)$ of Eq. (1) can be written:

if $j>i$,

$$l=(N-i)+\cdots+(N-1)+(j-i),$$

$$(\boldsymbol{r}_j-\boldsymbol{r}_i)=\boldsymbol{r}_l. \tag{2}$$

if $j<i$,

$$(\boldsymbol{r}_j-\boldsymbol{r}_i)=-(\boldsymbol{r}_i-\boldsymbol{r}_j),$$

$$l=(N-j)+\cdots+(N-1)+(i-j),$$

$$(\boldsymbol{r}_j-\boldsymbol{r}_i)=-\boldsymbol{r}_l.$$

Then, Eq. (1) will take the general form

$$\ddot{\boldsymbol{r}}_i=-\sum_{n=1}^{N'}\mu_{i,n}\frac{\boldsymbol{r}_n}{R_n^3}, \qquad N'=\frac{N(N-1)}{2}, \tag{3}$$

where $\mu_{i,n}$ are real constants and

$R_n^3=R_i^3$ when $n=i=[1,2,\ldots,(N-1)]$,

$R_n^3=\Delta_{j,i}^3$ when $n=l=[N,N+1,\ldots,N']$.

2.2. Matrix μ

In Eq. (3) the constants $\mu_{i,n}$ are a rectangular matrix with $(N-1)$ rows and N' columns in which on a same row i we have:

(a) for n varying into the range $[1,\ldots,(N-1)]$

if $n\neq i$ $\quad \mu_{i,n}=Km_n$,

if $n=i$ $\quad \mu_{i,n}=\mu_{i,i}=K(m_0+m_i)$.

(b) for n varying into the range $[N,\ldots,N']$

if $j>i$,

$\mu_{i,n}=\mu_{i,l}=-Km_j$,

if $j<i$,

$\mu_{i,n}=\mu_{i,l}=Km_j$,

All other elements of the row i are equal to zero.

3. Derivatives of the coordinates

Let $k\geq 0$ be an integer and $s^{(k)}$ being the kth derivative of a function s, let us differentiate Eq. (3) up to and including order k, the $(k+2)$th derivatives of $\boldsymbol{r}_i$ are linear combinations of the kth derivatives of the functions $\boldsymbol{r}_n/R_n^3$,

$$\boldsymbol{r}_i^{(k+2)}=-\sum_{n=1}^{N'}\mu_{i,n}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[\frac{\boldsymbol{r}_n}{R_n^3}\right]. \tag{4}$$

Let $\boldsymbol{r}=\boldsymbol{r}^{(0)}=(x,y,z)$ be a position vector, $\dot{\boldsymbol{r}}=\boldsymbol{r}^{(1)}=(\dot{x},\dot{y},\dot{z})$ a velocity vector and R a distance,

$$R=R^{(0)}=\sqrt{x^2+y^2+z^2}. \tag{5}$$

Let

$$G=G^{(0)}=\frac{1}{R^3}, \tag{6}$$

and

$$\boldsymbol{F}_r=\boldsymbol{F}_r^{(0)}=\frac{\boldsymbol{r}}{R^3}=G\boldsymbol{r} \tag{7}$$

be auxiliary functions of time, we see that it is sufficient to know the formal derivatives of Eq. (7) to obtain after substituting $\boldsymbol{r}_n$ for $\boldsymbol{r}$, those of the $\boldsymbol{r}_i$,

$$\boldsymbol{r}_i^{(k+2)}=-\sum_{n=1}^{N'}\mu_{i,n}\boldsymbol{F}_{r_n}^{(k)}. \tag{8}$$

Consequently,

$$j>i, \quad (\boldsymbol{r}_j^{(k+2)}-\boldsymbol{r}_i^{(k+2)})=\boldsymbol{r}_l^{(k+2)}. \tag{9}$$

Nevertheless, k being fixed and the derivatives of $\boldsymbol{r}_n$ being assumed known up to and including order k, we will use a recursive method to calculate the expressions of $\boldsymbol{F}_{r_n}^{(k)}$, rather than a direct analytical derivation of Eq. (7) which becomes quickly very difficult.

For that, we can easily get the following successive recurrent formulae, carried out with the aid of Leibniz's theorem, we have:

(a) for the functions R

if $k=1$

$$R^{(1)}=\frac{1}{R}[\boldsymbol{r}\cdot\boldsymbol{r}^{(1)}], \tag{10}$$

if $k\geq 2$

$$R^{(k)}=\frac{1}{R}\left[\boldsymbol{r}\cdot\boldsymbol{r}^{(k)}+\sum_{h=0}^{k-2}C_{k-1}^{h}[\boldsymbol{r}^{(k-1-h)}\cdot\boldsymbol{r}^{(h+1)} - R^{(k-1-h)}R^{(h+1)}]\right], \quad h\geq 0, \tag{11}$$

with

$$C_{k-1}^{h}=\frac{(k-1)!}{(k-1-h)!\,h!}.$$

(b) for the functions G

if $k=1$

$$G^{(1)}=-3G\frac{R^{(1)}}{R}, \tag{12}$$

if $k\geq 2$

$$G^{(k)}=-\frac{1}{R}\left[3\,G\,R^{(k)}+\sum_{h=0}^{k-2}C_{k-1}^{h}\times[R^{(k-1-h)}G^{(h+1)}+3G^{(k-1-h)}R^{(h+1)}]\right]. \tag{13}$$

(c) for the functions $\boldsymbol{F}_r$

if $k\geq 1$

$$\boldsymbol{F}_r^{(k)}=\sum_{h=0}^{k}C_k^h\,G^{(h)}\,\boldsymbol{r}^{(k-h)}, \tag{14}$$

with

$$C_k^h=\frac{k!}{(k-h)!\,h!}.$$

4. Solution

Given the initial conditions $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ at $t=t_0$, we can compute $\boldsymbol{r}_l, \dot{\boldsymbol{r}}_l$, by Eq. (2) and its first derivative respectively. We obtain thus all the $\boldsymbol{r}_n$ and $\dot{\boldsymbol{r}}_n$, $n=[1,2,\ldots,N']$.

Table 1. Positions and velocities of the nine major planets at the initial Julian ephemeris date of integration, JJ = 2 451 600.5, referred to the dynamical ecliptic and equinox J2000. Reciprocal of masses from the JPL DE200/LE200 ephemeris, mass of the Sun = 1

Planets and reciprocal of masses	Positions (AU)	Velocities (AU d^{-1})
Mercury	−0.2503322266997050	−0.0243880740618594
6 023 600.0000	0.2219933134940310	−0.0199006661918177
	0.0411116039958294	0.0006127790621238
Venus	0.0174780967869107	0.0200854700002338
408 523.5000	−0.7267426358190740	0.0004115973599546
	−0.0109417141664655	−0.0011537145705727
Earth + Moon (EMB)	−0.9091915624774220	−0.0070858429584318
328 900.5500	0.3916073804088610	−0.0158655216295262
	0.0000006337357788	0.0000000304051933
Mars	1.2030185340340200	−0.0071244586662406
3 098 710.0000	0.7867887615327550	0.0129051790708794
	−0.0130904757456262	0.0004454747030769
Jupiter	3.7330739975155200	−0.0050865465611740
1047.3500	3.2848131838485899	0.0060262786120076
	−0.0972162197545890	0.0000889035628590
Saturn	6.1644217605744700	−0.0044268306574236
3498.0000	6.7819637930593399	0.0037473494696568
	−0.3631445050232320	0.0001107931782759
Uranus	14.5796795804070001	0.0026475039005944
22960.0000	−13.5852040761606001	0.0027006548067743
	−0.2395026654553700	−0.0000242628753771
Neptune	16.9549722100605003	0.0025686464544125
19 314.0000	−24.8926233810363016	0.0017915021721705
	0.1218939840518710	−0.0000959701485907
Pluto	−9.7077482397720101	0.0030340544888208
130 000 000.0000	−28.0439936763418984	−0.0015216209707127
	5.8109566466828999	−0.0007156387594053
Energy integral [Mass × (AU d^{-1})^2]	−0.6654158715721031 10^{-7}	

(a) $k=0$, substituting $\boldsymbol{r}_n$ for $\boldsymbol{r}$ into Eqs. (5)–(7) we compute successively R_n, G_n and $\boldsymbol{F}_{r_n}$, whence $\boldsymbol{r}_i^{(2)}$ by Eq. (8), and $\boldsymbol{r}_l^{(2)}$ by Eq. (9).

(b) $k=1$, substituting $\boldsymbol{r}_n, \dot{\boldsymbol{r}}_n, R_n, G_n$ for $\boldsymbol{r}, \dot{\boldsymbol{r}}, R, G$, into Eqs. (10), (12), (14), we compute $R^{(1)}, G^{(1)}, \boldsymbol{F}_r^{(1)}$ whence $\boldsymbol{r}_i^{(3)}$ by Eq. (8), and $\boldsymbol{r}_l^{(3)}$ by Eq. (9).

(c) $k=2$, we can now obtain $\boldsymbol{r}_i^{(4)}$ by Eqs. (11), (13), (14) and (8), $\boldsymbol{r}_l^{(4)}$ by Eq. (9), etc. We will continue this process until all the required values at $t=t_0$ of the derivatives of the coordinates are obtained and then, substituting these into Taylor series expansions, we get finally the solution for $\boldsymbol{r}_i$ at time t.

$$\boldsymbol{r}_i(t)=\sum_{h=0}^{k+2} \boldsymbol{r}_i^{(h)}(t_0)\frac{(t-t_0)^h}{h!}. \tag{15}$$

A variable step-size Fortran program (264 statements) of this numerical integration method has then been written (Le Guyader 1990), in which, according to the studied problem, we have first to find a near constant integration step-size as large as possible using a suitable order derivation of the coordinates. In this way we obtain the smallest numerical drift in the results and the best speed of computation. Let us apply now solution (15) to some planetary problems and estimate its accuracy.

5. Comparison to the JPL ephemeris DE200

5.1. Initial conditions and ephemeris over 40 000 d of the nine major planets

To begin, we have used the complete integration of the JPL DE200 ephemeris (Standish 1982a) to which the relativistic perturbations (Lestrade & Bretagnon 1982) and those due to the Moon and minor planets (Bretagnon 1984) were first subtracted. For that the coordinates and velocities $\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i$ of the JPL ephemeris are convert, every 20 d, into the osculating elements a (semi major axis), λ (mean longitude) and into the functions k, h, q, p defined below,

$$k=e\cos(\Omega+\omega), \qquad q=\sin\frac{i}{2}\cos\Omega,$$

$$h=e\sin(\Omega+\omega), \qquad p=\sin\frac{i}{2}\sin\Omega,$$

where e, i, Ω, ω are the eccentricity, inclination, longitude of node, and argument of perihelion of the planets. The small perturbations above-mentioned are then subtracted, and thus we have a Newtonian reference ephemeris (E_R) to which we will compare our results. Conversely, we can easily return to the variables $\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i$.

The initial conditions $\boldsymbol{r}_i(t_0)$ and $\dot{\boldsymbol{r}}_i(t_0)$ of our first integration come from this ephemeris (E_R) at the date of 26th of February 2000 0^h TE(JJ = 2 451 600.5), they are given in Table 1 with the reciprocal of masses from the JPL DE200/LE200 ephemeris for every planets (Standish 1982b), the mass of the Sun being unity. This table gives also the value of the energy integral. Then, we have integrated over 40 000 d, the Newtonian system composed by the nine major planets in translation around the Sun, from the Julian Day JJ = 2 451 600.5 to the Julian Day JJ = 2 411 600.5. For that, we have used negative integration step-size of 4 d and developments into Taylor series up to derivative 25, this number being chosen in order to keep the integration step-size nearly constant all over the range of integration. Nevertheless, k being fixed and given a small number ε, the program divides this step-size $(t-t_0)$ by 2 as long as the quantity

$$\left[\sum_{i=1}^{N-1}\sum_{x_i,y_i,z_i}|\boldsymbol{r}_i^{(k+2)}(t_0)|\right]\frac{|(t-t_0)^{k+2}|}{(k+2)!}$$

is greater than ε.

Table 2. Differences between positions and velocities of the planets after two integrations (40 000 d each one), in a backward direction in time and its reverse from the initial Julian ephermeris date, JJ = 2 451 600.5

Planets	Positions (AU 10^{-10})	Vitesses (AU d^{-1} 10^{-10})
Mercury	2.80	−0.22
	2.29	0.20
	−0.07	0.04
Venus	−8.66	0.01
	−0.19	−0.24
	0.50	0.00
Earth + Moon (EMB)	3.97	−0.16
	8.93	0.07
	0.00	0.00
Mars	8.81	0.15
	−15.83	0.10
	−0.55	0.00
Jupiter	5.26	0.01
	−5.98	0.01
	−0.09	0.00
Saturn	4.47	0.00
	−3.38	0.00
	−0.12	0.00
Uranus	−1.88	0.00
	−2.87	0.00
	0.01	0.00
Neptune	−1.32	0.00
	−2.06	0.00
	0.07	0.00
Pluto	−1.50	0.00
	−0.08	0.00
	0.44	0.00

Difference on the energy integral 0.02 10^{-17} [Mass × (AU d^{-1})^2]

Table 3. Positions and velocities of the nine major planets after corrections by a least-square method at the initial Julian ephemeris date of integration, JJ = 2 451 600.5, referred to the dynamical ecliptic and equinox J2000

Planets	Positions (AU)	Velocities ($\mathrm{AU\,d^{-1}}$)
Mercury	−0.2503322272474241	−0.0243880740187149
	0.2219933130543761	−0.0199006662291828
	0.0411116040134522	0.0006127790550368
Venus	0.0174780969832841	0.0200854700008875
	−0.7267426357972082	0.0004115973655980
	−0.0109417141864230	−0.0011537145704358
Earth + Moon (EMB)	−0.9091915626221172	−0.0070858429546628
	0.3916073801866026	−0.0158655216304405
	0.0000006337269945	0.0000000304053757
Mars	1.2030185338987920	−0.0071244586683808
	0.7867887618949421	0.0129051790684479
	−0.0130904754207521	0.0004454747041100
Jupiter	3.7330739970653830	−0.0050865465617725
	3.2848131843439790	0.0060262786111905
	−0.0972162197513344	0.0000889035628536
Saturn	6.1644217608860861	−0.0044268306570807
	6.7819637928089400	0.0037473494696738
	−0.3631445050315127	0.0001107931782389
Uranus	14.5796795810394000	0.0026475039006449
	−13.5852040756255601	0.0027006548067234
	−0.2395026654509072	−0.0000242628754072
Neptune	16.9549722088488011	0.0025686464546824
	−24.8926233783199109	0.0017915021721450
	0.1218939840126886	−0.0000959701486240
Pluto	−9.7077482547662419	0.0030340544879385
	−28.0439936839154491	−0.0015216209715392
	5.8109566518888340	−0.0007156387590526
Energy integral [Mass × $(\mathrm{AU\,d^{-1}})^2$]	$-0.6654158716182916\ 10^{-7}$	

5.2. Reverse integration

In order to test the accuracy of the results, we have conversely integrated our system from the Julian date 2411 600.5 up to the initial Julian date 2 451 600.5; the final results obtained by this next integration are then subtracted to the initial coordinates and velocities of Table 1 and all the differences for the nine planets are collected in Table 2.

We can see first that the global drift of the results after 80 000 d remain small (relative accuracy on the energy integral = $3\ 10^{-12}$), and then that the relative accuracies (coordinates/semi major axis) are less than $1.2\ 10^{-19}$ (x-coordinate of Venus) for the inner planets, and $1.2\ 10^{-10}$ (y-coordinate of Jupiter) for the outer planets.

5.3. Comparison to DE200

Therefore, in order to compare the results of our integration to the reference ephemeris (E_R) previously defined, we begin to compute every 20 d the osculating elements $a, e, i, \Omega, \omega, \lambda$ of the planets and the functions k, h, q, p.

We get a first ephemeris (E_1) that is subtracted from (E_R). The differences are then treated by a least-square method in order to bring small corrections to the initial positions and velocities of each planet, and thus, after a new integration, we obtain an ephemeris (E_2) nearer (E_R) than (E_1), etc. We stop this process when the corrections become negligible for the required precision.

Table 3 gives the coordinates and velocities used for our last integration of 40 000 d and Table 4 its final results.

Table 4. Positions and velocities of the nine major planets after 40000 d at the final Julian ephemeris date of integration, JJ = 2411600.5, referred to the dynamical ecliptic and equinox J2000

Planets	Positions (AU)	Velocities ($AU\,d^{-1}$)
Mercury	−0.2576687518872368	0.0175868799212131
	−0.3791513241443215	−0.0144665374801825
	−0.0071950703512374	−0.0027988495000633
Venus	−0.0478396027134483	0.0200455386154473
	−0.7251830666565513	−0.0014100355266644
	−0.0069246827363900	−0.0011776646939453
Earth + Moon (EMB)	0.8703804807184863	0.0084796600188988
	−0.5149679396699509	0.0147442829692835
	−0.0001096032600599	0.0000038603797076
Mars	0.6362707422639920	0.0130220279289513
	−1.2551195580973074	0.0075226527499204
	−0.0420319593297117	−0.0001658371106017
Jupiter	3.3079184999203806	0.0056401087405234
	−3.8541443085480818	0.0052787064953480
	−0.0585602232678248	−0.0001480045741013
Saturn	−8.6498328693881499	−0.0023396860855807
	3.4092364104726043	−0.0052006445435954
	0.2828278405758518	0.0001845459284167
Uranus	−16.3165791780625433	0.0018104636390494
	−8.6194029169052933	−0.0036590291030088
	0.1801147430157560	−0.0000372832508783
Neptune	11.9722583886743688	−0.0028885982212621
	27.3074939138436470	0.0012836838050364
	−0.8380811427838262	0.0000401984022563
Pluto	17.5090297675820956	−0.0020324972718968
	43.6921163840676954	0.0006366390838723
	−9.7426428668190121	0.0005213576991192
Energy integral [Mass × $(AU\,d^{-1})^2$]	$-0.6654158716193928\ 10^{-7}$	

Their comparisons with (E_R) have given differences of about 10^{-10} AU on the position coordinates on the whole range of integration.

6. Comparisons to Schubart and Stumpff's numerical integration

6.1. Ephemeris over 1000 yr of the eight first major planets

After the good accuracy obtained in our first problem and in order to know better the behaviour of our program, we wanted to increase greatly the range of integration and to compare our results to those got by Schubart & Stumpff integration program (1966, here referred to also as *1966-program*), which is based on a Stormer–Cowell method, including differences up to order 11 in the numerical integration algorithm. Thus we have integrated the planetary problem of the eight first major planets (Pluto is excluded) in translation around the Sun over intervals of 1000 yr, in a backward direction in time and its reverse, the initial conditions r_i & $\dot{r}_i$ of this problem coming from the ephemeris VSOP87 (Bretagnon & Francou 1988) and being taken at the epoch J2000 (JJ = 2451545.0). As before, we have used negative or positive integration step-size of 4 d and 25 derivatives into the Taylor series expansions of Eq. (15), in comparison with the *1966-program* which uses for this problem a fixed step-size of 0.5 d.

Then, after returning to the initial point of integration, we have computed the differences on the mean longitude of each planet. Table 5 compares the results obtained by our program and Schubart & Stumpff's and shows important improvement by factors going from about 300 for

Table 5. Comparisons of results got by Le Guyader's numerical integration program and Schubart & Stumpff's. The differences on the mean longitudes are computed at the initial Julian ephemeris date of integration JJ = 2 451 545.0 after two integrations over 1000 yr each one, in a backward direction in time and its reverse

differences on the mean longitudes in arc second		
Planets	Le Guyader	Schubart and Stumpff
Mercury	0″005	3″00
Venus	0″0008	0″24
Earth + Moon (EMB)	0″0003	0″18
Mars	0″00008	0″16
Jupiter	0″000002	0″006
Saturn	0″0000003	0″002
Uranus	0″00000006	0″002
Neptune	0″00000006	0″001

Venus up to 16 000 for Neptune. We see that these results are particularly interesting for the inner planets and especially for Mercury whose presence is better controlled by our program.

6.2. Ephemeris over 6000 yr of the four outer planets

Increasing always our range of integrations we have integrated the problem of the four outer planets (Jupiter, Saturn, Uranus, Neptune) in translation around the Sun over intervals of 6000 yr, in a backward direction in time and its reverse, the initial conditions $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ (JJ = 2 451 545.0) coming from an analytical theory of the four outer planets by Simon et al. (1992). In this problem, we have used negative or positive near constant integration step-size of 400 d and 25 derivatives into the Taylor series expansions of Eq. (15), in comparison this time, with a fixed step-size of 40 d in the *1966-program.*

After returning to the initial point of integration, we have computed the differences on the functions a, λ, k, h, previously defined for the planets. Table 6 compares the results obtained by our program and Schubart & Stumpff's and shows that the mean longitude of the most difficult planet, Jupiter, is here improved by a factor 15. In this example, in spite of a great increase in the integration range, the differences between the two programs concerning the mean longitudes are less than those of Table 5. In this case, without the inner planets, the *1966-program* remains therefore sufficiently accurate.

7. Conclusion

The examples given in Sects. 5 and 6 illustrate the precision of our program and the comparisons with the results given by Schubart & Stumpff's program for the major planets are particularly interesting; the gain in precision is very important specially when the inner planets are taken into account. That is due to the analyticity of the formulae used and to the high orders Taylor series which are able to keep a large integration step-size even in the neighbourhood of the perihelia of the planets. These two facts decrease greatly the rounding errors in the integrations.

Our program is fluently used to test analytical theories or to study numerically different problems, e.g. it has been used by Simon et al. (1992) to test the accuracy of their theory of the motion of the four outer planets in terms of only one angular variable. Currently it is used to check and estimate secular perturbation theories of the four outer planets over 500 000 yr, and also to study numerically Neptune's satellite system, specially Nereid (eccentricity $\simeq 0.75$) and the satellite Proteus recently discovered by Voyager 2 (mean motion $\simeq 320^\circ\,\mathrm{d}^{-1}$), which needs a 15th

Table 6. Comparisons of results got by Le Guyader's numerical integration program and Schubart & Stumpff's. The differences on the a, λ, k, h, are computed at the initial ephemeris Julian date of integration JJ = 2 451 545.0 after two integrations over 6000 yr each one, in a backward direction in time and its reverse

Le Guyader's numerical integration program				
Planets	a (AU)	λ (″)	k (″)	h (″)
Jupiter	$0.15\,10^{-10}$	$8.0\,10^{-4}$	$0.3\,10^{-6}$	$0.3\,10^{-6}$
Saturn	$1.2\,10^{-10}$	$3.0\,10^{-4}$	$3.0\,10^{-6}$	$3.0\,10^{-6}$
Uranus	$4.2\,10^{-10}$	$1.5\,10^{-4}$	$3.0\,10^{-6}$	$3.0\,10^{-6}$
Neptune	$9.0\,10^{-10}$	$0.7\,10^{-4}$	$3.0\,10^{-6}$	$3.0\,10^{-6}$
Schubart and Stumpff's numerical integration program				
Jupiter	$0.3\,10^{-9}$	$12.0\,10^{-3}$	$2.0\,10^{-5}$	$2.0\,10^{-5}$
Saturn	$2.4\,10^{-9}$	$2.0\,10^{-3}$	$6.0\,10^{-5}$	$5.0\,10^{-5}$
Uranus	$4.8\,10^{-9}$	$1.7\,10^{-3}$	$4.0\,10^{-5}$	$4.0\,10^{-5}$
Neptune	$9.6\,10^{-9}$	$1.0\,10^{-3}$	$4.0\,10^{-5}$	$4.0\,10^{-5}$

order derivation with an integration step-size of less than one hour.

Nevertheless, let us note that our program is about 8 times slower than Schubart & Stumpff's, e.g. in Sect. 5, the integration over 40000 d of the nine major planets, carried out in "double precision" by the IBM 3090 computer of the *Centre Inter Regional de Calcul Electronique du CNRS*, took about 38^s by the *1966-program* and 5^{mn} 10^s by ours to be computed. But also to get a higher accuracy, a user of the *1966-program* can go down to smaller values of the step-size and change from "double precision" to "quadruple precision", that causes a great increase in the necessary computer time.

Our Fortran program with its main directions for use can be asked from the Bureau des Longitudes.

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