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THE POTENTIAL ENERGY TENSORS FOR SUBSYSTEMS

R. CAIMMI AND L. SECCO
Dipartimento di Astronomia—Universita di Padova, Vicolo Osservatorio 5, I-35122 Padova, Italy
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ABSTRACT

An extension of Chandrasekhar's tensor virial theorem for one subsystem distorted by the tidal potential induced by another subsystem is formulated, with the possibility to extend the results to the tidal potential induced by any number of subsystems. To this aim, the self-energy tensor, the interaction-energy tensor, the tidal-energy tensor, and the residual-energy tensor are defined for each subsystem. The above mentioned quantities are evaluated for the special case of homogeneous, coaxial ellipsoids, one lying completely within the other. Concerning the special case of spheroids, a comparison is made with the results of previous approaches.

Subject headings: celestial mechanics, stellar dynamics — galaxies: kinematics and dynamics — methods: analytical

1. INTRODUCTION

The concept of a potential-energy tensor may be—and in fact turns out to be—very useful in dealing with astrophysical problems involving ellipsoidal figures of equilibrium (e.g., Chandrasekhar & Lebovitz 1962, and references therein), as it allows a formulation of the virial theorem in tensor form (e.g., Chandrasehkar 1969, chap. 2, § 11*b*). For this reason, an extension of the theory to mass distributions distorted by a tidal potential is expected to be highly rewarding and necessarily has to be performed. In addition, large-scale celestial bodies (such as galaxies) appear to be made of at least two subsystems which link only via gravitational interaction, in such a way that each component may be thought of as distorted by the tidal potential induced by the others. Then the application of the virial theorem in tensor form to each component separately allows us to set more of an amount of information than the application of the theorem to the system as a whole.

We shall limit ourselves, for sake of simplicity, to the case of two subsystems, keeping in mind that all the results may be generalized to *n* subsystems. The extension of Chandrasekhar's theory to the case under discussion is performed in § 2, and a deeper analysis is continued in § 3 in connection with a special case of astrophysical interest, i.e., homogeneous, coaxial ellipsoids, one lying completely within the other. In respect to previous attempts (Brosche, Caimmi, & Secco 1983; Caimmi, Secco, & Brosche 1984) it makes: (1) an extension from spheroidal to ellipsoidal configurations, and (2) a different expression for the tensor components related to subsystems with nonsimilar boundaries. The reasons for this discrepancy are explained in § 4, where some conclusions are also reported.

2. BASIC CONSIDERATIONS

The Newtonian potential at the point $P(x_1, x_2, x_3)$ and the potential energy for a generic mass distribution are expressed as (e.g., Chandrasekhar 1969, chap. 2, § 10)

$$\mathscr{V}(x_1, x_2, x_3) = G \int_S \rho(x'_1, x'_2, x'_3) \frac{d^3S'}{|\boldsymbol{R} - \boldsymbol{R}'|}, \qquad d^3S' = dx'_1\, dx'_2\, dx'_3 , \tag{1}$$

$$\Omega = -\frac{1}{2} \int_S \rho(x_1, x_2, x_3) \mathscr{V}(x_1, x_2, x_3) d^3S , \qquad d^3S = dx_1\, dx_2\, dx_3 , \tag{2}$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$, S is the volume of the body, and G is the constant of gravitation. Concerning the former integral, a change of frame by translation with $P(x_1, x_2, x_3)$ as the new origin leads to (see Fig. 1)

$$\mathscr{V}(x_1, x_2, x_3) = G \int_S \rho(X_1 + x_1, X_2 + x_2, X_3 + x_3) \frac{d^3S'}{|\boldsymbol{r}|}, \qquad d^3S' = dX_1\, dX_2\, dX_3 = dx'_1\, dx'_2\, dx'_3 . \tag{3}$$

The tensor potential and the potential-energy tensor are defined by Chandrasekhar (1969, chap. 2 § 10) as

$$\mathscr{V}_{pq}(x_1, x_2, x_3) = G \int_S \rho(X_1 + x_1, X_2 + x_2, X_3 + x_3) \frac{X_p X_q}{|\boldsymbol{r}|^3}\, d^3S' , \tag{4}$$

$$\Omega_{pq} = -\tfrac{1}{2} \int_S \rho(x_1, x_2, x_3) \mathscr{V}_{pq}(x_1, x_2, x_3) d^3S . \tag{5}$$

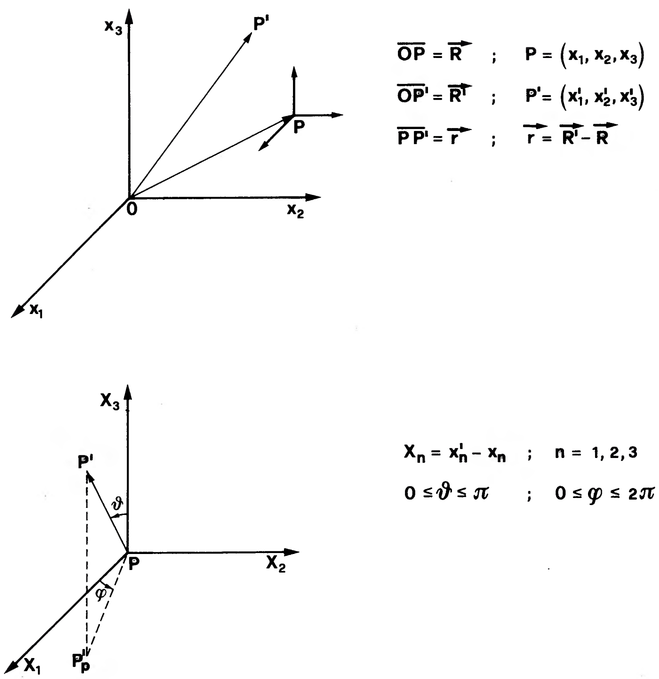

FIG. 1.—Change of frame by translation, with P as new origin

These tensors are manifestly symmetric in respect to their indices, and it is easy to see that

$$\sum_{r=1}^{3} \mathcal{V}_{rr}(x_1, x_2, x_3) = \mathcal{V}(x_1, x_2, x_3) , \tag{6}$$

$$\sum_{r=1}^{3} \Omega_{rr} = \Omega . \tag{7}$$

To gain more insight, we think of the body under consideration as made of two components, denoted as i and j. The potential of each component as the point $P(x_1, x_2, x_3)$ is expressed according to equation (3):

$$\mathcal{V}_u(x_1, x_2, x_3) = G \int_{S_u} \rho_u(X_1 + x_1, X_2 + x_2, X_3 + x_3) \frac{d^3S'}{|\boldsymbol{r}|} , \qquad u = i, j , \tag{8}$$

and the tensor potential of each component at the same point is expressed according to equation (4):

$$(\mathcal{V}_u)_{pq}(x_1, x_2, x_3) = G \int_{S_u} \rho_u(X_1 + x_1, X_2 + x_2, X_3 + x_3) \frac{X_p X_q}{|\boldsymbol{r}|^3} d^3S' \qquad u = i, j . \tag{9}$$

Equations (6), (8), and (9), lead to

$$\sum_{r=1}^{3} (\mathcal{V}_u)_{rr}(x_1, x_2, x_3) = \mathcal{V}_u(x_1, x_2, x_3) , \qquad u = i, j . \tag{10}$$

Keeping in mind that $\rho_u = 0$ outside S_u and $\rho = \rho_i + \rho_j$, it is easy to see that both the potential and the tensor potential are additive:

$$\mathcal{V}(x_1, x_2, x_3) = \mathcal{V}_i(x_1, x_2, x_3) + \mathcal{V}_j(x_1, x_2, x_3) , \tag{11}$$

$$\mathcal{V}_{pq}(x_1, x_2, x_3) = (\mathcal{V}_i)_{pq}(x_1, x_2, x_3) = (\mathcal{V}_j)_{pq}(x_1, x_2, x_3) . \tag{12}$$

The potential energy may be expressed as (e.g., MacMillan 1930, chap. III, § 76):

$$\Omega = \Omega_i + W_{ij} + W_{ji} + \Omega_j , \tag{13}$$

$$\Omega_u = -\tfrac{1}{2} \int_{S_u} \rho_u(x_1, x_2, x_3) \mathcal{V}_u(x_1, x_2, x_3) d^3S , \qquad u = i, j , \tag{14}$$

$$W_{uv} = -\tfrac{1}{2} \int_{S_u} \rho_u(x_1, x_2, x_3) \mathcal{V}_v(x_1, x_2, x_3) d^3S , \qquad u = i, j , \qquad v = j, i , \tag{15}$$

where Ω_u is the self energy and W_{uv} is the interaction energy. If we put the explicit expression of $\mathscr{V}_v$ into equation (15), the symmetry of W_{uv} is readily shown:

$$W_{ij} = W_{ji} \tag{16}$$

(e.g., MacMillan 1930, chap. III, § 76).

Then the self-energy tensor can be defined according to equation (5):

$$(\Omega_u)_{pq} = -\frac{1}{2}\int_{S_u} \rho_u(x_1, x_2, x_3)(\mathscr{V}_u)_{pq}(x_1, x_2, x_3)d^3S , \qquad u = i, j , \tag{17}$$

and the interaction-energy tensor can be defined by generalization of equation (15):

$$(W_{uv})_{pq} = -\frac{1}{2}\int_{S_u} \rho_u(x_1, x_2, x_3)(\mathscr{V}_v)_{pq}(x_1, x_2, x_3)d^3S , \qquad u = i, j , \qquad v = j, i , \tag{18}$$

where equations (10), (14), (17), and (18), lead to

$$\sum_{r=1}^{3} (\Omega_u)_{rr} = \Omega_u , \qquad u = i, j , \tag{19}$$

$$\sum_{r=1}^{3} (W_{uv})_{rr} = W_{uv} , \qquad u = i, j, \quad v = j, i , \tag{20}$$

and keeping in mind that $\rho_u = 0$ outside S_u and $\rho = \rho_i + \rho_j$, equations (5), (12), (14), and (15), lead to

$$\Omega_{pq} = (\Omega_i)_{pq} + (W_{ij})_{pq} + (W_{ji})_{pq} + (\Omega_j)_{pq} . \tag{21}$$

If we put the explicit expression of $(\mathscr{V}_v)_{pq}$ into equation (18), the symmetry of $(W_{uv})_{pq}$ is readily shown:

$$(W_{ij})_{pq} = (W_{ji})_{pq} . \tag{22}$$

In summary, the tensors $(\mathscr{V}_u)_{pq}$, $(\Omega_u)_{pq}$, $(W_{uv})_{pq}$, are symmetric and their traces equal $\mathscr{V}_u$, Ω_u, and W_{uv}, respectively; in addition, the last tensor is symmetric also in respect to the exchange of one component with the other.

At this stage, let us define the tidal energy as the virial for a given component in connection with the tidal field induced by the other:

$$V_{uv} = \int_{S_u} \rho_u(x_1, x_2, x_3) \sum_{r=1}^{3} x_r \frac{\partial \mathscr{V}_v}{\partial x_r} d^3S , \qquad u = i, j , \qquad v = j, i , \tag{23}$$

where $\rho_u(\partial \mathscr{V}_v/\partial x_r)d^3S$ represents the rth component of the force exerted by the vth subsystem on the mass element $\rho_u d^3S$ of the uth subsystem (e.g., Brosche et al. 1983).

Then the tidal-energy tensor can be defined by generalization of equation (23):

$$(V_{uv})_{pq} = \int_{S_u} \rho_u(x_1, x_2, x_3)x_p \frac{\partial \mathscr{V}_v}{\partial x_q} d^3S , \qquad u = i, j , \qquad v = j, i , \tag{24}$$

where equations (23) and (24) lead to

$$\sum_{r=1}^{3} (V_{uv})_{rr} = V_{uv} , \qquad u = i, j , \qquad v = j, i , \tag{25}$$

and the virial theorem in tensor form may be expressed as (e.g., Caimmi et al. 1984)

$$2(T_u)_{pq} + (\Omega_u)_{pq} + (V_{uv})_{pq} = 0 , \qquad u = i, j , \qquad v = j, i \tag{26}$$

for pressure-free subsystems.

On the other side, it has to be for the whole system (e.g., Chandrasekhar 1969, chap. II, § 11):

$$2T_{pq} + \Omega_{pq} = 0 . \tag{27}$$

Equations (21), (26), and (27) lead to

$$(V_{ij})_{pq} + (V_{ji})_{pq} = (W_{ij})_{pq} + (W_{ji})_{pq} , \tag{28}$$

or, without loss of generality,

$$(V_{uv})_{pq} = (W_{uv})_{pq} + (Q_{uv})_{pq} , \qquad u = i, j , \qquad v = j, i , \tag{29}$$

$$(Q_{ij})_{pq} = -(Q_{ji})_{pq} . \tag{30}$$

Equations (29), (30), and (22) show that the tidal-energy tensor is made by a symmetric part, W_{uv}, and an antisymmetric part, Q_{uv}, in respect to the exchange of one component with the other. Equations (18), (24), and (29), allow us to define the residual-energy tensor:

$$(Q_{uv})_{pq} = \int_{S_u} \rho_u(x_1, x_2, x_3)\left[x_p \frac{\partial \mathscr{V}_v}{\partial x_q} + \frac{1}{2}(\mathscr{V}_v)_{pq}(x_1, x_2, x_3)\right] d^3S , \quad u = i, j , \quad v = j, i . \tag{31}$$

Equations (15), (20), (23), and (25) allow us to define the residual energy:

$$Q_{uv} = \int_{S_u} \rho_u(x_1, x_2, x_3)\left[\sum_{r=1}^{3} x_r \frac{\partial \mathscr{V}_v}{\partial x_r} + \frac{1}{2}\mathscr{V}_v(x_1, x_2, x_3)\right] d^3S , \quad u = i, j , \quad v = j, i , \tag{32}$$

or equivalently,

$$Q_{uv} = \sum_{r=1}^{3} (Q_{uv})_{rr} . \tag{33}$$

Keeping in mind equations (20), (25), and (33), it is easy to see that a summation over the diagonal terms in equations (29) and (30) leads to

$$V_{uv} = W_{uv} + Q_{uv} , \quad u = i, j , \quad v = j, i , \tag{34}$$

$$Q_{ij} = -Q_{ji} \tag{35}$$

in connection with the related trace.

3. A SPECIAL CASE

We particularize the above results to the special case of astrophysical interest, i.e., homogenous, coaxial ellipsoids, one lying completely within the other. Let us denote with i the inner subsystem and denote with j the outer.

The explicit expressions of the potential and the potential tensor are calculated by performing the integrations at the right-side members of equations (8) and (9), respectively; the final result turns out to be (e.g., Chandrasekhar 1969, chap. 3, §§ 17, 18, 21, 22)

$$\mathscr{V}_u(x_1, x_2, x_3) = \pi G \rho_u \sum_{r=1}^{3} (A_u)_r[(a_u)_r^2 - x_r^2] , \quad u = i, j , \tag{36}$$

$$(\mathscr{V}_u)_{pq}(x_1, x_2, x_3) = \pi G \rho_u \{2[(A_u)_q - (a_u)_p^2 (A_u)_{pq}] x_p x_q + \delta_{pq}(a_u)_p^2 [(A_u)_p - \sum_{r=1}^{3} (A_u)_{pr} + x_r^2]\} , \quad u = i, j , \tag{37}$$

where, in general,

$$A_p = a_1 a_2 a_3 \int_0^{+\infty} \hat{\Delta}^{-1}(a_p^2 + s)^{-1} ds , \tag{38}$$

$$A_{pq} = a_1 a_2 a_3 \int_0^{+\infty} \hat{\Delta}^{-1}(a_p^2 + s)^{-1}(a_q^2 + s)^{-1} ds , \tag{39}$$

$$\hat{\Delta} = (a_1^2 + s)^{1/2}(a_2^2 + s)^{1/2}(a_3^2 + s)^{1/2} , \tag{40}$$

and $a_1 \geq a_2 \geq a_3$ are the semi-axes of the ellipsoid. In the case under consideration, we suppose this inequality to hold for the inner subsystem, while all the possibilities remain open for the outer subsystem.

The explicit expressions for the self energy and the self-energy tensor are calculated by performing the integrations at the right-side member of equations (14) and (17), respectively; the final result turns out to be (e.g., Chandrasekhar 1969, chap. 3, § 22):

$$\Omega_u = -\frac{3}{10}\frac{GM_u^2}{(a_u)_1} \sum_{r=1}^{3} (\epsilon_u)_{r2}(\epsilon_u)_{r3}(A_u)_r , \quad u = i, j , \tag{41}$$

$$(\Omega_u)_{pq} = -\frac{3}{10}\frac{GM_u^2}{(a_u)_1} \delta_{pq}(\epsilon_u)_{p2}(\epsilon_u)_{p3}(A_u)_p , \quad u = i, j , \tag{42}$$

where δ_{pq} represents the Kronecker symbol, and, in general,

$$\epsilon_{pq} = \frac{a_p}{a_q} . \tag{43}$$

The explicit expressions of the interaction energy and the interaction-energy tensor in connection with the inner subsystem, are calculated by performing the integrations at the right-side member of equations (15), and (18), respectively, with $u = i$ and $v = j$; the final result turns out to be

$$W_{ij} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3} \sum_{r=1}^{3} (\epsilon_i)_{r2}(\epsilon_i)_{r3}(A_j)_r\left[\frac{5}{4} y_r^2 - \frac{1}{4}\right] , \tag{44}$$

$$(W_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[\frac{1}{2}(A_j)_q - \frac{1}{2}(a_j)_p^2(A_j)_{pq} + \frac{5}{4}y_p^2(A_j)_p - \frac{1}{4}y_p^2\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2\right], \tag{45}$$

$$m = \frac{M_j}{M_i}, \qquad y_p = \frac{(a_j)_p}{(a_i)_p}; \tag{46}$$

with the same expression for W_{ji}, $(W_{ji})_{pq}$, according to equations (16) and (22), respectively.

Owing to the presence of the Kronecker symbol at the right-hand side of equation (45), one can replace $(A_j)_q$, $(A_j)_{pq}$, with $(A_j)_p$, $(A_j)_{pp}$, respectively, and use the general relation (e.g., Chandrasekhar 1969, chap. 3, § 21),

$$\sum_{r=1}^{3} A_{pr}a_r^2 = 3A_p - 2A_{pp}a_p^2, \tag{47}$$

to obtain a more compact expression of the interaction-energy tensor:

$$(W_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[(A_j)_p\left(\frac{5}{4}y_p^2 - \frac{1}{4}\right) - \frac{1}{4}\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2(y_p^2 - y_r^2)\right]; \tag{48}$$

for subsystems with similar boundaries equation (46) discloses that $y_p = y_r$ and the last term within parenthesis equals zero; it is also easy to see that the related contribution to the trace of the tensor is null.

The explicit expressions of the tidal energy and the tidal-energy tensor in connection with the inner subsystem, are calculated by performing the integrations at the right-hand side of equations (23) and (24), respectively, with $u = i$ and $v = j$; the final result turns out to be

$$V_{ij} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\sum_{r=1}^{3}(\epsilon_i)_{r2}(\epsilon_i)_{r3}(A_j)_r, \tag{49}$$

$$(V_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}(A_j)_q. \tag{50}$$

Owing to the presence of the Kronecker symbol at the right-hand side of equation (50), one can replace $(A_j)_q$ with $(A_j)_p$ to obtain

$$(V_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}(A_j)_p, \tag{51}$$

which is directly comparable with equation (48).

The explicit expression of the residual energy and the residual-energy tensor in connection with the inner subsystem, come from equations (34), (44), (49), (29), (45), and (50), respectively, as

$$Q_{ij} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\sum_{r=1}^{3}(\epsilon_i)_{r2}(\epsilon_i)_{r3}\frac{5}{4}(A_j)_r(1 - y_r^2), \tag{52}$$

$$(Q_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[\frac{1}{2}(A_j)_q + \frac{1}{2}(a_j)_p^2(A_j)_{pq} - \frac{5}{4}y_p^2(A_j)_p + \frac{1}{4}y_p^2\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2\right], \tag{53}$$

with the opposite expressions for Q_{ij}, $(Q_{ji})_{pq}$, according to equations (35) and (30), respectively.

Owing to the presence of the Kronecker symbol at the right-hand side of equations (45) and (50), one can use equations (48) and (51) to obtain a more compact expression of the residual-energy tensor:

$$(Q_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[\frac{5}{4}(A_j)_p(1 - y_p^2) + \frac{1}{4}\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2(y_p^2 - y_r^2)\right]. \tag{54}$$

For subsystems with similar boundaries, $y_p = y_r$ according to equation (46), and the last term within parentheses equals zero; it is also easy to see that the related contribution to the trace of the tensor is null.

The explicit expressions of the tidal energy and the tidal-energy tensor in connection with the outer subsystem, come from equations (16), (34), (35), (44), (52), (22), (29), (30), (45), and (53), respectively, as

$$V_{ji} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\sum_{r=1}^{3}(\epsilon_i)_{r2}(\epsilon_i)_{r3}(A_j)_r\left(\frac{5}{2}y_r^2 - \frac{3}{2}\right), \tag{55}$$

$$(V_{ji})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[-(a_j)_p^2(A_j)_{pq} + \frac{5}{2}y_p^2(A_j)_p - \frac{1}{2}y_p^2\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2\right]. \tag{56}$$

Owing to the presence of the Kronecker symbol at the right-hand side of equations (45) and (50), one can use equations (48) and (54) to obtain a more compact expression of the tidal-energy tensor:

$$(V_{ji})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\left[(A_j)_p\left(\frac{5}{2}y_p^2 - \frac{3}{2}\right) - \frac{1}{2}\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2(y_p^2 - y_r^2)\right]; \tag{57}$$

for subsystems with similar boundaries, $y_p = y_r$ according to equation (46), and the last term within parentheses equals zero; it is also easy to see that the related contribution to the trace of the tensor is null.

Equations (41), (51), and (57) allow an explicit expression to equations (26), i.e., the virial therom in tensor form for pressure-free subsystems, in connection with the special case under discussion, of homogeneous, coaxial ellipsoids, one lying completely within the other.

4. DISCUSSION AND CONCLUSION

Neutsch (1979) calculated the self-energy tensor and the interaction-energy tensor for ellipsoidal-symmetric distributions of matter; in his notation, potential energies are positive and interaction energies include the contribution of both components. The above results may be compared with Neutch's results in the special case of homogeneous and coaxial spheroids; mutatis mutandis, they turn out to coincide. It is worth recalling that the density distribution of matter at the interface is continuous in the former alternative (Chandrasekhar 1969, chap. 3, §§ 20–22) and discontinuous in the latter.

On the other side, this is not the case with respect of previous approaches (Brosche et al. 1983; Caimmi et al. 1984); this is why, instead of the tensor potential expressed by equation (37), the following has been used:

$$(\mathscr{V}_u)_{pq}(x_1, x_2, x_3) = \pi G\rho_u\left\{2[(A_u)_q - (a_u)_p^2(A_u)_{pq}]x_p x_q + \delta_{pq}\left[(a_u)_p^2(A_u)_p - \sum_{r=1}^{3}(A_u)_{pr}(a_u)_r^2 x_p^2\right]\right\}, \tag{58}$$

which does not come from the general definition (4), and the related expressions of $(W_{ij})_{pg}$, $(Q_{ij})_{pq}$ differ by a traceless tensor:

$$(W_{ij})_{pq} = (\tilde{W}_{ij})_{pq} - (\Delta_{ij})_{pq}, \tag{59a}$$

$$(Q_{ij})_{pq} = (\tilde{Q}_{ij})_{pq} + (\Delta_{ij})_{pq}, \tag{59b}$$

$$(\Delta_{ij})_{pq} = -\frac{3}{10}\frac{GM_i^2}{(a_i)_1}\frac{m}{y_1 y_2 y_3}\delta_{pq}(\epsilon_i)_{p2}(\epsilon_i)_{p3}\frac{1}{4}\sum_{r=1}^{3}(A_j)_{pr}(a_i)_r^2(y_p^2 - y_r^2), \tag{59c}$$

where $\tilde{W}$ and $\tilde{Q}$ are calculated according to previous attempts (Brosche et al. 1983; Caimmi et al. 1984) and Δ_{ij} is null in the special case of subsystems with similar boundaries. Accordingly, the related expressions of $(V_{ij})_{pq}$ coincide, with no change on the tidal action of the outer component over the inner, and the related expressions of $(V_{ji})_{pq}$ differ by $2(\Delta_{ij})_{pq}$, with some change on the tidal action of the inner component over the outer. In addition, it may be seen that the expressions of $(W_{ij})_{pq}$, $(Q_{ij})_{pq}$, calculated using equations (18), (31), and (58), no longer fulfill equations (22) and (30), respectively. On the other side, it has been assumed in previous approaches (Brosche et al. 1983; Caimmi et al. 1984), which leads to the above mentioned difference.

For copolar spheroids, equation (59c) reduces to

$$(\Delta_{ij})_{xx} = (\Delta_{ij})_{yy} = -\frac{3}{10}\frac{GM_j^2}{a_j}\frac{1}{m}\frac{1}{4}\frac{1}{y^2}\frac{1-\eta^2}{\eta^2}\frac{\epsilon_j}{1-\epsilon_j^2}(\gamma_j - \alpha_j), \tag{60a}$$

$$(\Delta_{ij})_{zz} = -\frac{3}{10}\frac{GM_j^2}{a_j}\frac{1}{m}\frac{1}{4}\frac{1}{y^2}2\frac{\eta^2-1}{\eta^2}\frac{\epsilon_j}{1-\epsilon_j^2}(\gamma_j - \alpha_j), \tag{60b}$$

$$y_1 = y_2 = y \geq 1, \qquad y_3 = \eta y \geq 1. \tag{60c}$$

By use of equations (48), further analysis shows that

$$-\frac{1}{5} \leq -\frac{1}{5y^2} \leq \frac{(\Delta_{ij})_{pp}}{(W_{ji})_{pp}} \leq \frac{y^2-1}{4y^2} \leq \frac{1}{4}, \qquad p = x, y \tag{61a}$$

$$-\frac{1}{3} \leq \frac{1-y^2}{3y^2-1} \leq \frac{(\Delta_{ij})_{zz}}{(W_{ji})_{zz}} \leq \frac{2}{5y^2-2} \leq \frac{2}{3}, \tag{61b}$$

$$-\frac{1}{6} \leq -\frac{1}{2}\frac{1}{5y^2-2} \leq \frac{(\Delta_{ij})_{pp}}{(V_{ji})_{pp}} \leq \frac{1}{4}\frac{y^2-1}{2y^2-1} \leq \frac{1}{8}, \qquad p = x, y \tag{62a}$$

$$-\frac{1}{4} \leq -\frac{1}{2}\frac{y^2-1}{2y^2-1} \leq \frac{(\Delta_{ij})_{zz}}{(V_{ji})_{zz}} \leq \frac{1}{5y^2-2} \leq \frac{1}{3}, \tag{62b}$$

where the equalities hold only for special limiting configurations. We expect that, for typical configurations, the values of the above ratios probably do not exceed a few percent. In addition, the self energy turns out to be dominant with respect to the tidal energy for the outer subsystem, provided $m \gg 1$; if this is the case, the term $(\Delta_{ij})_{pq}$ should yield a negligible contribution for any configuration.

The results of the present paper are valid for two subsystems but they may be generalized to n subsystems, by considering both the interaction-energy tensor, the tidal-energy tensor, and the residual-energy tensor, as made by the sum of the contributions related to all components other than the one under consideration.

In summary, a formulation has been built up concerning the virial theorem for subsystems, where the tidal-energy tensor is the sum of the interaction-energy tensor and the residual-energy tensor, the former symmetric and the latter antisymmetric, with respect to the exchange of one component with the other. Explicit expressions for the above mentioned quantities have been determined in the special case of homogeneous, coaxial ellipsoids, one lying completely within the other. Accordingly, an explicit expression to the virial theorem in tensor form for pressure-free subsystems of the kind considered, has been allowed. The comparison with previous results related to the special case of spheroids has shown a discrepancy, owing to (1) a different formulation of the potential tensor in the two cases, and (2) an assumed, but not actual, symmetry of the interaction-energy tensor with respect to the exchange of one component with the other, in connection with the previous case. However, the two approaches have turned out to yield the same results in the special case of subsystems with similar boundaries, and a correction not exceeding a few percent for typical configurations.

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