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# A NUMERICAL EXPLORATION OF THE EVOLUTION OF TROJAN-TYPE ASTEROIDAL ORBITS

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## ABSTRACT

Numerical $N$-body integrations have been conducted to study the stability of orbits of asteroids, which are co-orbital with the major planets. In Jupiter's case such celestial bodies are the well-known Trojan asteroids, while the newly discovered asteroid 1990MB is the first known "Martian Trojan." In the outer solar system our 20 million year integrations indicate the possible existence of stable Trojan-type tadpole orbits for all the other giant planets. Horseshoe orbits were found to be unstable on the timescale of several million years. Since the evolution of the unstable orbits is likely to be descriptive of the general evolutionary patterns of asteroidal orbits we describe briefly the observed behavior of these orbits, too. The unstable orbits have typically two ways of possible evolution: either to move up in the solar system to greater semimajor axis beyond Neptune's orbit, sometimes to orbits resembling that of Pluto (over millions of years), or to be ejected out of the solar system, most often by Jupiter. In many cases the asteroids are trapped into resonant orbits (resonance with Neptune) for times up to millions of years. These and many other evolved orbits have perihelion distances near the distance of Neptune. For the stable Trojan-type orbits the probability density of the planet-Trojan (libration) angle is discussed to show where observational searches should be directed. In the inner solar system our two million year calculations indicate the existence of long-term stability for both tadpole and horseshoe type orbits of asteroids co-orbital with the planets Venus, Earth, and Mars. Comparison of the numerical integrations with a simple analytic theory shows good agreement. This suggests that analytical theories can provide good approximations for the Trojan-type orbits in the inner solar system. The Liapunov exponents were calculated for some of the orbits. These, however seem not to provide any clear answer to the question of very long term persistence of the types of motion.

## 1. INTRODUCTION

This work is a continuation of previous papers [Zhang & Innanen 1988 a,b,c; Innanen & Mikkola 1989 (Paper I); Mikkola & Innanen 1990a (Paper II); Mikkola & Innanen 1990b], where numerical results from less extensive experiments were described for the behavior of Trojan-like orbits. Similar exploratory calculations were made by Everhart (1973) and by Weissman & Wetherill (1974), while Rabe (1967) has studied theoretically the stability of Trojan orbits. A comprehensive theoretical treatment of the orbits in the framework of the restricted three-body problem can be found in Szebehely (1967). More recent theoretical studies have been published, e.g., by Garfinkel (1976, 1977, 1978, 1980).

For non-Jovian Trojans the external perturbative acceleration (due to other planets) is typically larger than that due to the primary planet itself. This has been considered as a serious *a priori* objection against the existence of non-Jovian Trojans. However, there is now observational evidence for a co-orbital asteroid in Mars' orbit (IAUC 5067, 5075). This fact at once removes the above argument and shows that the magnitude of the perturbation, compared with the "regulating force" due to the primary planet, is not decisive for stability. (As in the previous papers, we use the word "stable" in the meaning "the test particle orbit preserves its Trojan-type character," which is the only stability definition having, possibly, observational relevance.)

In this paper we discuss the evidence for stability of Trojan-like orbits for all the major planets in the solar system. For Saturn, Uranus, and Neptune the results are derived from 20 million year numerical orbit integrations, which demonstrate the stability of tadpole-type Trojan orbits for at least the time quoted. The evolution of many unstable orbits is also followed. Several of these show an interesting tendency to spend long periods of time in orbits resembling that of Pluto. In addition, it is evident that there are many (other) resonances (mainly with Neptune) which can trap the asteroid for prolonged periods of time.

For the inner solar system planets we examine the stability in terms of the results from a 2.25 million year numerical integration. Although the time span covered by our integration is only $0.5\times10^{-3}$ times the age of the solar system, it nevertheless simulates about 10 million periods around the Sun for Mercury and more than a million for Mars. The results are in agreement with the earlier much less extensive calculations (Mikkola & Innanen 1990, Paper II) as well as with a simple analytic presentation, which is essentially based on the restricted three-body problem. The recent discovery of the first known Mars' Trojan 1990MB (IAUC 5067, 5075, Mikkola & Innanen

1990b) gives an observational confirmation for our conclusions.

Due to the great computational effort required by these calculations only a modest number of orbits was studied. Consequently there are many questions left open by this work, including especially the effect of inclination. (Here it is worth noting that the inclinations of many known Trojans, including that of the newly found Mars' Trojan, are quite large.) Preliminary results in a simple model (Zhang & Innanen 1988c) show that stability may persist to high inclinations.

It would be interesting to have a definite test for the stability of a Trojan orbit. The Liapunov exponents have been generally suggested as a possibility to confirm stability. However, this is most useful for periodic orbits and naturally tends to double the computational overhead. In the present calculations it is likely to be difficult to get any definitive stability criterion by calculating the Liapunov exponents (compare, e.g., the controversy concerning Pluto's chaoticity [Milani *et al.* (1989), Sussman & Wisdom (1988)]. One cannot know when the limit is approached, and also it is not possible to say whether the limit is zero or a very small positive quantity [for a discussion see Marchal (1988)].

We have nevertheless calculated the Liapunov exponents for many of the stable looking orbits. In most cases it is hard to say with certainty whether or not the Liapunov exponent is different from zero. It is also evident that even a "chaotic" Trojan may preserve its orbit type for astronomically long periods of time. In a study of inclined Trojan orbits [presently in progress] this has been confirmed for a chaotic Saturnian Trojan for a period of time of more than 200 million years.

## 2. NUMERICAL EXPERIMENTS

In the integrations we used a $N$-body point mass model for the solar system. When studying the outer solar system the inner planets from Mercury to Mars were neglected but the giants from Jupiter to Neptune were treated self-consistently, i.e., their orbits were numerically integrated together with the motions of the hypothetical asteroids (our test particles).

The model in the inner solar system calculations includes the planets from Mercury to Saturn, while Uranus and Neptune were neglected to reduce the numerical overhead. The neglected planets have indirect effects on the motions in the inner solar system (mainly by affecting the motions of Saturn and Jupiter), but these effects are not expected to influence the stability of the Trojan type orbits considered here.

Our integration routine solves directly the Newtonian equations of motion

$$\ddot{\mathbf{r}}_j = G \sum_i m_i \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}^3}, \tag{1}$$

where the notation is standard, by means of a Bulirsch–Stoer (1966) routine for second-order differential equations. When calculating the Liapunov exponents we solved simultaneously the equations for the variations of the coordinates of the asteroid

$$\delta\ddot{\mathbf{r}}_j = G \sum_i m_i \left( \frac{-\delta\mathbf{r}_j}{r_{ij}^3} - \frac{3\mathbf{r}\mathbf{r}\cdot(-\delta\mathbf{r}_j)}{r_{ij}^5} \right). \tag{2}$$

Arbitrary initial conditions can be used for the variations $\delta\mathbf{r}_j$, because any strong instability in the motion will soon dominate the solution for these variations. Finally, the Liapunov exponent is defined by

$$\lambda_j = \lim_{t\to\infty} \log(|\mathbf{r}_j|)/t. \tag{3}$$

### *2.1 An Accuracy Test*

In long-term solar system computations it is customary to use as high a computational precision as possible, while our computations are of modest precision only. This might not be justified if we were interested in the evolution of the orbits of the major planets. However, to obtain qualitatively correct results (typical evolution) for asteroidal orbits a modest precision is acceptable. For stable Trojan-type orbits it is also evident that actually a much higher precision is maintained than for other orbit types. The reason is that a Trojan may be viewed as a (nonlinear) oscillator, and such systems are easy to integrate numerically (if they are stable). The oscillator property is well known from the usual treatment of the small oscillations around the triangular Lagrangian points. More generally, one can see that the oscillator treatment is applicable for the librational motion of the Trojans by considering the equations of motion in terms of the orbital elements (see, e.g., Garfinkel 1976, 1977, 1978, 1980). The fact that we use directly Newtonian equations formulated in rectangular coordinates does not affect the numerical stability properties of the Trojan-like orbits.

The above heuristic notes were confirmed in a test calculation designed as follows: To obtain an estimate for the accuracy of the most difficult-to-calculate Trojan-like orbit in our simulation we calculated the orbits of some Saturnian Trojans in the restricted three-body framework. We thus replaced Saturn's orbit by a circle (initially), Jupiter by a massless particle and used the same error tolerance as in the actual simulations. In this way we can expect the step-size selection algorithm to work similarly to the actual simulation, thus giving a comparable precision. This calculation was carried out for 10 Myr. As expected the semimajor axis of "Saturn" shows a practically linear relative error propagation of $3\times10^{-16}$/yr thus amounting to a total of $3\times10^{-9}$ relative error in the 10 Myr time. Also, the initially circular orbit developed a small eccentricity of $7\times10^{-10}$ during the first 4 Myr after which it remained essentially constant. However, a very important point observed is that the Trojans follow "adiabatically" this evolution of the planet's orbit. The Jacobian constant, when calculated in the system in which Saturn's semimajor axis is $=1$ (using the present value at every time for scaling the Jacobian constant), did not show any secular error. Instead it fluctuated between limits which were

$\pm 7 \times 10^{-13}$ for $\Delta a/a = 0$

and

$\pm 4 \times 10^{-12}$ for $\Delta a/a = 0.015$.

Examination of a graph of these errors showed us that most of the fluctuation in the case $\Delta a/a = 0$ is actually not error, but is due to the (itself erroneous) development of the small eccentricity of the "Saturn's" orbit. (We see obviously regular increasing fluctuation during the first 4 Myr after which the fluctuation remains at a constant level, just like the erroneous eccentricity.) The corresponding error in the Jacobian constant of the particle in the orbit of Jupiter grew linearly with time, amounting to a value of $2 \times 10^{-4}$ at the end of the calculation.

From these we conclude that the orbits of the (stable) Trojans, relative to the primary planets, are much more accurate than the orbits of "free" asteroids or the major planets themselves. This applies to the most important orbital elements only: the phases of the oscillations are undoubtedly unstable, but they are irrelevant for the stability problem studied in this paper.

Here we point out that the agreement between the numerical results and a simple analytic model (see below) is notable which also provides an indirect confirmation of the sufficiency of the numerical method.

### 2.2 *Initial Conditions*

The initial coordinates and velocities for the planets were taken from Oesterwinter & Cohen (1971), while initial values for the Trojans were produced by rotating each planet's orbit 60° in the ecliptic plane (which was our *xy* plane). After the rotation, the initial osculating semimajor axes were increased by various small amounts (listed below), a procedure which provides us a set of initial orbits with shapes varying from tadpoles to horseshoes.

Outer solar system: Two test particles were placed in Saturn's orbit and 11 for each of Uranus and Neptune. In what follows we will call these test particles S1, S2 (Saturnian Trojans), U1, U2,...,U11 (Uranian Trojans), N1,...,N11 (Neptunian Trojans). The values of the initial semimajor axis shifts were as follows:

(i) $\Delta a/a = 0.01$, and 0.015 for the Saturnian Trojans S1 and S2,

(ii) 0, 0.0025, 0.005,..., 0.025 for the Uranian Trojans U1, U2,..., U11, and similarly,

(iii) 0, 0.0025, 0.005,..., 0.025 for the Neptunian Trojans N1, N2,...,N11 [i.e., for *Un*, $\Delta a/a = 0.0025(n-1)$ and similarly for N*n*]. During the Trojan's motion, its heliocentric semimajor axis varies such that the average mean motion equals that of the primary planet (see Paper I, Fig. 3). The maximum deviations in semimajor axis occur near the Lagrangian points.

The results of the computation were saved by taking "snapshots" of the system every 10 000 yr.

Inner solar system: Five test particles were placed at or near the triangular Lagrangian points of each of the terrestrial planets from Mercury to Mars. The shifts $\Delta a/a$ used for the initial osculating heliocentric semimajor axis were $\Delta a/a = 0.0$, 0.001, 0.002, 0.003, 0.004. To save the results, snapshots were taken every 500 yr. The rather long time intervals between the consecutive snapshots were necessary to avoid the production of excessive amounts of data.

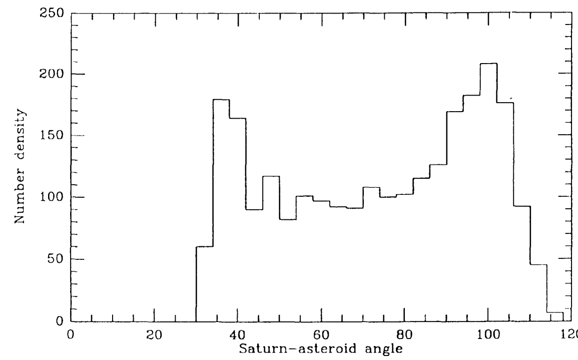


FIG. 1. Probability density of the Saturn-Sun-Trojan angle $\theta$ for the experiment S2. We note that the most likely place to observe the Trojan is near the turning points, i.e., near $\theta = 40°$ and $\theta = 100°$, rather than at the Lagrangian triangular points.

## 3. RESULTS

### 3.1 *Outer Solar System*

Only tadpole orbits survived as Trojan-like orbits over the 20 Myr integration period. For Saturn the situation was discussed in Paper I, where the immediate neighborhood of the triangular Lagrange point was found to be unstable, but there was evidence for the stability of wide tadpole orbits. Their stability region was estimated to be (roughly) $0.01 < \Delta a/a < 0.02$. Here, however, the orbit S1 with $\Delta a/a = 0.01$ became unstable after the long time of $\approx 11$ Myr. This result may be due to accumulation of numerical error but in any case it is a demonstration of the presence of a destabilizing mechanism (which, as noted in Paper I, is still unexplained in detail). On the other hand, the wider orbit S2 survived the integration, remaining a tadpole for the entire 20 Myr.

An interesting question concerning the behavior of the stable orbits is the angular width of the libration around the Lagrange point. Since this is also important from an observational point of view, we present results in the form of probability density histogram for the angle $\theta$ between the planet Saturn and the Trojan (as seen from the Sun) in Fig. 1. For the widest stable Trojan orbits of the planets Uranus and Neptune the corresponding histograms are very similar. One concludes from this that possible Trojans may be found anywhere in the range $30° < \theta < 100°$. Thus the region is quite wide and, when inclination effects are added, one finds that observational searches must be conducted over large areas. Clearly it is not satisfactory to look just at the Lagrange point neighborhoods.

A brief summary of the evolution of the various test particle orbits, including the unstable ones, is given as follows.

(i) S1: Tadpole motion (stable appearance) for 9.5 Myr at which time the eccentricity began to grow. A close approach to Jupiter occurred at time $\approx 10.5$ Myr with a consequent ejection out of the solar system.

(ii) S2: Stable tadpole motion for the entire 20 Myr covered by the integration. The libration angle Saturn-Sun-Trojan varies between 30° and 120°.

(iii) U1: Stable tadpole. The libration angle Uranus-Sun-Trojan varies between 35° and 95°.

(iv) U2: Stable tadpole. The libration angle Uranus-Sun-Trojan varies between 27° and 120°.

(v) U3: Horseshoe for $\approx 1$ Myr, after which semimajor axis increased to near-Plutonian orbit (during a half a million years), followed by decline in $a$. For several million years the particle then had semimajor axis between Uranus and Neptune, while the perihelion distance approached Jupiter's distance. Ejection by Jupiter occurred at $\approx 4.6$ Myr.

(vi) U4: Horseshoe for $\approx 1.2$ Myr, then ascent to an orbit with semimajor axis between Neptune and Pluto and perihelion distance close to Neptune. After half a million years, this particle returned to an Uranus horseshoe orbit in which it persisted for about one million years. Then a gradual growth of the semimajor axis began leading eventually to a Pluto-like orbit at time $\approx 7$ Myr. Here the particle spent the next 10 Myr before descending again. The orbital elements for this particle are plotted in Fig. 2.

(vii) U5: Horseshoe for 4.5 Myr, after which eccentricity growth lead to a close encounter with Saturn. Ejection by Saturn lead to a very long cometlike orbit at 5.5 Myr.

(viii) U6: Horseshoe for 4 Myr. Jump to a Pluto-like orbit where the particle spent 2.5 Myr after which Neptune ejected it to a cometlike orbit with semimajor axis near 100 AU.

(ix) U7: Horseshoe for 6.5 Myr followed by a 2 Myr "chaotic" motion in the Saturn-Uranus-Neptune region. Ejection out of the solar system by Saturn occurred at about 8.5 Myr.

(x) U8: This particle destabilized immediately from the Trojan-like orbit, ascended to the Neptune region and visited several times a Pluto-like orbit.

(xi) U9: During the first half a million years this particle went up to an orbit with semimajor axis varying at and beyond that of Pluto. The perihelion distance was persistently close to Neptune's orbit.

(xii) U10: Chaotic from the beginning. Ejected out of the solar system by Jupiter at the time of about 4.7 Myr.

(xiii) U11: Semimajor axis grew to a value nearly equal to that of Pluto in about 6 Myr, while the perihelion remained close to Uranus. After visiting the Pluto region, this particle came back to the inner parts of the solar system. Encounter with Jupiter at the time of 10.5 Myr lead to ejection out of the solar system.

(xiv) N1: Stable tadpole motion for all the 20 Myr covered by the integration. The libration angle Neptune-Sun-Trojan varies between 45° and 80°.

(xv) N2: Stable tadpole motion for all the 20 Myr covered by the integration. The libration angle Neptune-Sun-Trojan varies between 37° and 94°.

(xvi) N3: Stable tadpole motion for all the 20 Myr covered by the integration. The libration angle Neptune-Sun-Trojan varies between 32° and 112°.

(xvii) N4: This orbit varied between a wide tadpole and a horseshoe. It remained co-orbital with Neptune for 15.5 Myr after which the particle began an erratic motion between Uranus and Neptune.

(xviii) N5: Horseshoe for 4.2 Myr, followed by descent to Uranus' neighborhood for a couple of hundred thousand years. This was followed by a two-million year sojourn of near-circular motion between Uranus and Neptune. The perihelion reached Uranus' orbit and the particle's orbit evolved to one with semimajor axis at and beyond Pluto's orbit, while the perihelion remained near Neptune's distance from the Sun. This behavior persisted for the rest of the calculation.

(xix) N6: This is very similar to N5. The only essential difference is that the destabilization of the initial Trojan-type orbit takes place at about 6.5 Myr.

(xx) N7: Here the Trojan-type motion persisted for 12.5 Myr. After this the particle developed a Pluto-like orbit for a period of about a half a million years. Finally, the object descended and was ejected out of the solar system (at 17.5 Myr) by Jupiter.

(xxi) N8: After an initial period of 2.5 Myr as a horseshoe Trojan, this orbit descended and the particle was ejected by Jupiter at the time of 4 Myr.

(xxii) N9: This persisted as a wide-horseshoe Trojan for 12 Myr. Visited Pluto region (2 Myr), descended and was ejected by Jupiter at 16.6 Myr.

(xxiii) N10: This orbit is a "planet crossing grand tourist." The semimajor axis fluctuates around Neptune's region for the first 5 Myr while the perihelion descends to near Saturn's orbit. After this the semimajor axis increased and varied around that of Pluto. The inclination seems to increase secularly during the first 14 Myr, after which its value fluctuates around 30°.

(xxiv) N11: This particle "jumps" in a short time to an orbit with semimajor axis varying somewhat beyond that of Pluto's, while the perihelion distance is near Neptune's orbit. At about 10 Myr the particle sinks down to an orbit between Uranus and Neptune. Then around 16 Myr it encounters Uranus, jumps up (at 17 Myr) and spends more than a million years as a Neptunian Trojan again!

Figures 2 and 3 describe in more detail the evolution of the orbital elements $a$, $q$, $i$ (semimajor axis, perihelion distance, and inclination) for the orbits U4, U8, U9, N5, N9, and N11. We draw attention especially to the Fig. 2 where the particle U4, a Uranian Trojan during the first million years, jumps close to Neptune's orbit, comes back, and remains a Uranian Trojan for another million years, after which it gradually ascends to an orbit very similar to that of Pluto. Note that all the three elements $a$, $e$, $i$ are close to the corresponding values for Pluto. This particle occupies a Pluto-like orbit for about 10 Myr before another descent.

As can be found by examining the orbital evolutions from Figs. 1 and 2, these simulations demonstrate that asteroids in the outer solar system evolve in a qualitatively similar way. They generally tend to develop fairly high inclinations (10°–20°), their semimajor axes often increase,

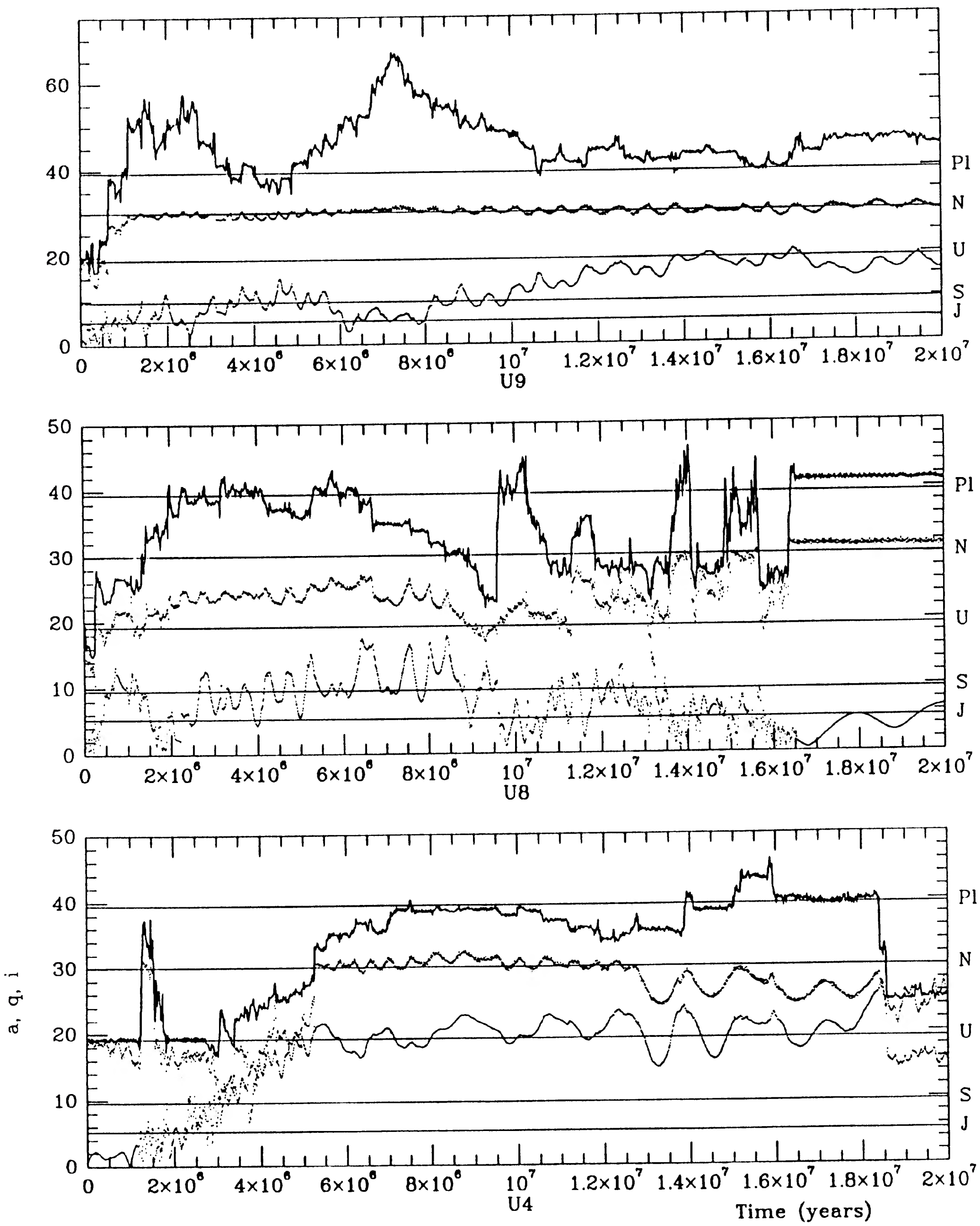


FIG. 2. Evolution of the orbital elements *a*, *q*, *i* of the experiments U4, U8, and U9. The horizontal lines indicate the mean distances of the major planets. The uppermost curve plots the behavior of the semimajor axis. The dotted curve below that for the semimajor axis indicates the evolution of the perihelion distance, while the lowermost curve is for the inclination. The coordinate scale for *a* and *q* is in astronomical units while the reading also gives the inclination in degrees. Bottom: Evolution of the orbit U4. We note that, after about 1.5 Myr, the original Trojan orbit destabilizes, the asteroid jumps above Neptune's orbit, and descends until Uranus again captures it to a Trojan orbit for about a million years. After this a gradual increase in the semimajor axis occurs, and at about 7 Myr the orbit is close to that of Pluto. At this time the inclination has increased to a value around 20°. We also note a resonance pattern (this is the 3:2 resonance with Neptune) between 16 and 18.5 Myr. Middle: Evolution of the orbit U8. We note an erratic evolution with frequent visits to the Pluto region. From the time 16.5 Myr an interesting resonance pattern appears. This is actually the 8:5 resonance with Neptune. Top: Evolution of the orbit U9. This orbit moves persistently near and above Pluto's orbit and has its perihelion at Neptune's distance.

sometimes being captured into resonances for prolonged times, while the perihelia typically remain in the neighborhood of Neptune. Such an evolution may have been common at the time of the formation of the solar system. Thus some of the asteroids ejected outside Neptune's orbit may still be there forming a possible source of comets, as suggested by cometary theorists (Duncan *et al.* 1988, 1989; Gladman & Duncan 1990).

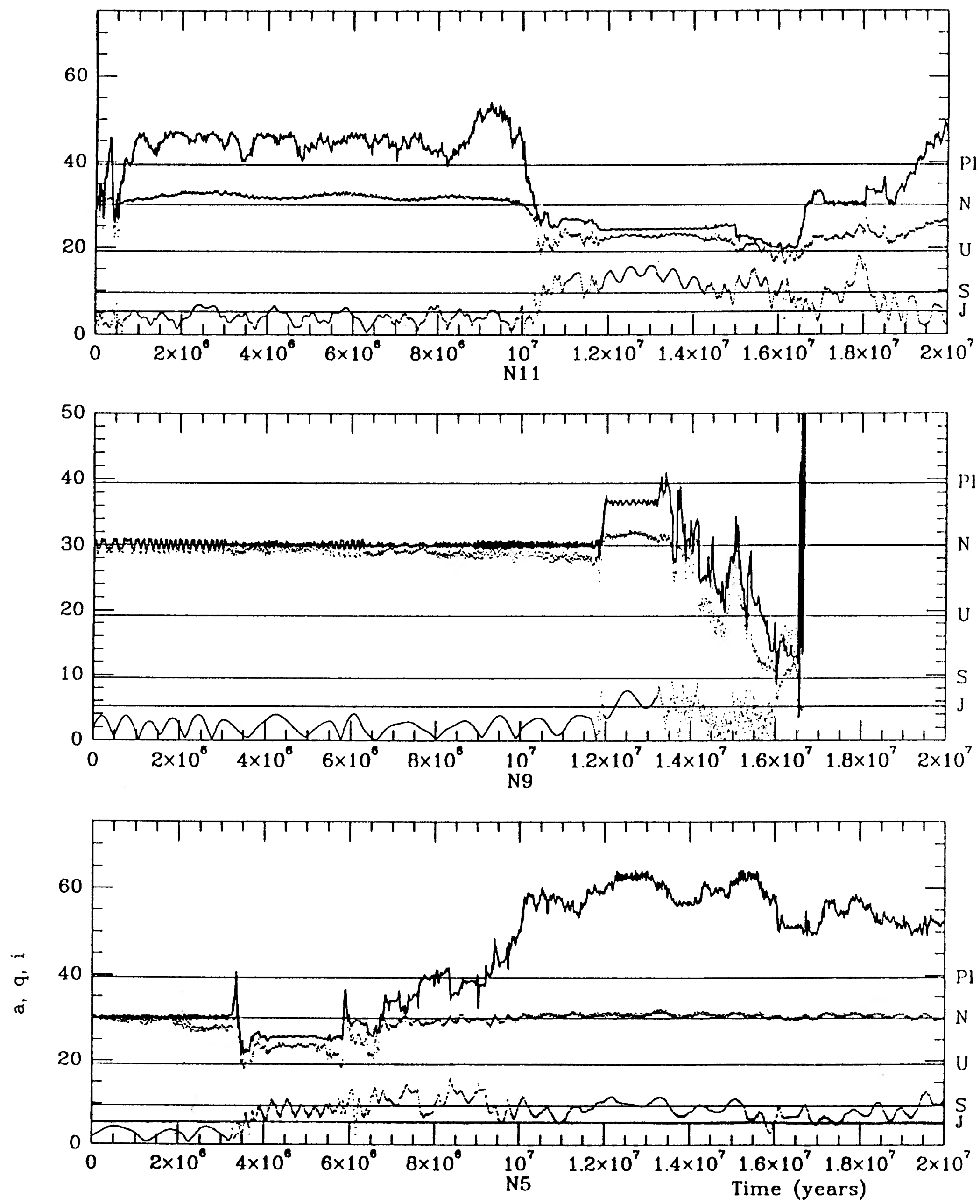


FIG. 3. Bottom: Evolution of the orbit N5. We note two resonant patterns at 12 to 13 and 15 to 16 Myr. The resonance is 3:1 with Neptune. Middle: Evolution of the orbit N9. After 12 Myr the Trojan-type orbit destabilizes and jumps to a 4:3 resonance orbit with Neptune. After a brief visit to a Pluto-like orbit, chaotic behavior begins. Finally an ejection by Jupiter occurs at about 16.6 Myr. Top: Evolution of the orbit N11. After 10 Myr in an orbit beyond Pluto, the asteroid descends to a nearly circular orbit between Uranus and Neptune. Between 17 and 18 Myr the asteroid is again a Neptunian Trojan but the eccentricity is large, allowing the perihelion to be near Uranus orbit.

### *3.2 Inner Solar System*

Except for the cases of Mercury and Mars, there is not much evolution in the calculated orbits, and they remain similar for all the 2.25 Myr of integration. In the case of Mercury the particles' orbits lost their Trojan character fairly rapidly, except the one with $\Delta a/a=0$ initially. In this particular orbit the amplitude of the libration angle, however, increased continuously during the integration. At the time of about 2.1 Myr the original tadpole orbit turned to a horseshoe. At this time, however, the orbital eccentricity had become sufficiently high (near 0.7) to cause encounters with Venus. The average semimajor axis seems still to be the same as that of Mercury itself, but the orbit is likely to be unstable in the sense that the behavior may not be

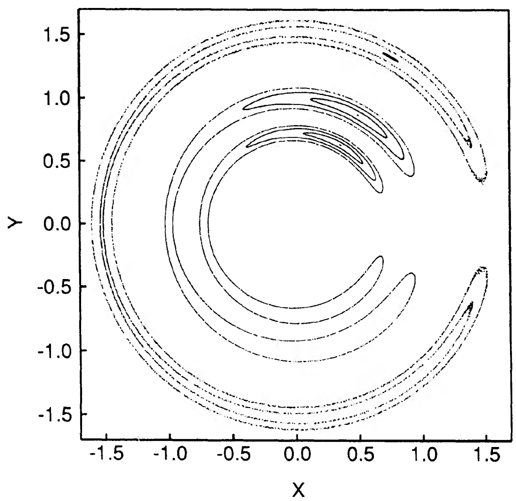


FIG. 4. The mean motions of the test particles in the inner solar system (radial coordinate=semi-major axis, angular coordinate=difference of the mean longitudes of the asteroid and the primary planet), for 2.25 million years. The deviations of the particles' semimajor axes from those of the primary planet itself are magnified by a factor of 30. For Venus and Earth all the integrated orbits (initial $\Delta a/a=0.0$, 0.001, 0.002, 0.003, 0.004) were stable and are shown in the diagram. In the case of Mars only the three orbits with $\Delta a/a=0.0$, 0.001 0.002 preserved their Trojan-type character and are included in the plot. (The $xy$ tic intervals are=0.5 astronomical units.)

considered as one of a Trojan character anymore. For Venus and Earth, all the integrated orbits remained stable with no essential evolution, while for Mars only the three orbits with $\Delta a/a=0.0$, 0.001, and 0.002 remained stable. These results are summarized in Fig. 4, where the "mean motions" of the Trojans are plotted. We use the semimajor axis as the radial coordinate and the mean longitude $\lambda=M+\omega+\Omega$ (here $M$, $\omega$, and $\Omega$ have their usual meaning) is used for the angular coordinate. The 'mean motion' relative to the primary planets is defined in terms of the differences of the mean longitudes of the Trojans and the planets. For ease of visualization the deviations of the particles' semimajor axes from that of the primary planet are magnified by a factor of 30, i.e., the quantity plotted is $a_{\text{planet}}+30(a_{\text{particle}}-a_{\text{planet}})$. One notes the general stability of the semimajor axis—libration angle relation. No sign of evolution is detectable—apart from a minor scatter of the plotted points (presumably due to the small periodic variations in the orbital eccentricities)—the mean motions seem steady. In the case of Mars, the approaches to the planet are quite close but still, presumably due to the smallness of Mars' mass, the orbit clearly preserves its Trojan-type character despite the fairly large eccentricity of the orbits.

Because of the apparent stability of the mean librational motion, it seems clear that only the development of a large eccentricity could destabilize these tentatively stable librators. However, thus far we have seen only quasiperiodic fluctuations in the eccentricities (excepting the unstable particles of Mercury and Mars). It seems that eccentricities of the Trojans essentially mimic the respective planetary eccentricities, with some quasiperiodic variations superimposed.

### *3.3 Comparison with a Simple Theory*

In the inner solar system the eccentricities of the orbits and the masses of the planets are very small. Therefore one expects that analytical models may be useful in describing the orbital behavior. Here we present a comparison of the numerical results with a simple analytic approximation. This basic approximation is a standard starting point in the analytical theories of Trojan asteroids. We present a brief derivation of the model and compare it with the numerical integrations (for a complete theory see, e.g., Garfinkel 1976, 1977, 1978, 1980).

The heliocentric Hamiltonian for a Trojan may be written

$$H=-\frac{\mathcal{M}_\odot+m}{2a}-mU-R, \tag{4}$$

where $\mathcal{M}_\odot$ and $m$ are the masses of the Sun and the planet, $a$ is the Trojan's heliocentric semimajor axis, $U$ is the disturbing function due to the primary planet while $R$ represents the effects of other planets.

Let us write $a=a_1(1+\epsilon)$ ($a_1$=the semimajor axis of the primary planet), expand $H$ in terms of the small parameters: eccentricities, inclinations, and $\epsilon$, neglect $R$ and retain only the leading terms. Then, in the rotating coordinate system (which rotates at the angular velocity equal to the mean motion of the primary planet), we have the time independent Hamiltonian

$$\tilde{H}=-\frac{\mathcal{M}_\odot+m}{a_1}\left(\frac{8}{3}\epsilon^2+\mu\tilde{U}\right), \tag{5}$$

where $\mu=m/(\mathcal{M}_\odot+m)$ and

$$\tilde{U}=\frac{1}{2\sin(\theta/2)}-\cos(\theta). \tag{6}$$

Here $\theta$ is the difference of the mean longitudes $\lambda$ ($=M+\omega+\Omega$) of the Trojan and the primary planet. Because this approximate Hamiltonian is independent of time, it is a conserved quantity. Thus we have a relation between $\epsilon$ and $\theta$ from which we obtain the approximation

$$\epsilon^2=\epsilon_0^2-\frac{8\mu}{3}\left(\frac{1}{2\sin(\theta/2)}-\cos(\theta)-\frac{1}{2}\right) \tag{7}$$

into which we must substitute $\epsilon_0=\Delta a/a$ from the initial values (Sec. 2) to get the analytic approximations to the curves in Fig. 4. In Fig. 5 we compare the curves defined by the above equation with the results of our numerical integration (in the case of Earth's Trojans). The surprisingly good agreement argues for the reliability of the computations and suggests that simple analytic theory may be used to describe the motions of Trojan-like asteroids in the inner solar system, at least to a good first approximation.

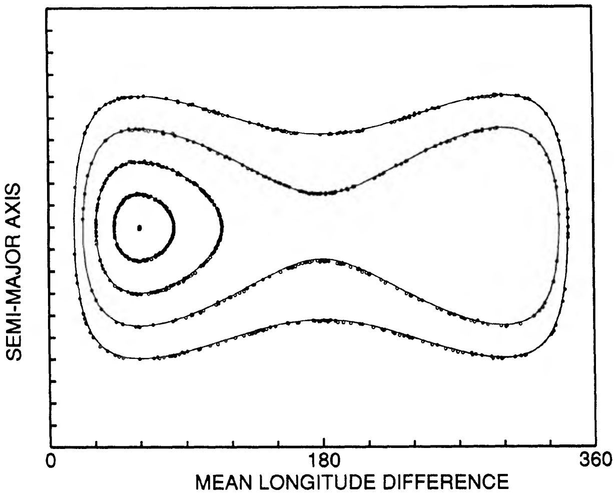


FIG. 5. A comparison of the numerical integrations and the simple theory for the Earth's Trojans. The semimajor axis is plotted against the difference of mean longitudes. The full lines are from the simple theory and the circles are from the numerical integrations (with a time interval of 5000 yr). One notes a surprisingly good agreement, an observation which, in our view, confirms both the reliability of the computations and the usefulness of the analytic approximations. (The vertical tic interval is =0.001.)

### *3.4 Liapunov Exponents*

We calculated the Liapunov exponents for some of the stable looking orbits: In the outer solar system the asteroidal orbits with $\Delta a/a=0.015$ for Saturn, $\Delta a/a=0$ for Uranus and Neptune were included. These are illustrated in Fig. 6, where the maxima of the quantity $\log_{10}(|\delta \mathbf{r}|)$ over periods of time of 100 000 yr are plotted as a function of time. One notes the linear increase of the curve for Saturn's Trojan. This indicates sensitive dependence on initial conditions ("chaos") with an $e$-folding time of $\approx$217 000 yr

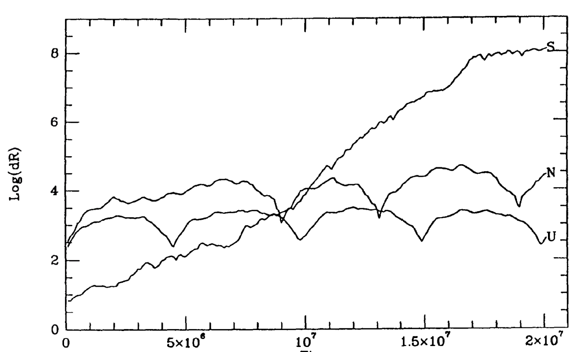


FIG. 6. Evolution of the logarithm of a small initial displacement $\log_{10}(|\delta \mathbf{r}|)$ for the Trojans of the planets Saturn, Uranus, and Neptune. (For Saturn the illustrated quantity is $\log_{10}(|\delta \mathbf{r}|)/5$.) The chaoticity of the Saturn Trojan is unmistakable, while for the other cases we note only periodic fluctuations. The Saturn Trojan had initially $\Delta a/a=0.015$, while the particles on the orbits of Uranus and Neptune started at the Lagrange points. The curves actually plot the maxima of the illustrated quantities over intervals of time of 100 000 yr.

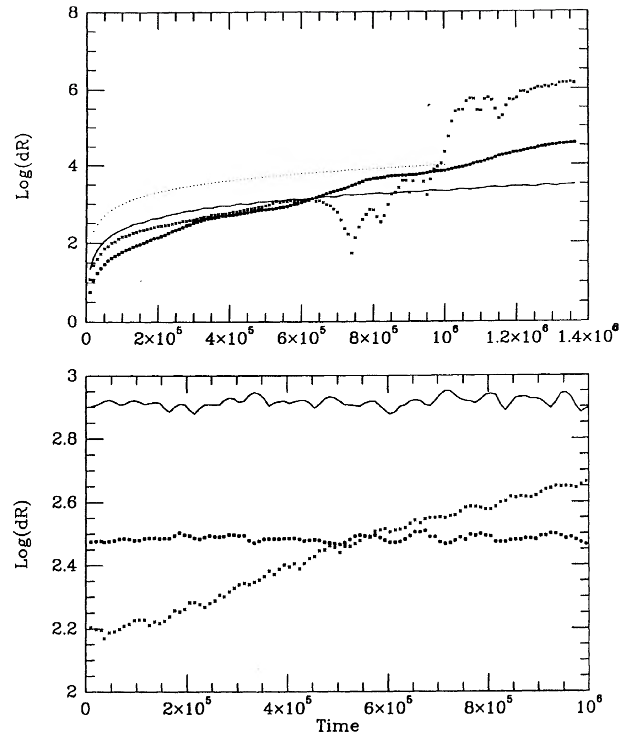


FIG. 7. Evolution of the logarithm of a small initial displacement for the Trojans of the planets Venus, Earth, and Mars. The full drawn curves are for Earth's Trojans while the squares are for Venus' and crosses for Mars' Trojans. Bottom: The particles with $\Delta a/a=0$. It seems clear that Earth's and Venus' Trojans are tightly locked into the resonance and the deviations do not show any secular increase. In the case of Mars the rate of increase indicates an $e$-folding time of $\approx 10^6$ yr. This apparent chaoticity may, however, be due to the beginning of a very long period variation. Top: The particles with $\Delta a/a=0.001$. Here we see an essentially $\log_{10}(t/100)$ behavior (the dotted line, plotted for comparison). However, again the Mars' Trojan (crosses) shows signs of sensitive dependence. This is, however, expected because the particular orbit considered here is very close to the critical one which goes through the Lagrangian point $L_3$. (At $L_3$ there is the bifurcation between a tadpole and a horseshoe orbit.) Similarly to Fig. 6 the plots are for maxima of the illustrated quantities over intervals of time of 10 000 yr.

for small displacements. (Note that the values for this Saturn Trojan were reduced by a factor of 5 to fit the curve into the same figure with the other curves.) Despite this chaoticity the orbit does not show any actual signs of destabilization. On the contrary we have calculated the orbit of a Saturnian Trojan (in a study of inclined orbits, presently in progress) for more than 230 Myr and the orbit has preserved its original character for all that time.

In the inner solar system the Trojans of the planets Venus, Earth, and Mars were included. Here two values of $\Delta a/a$ (=0.0, 0.001) were used. Figure 7 illustrates the behavior $\log(|\delta \mathbf{r}|)$ (maxima over 10 000 year periods) for these particles. It seems that the Mars' Trojans may be chaotic, however, for the Trojan near the Lagrange point the observed behavior may as well be just a beginning of a very long wave. The random looking behavior of the deviation for the horseshoe orbit of the Mars' Trojan with $\Delta a/a=0.001$ is expected because this orbit passes very closely the point $L_3$. Thus a very small displacement may turn this orbit from a horseshoe to a tadpole. The sensitive dependence on initial conditions in this orbit is thus understandable in terms of a phenomenon which is not necessarily related to instability in our sense of the word.

## 4. CONCLUSIONS

This work confirms the conclusion of Paper I (which was based on much shorter integrations): long-term orbital stability exists in some neighborhood of the classical Lagrangian triangular points for all of the outer planets. For Saturn the semimajor axes of the stable Trojans differ from that of Saturn by $\approx 1.5\% \pm 0.5\%$, when the Trojan is near the triangular point. For Uranus and Neptune this stable region, in terms of $\Delta a/a$, is around the value$=0$, and extends to $<0.5\%$ for Uranus and to $<1\%$ for Neptune. The stable orbits are of the tadpole type and the libration angle planet-Sun-Trojan may vary between the limits of (roughly) 30° and 110° for all of the planets considered.

Horseshoe orbits seem to be unstable in the long run. In our experiments the destabilization times of initially horseshoe type orbits were typically millions of years.

A byproduct of this stability study is the observation that asteroids, which are interplanetary grand tourists in the outer solar system, tend to populate orbits beyond that of Neptune. The perihelia are typically near Neptune's orbit and we also observe that, quite frequently, the semimajor axis and inclination of the orbits are not far from the values of Pluto's orbit. This seems to suggest that Pluto's orbit may be a kind of "attractor" for asteroidal orbits in the outer solar system in general. This property of the orbits seems to be associated with the 3:2 resonance between Pluto and Neptune. Several other resonances, such as 8:5, 3:1, and 4:3, were also found to tend to capture the asteroidal orbits for long periods of time. This phenomenon may be also associated with the recent results of Sussman & Wisdom (1988) which show that Pluto's orbit is, in fact, chaotic in the longer term. Further investigations are clearly desirable, and are in progress.

In the inner solar system the main conclusion from this work is to confirm the existence of stable Trojan-type asteroidal orbits in the inner solar system for the planets Venus, Earth, and Mars for at least several million years.

The results clearly suggest that orbits which are of the tadpole type preserve their position near the triangular Lagrangian points for millions of years without any essential changes in their orbital behavior. Our conjecture is that this tadpole behavior will persist for (at least) a significant fraction of the age of the solar system.

Horseshoe orbits behave very similarly to the tadpoles when stability over the present several million years timescale is considered. Since there are no evolutionary trends observable in these orbits thus far, we conclude that stability should persist for much longer timescales. However, we found horseshoe orbits to be unstable in the outer solar system when the timescale is tens of millions of years. Thus it is not possible to draw conclusions, with any certainty, concerning the stability of horseshoe orbits over very long times.

As in Paper II, we stress that analytic theories may be valid for very long times in the inner solar system, as suggested by the very slow evolution of the orbits and the success of the simplest possible theory.

The calculated Liapunov exponents suggest that the Saturnian and Martian Trojans are chaotic. The evolution of the orbital elements in our calculations, however, do not suggest instability in our sense of the word. (Recall also the existence of the Martian Trojan 1990MB!) Of course it is possible that 1990MB is in some sort of transitory dynamical phase in the 10 Myr time-frame setting. In that sort of time frame, the Liapunov exponents do not appear to provide conclusive evidence for stability, but hint at instability. As we have noted earlier, the orbital inclination may play a significant role in the stability question. This aspect is under investigation. Thus we must (at this time) conclude that the only way to study the stability of Trojan orbits is very longterm brute-force calculation.

This research has been supported, in part, by a grant from the Natural Science and Engineering Research Council of Canada. A great deal of impetus to our further computations on planetary Trojans has been given by the discovery of asteroid 1990MB. Its identification as a Mars Trojan owes much to the perspicacity of Dr. Ted Bowell. We are grateful to him for communicating to us his findings.

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