ON THE ORIGIN OF THE BLUE STRAGGLERS IN THE GLOBULAR CLUSTER NGC 5053

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ABSTRACT

The hypothesis that the blue stragglers in the low-density globular cluster NGC 5053 are merged stars formed as a result of physical stellar collisions during dynamical interactions involving primordial binary stars has been investigated in detail. This was done by carrying out a large number of binary-binary and binary-single scattering experiments in which the stars were allowed to instantly merge if they touched. Uniform distributions of binary periods P in $\log P$ space between various limits were considered. It has been found that collisions between binaries with periods exceeding 10^2 days are the best at producing merged stars that remain bound to NGC 5053. The distribution of binary mass ratios is much less important than the distribution of periods in determining the rate at which stars collide. Binary-binary interactions are almost an order of magnitude more effective than binary-single collisions at producing mergers. In the limiting case of a binary frequency of 100% in the core of NGC 5053, we find that, depending on the period range considered, 28%–55% of the cluster's blue stragglers can be accounted for by physical stellar collisions during binary-binary interactions. With the binary frequency in the core reduced to 50%, binary-binary and binary-single collisions can still account for 9%–18% of the blue stragglers. These percentages are roughly doubled if one includes tidal capture binaries formed during dynamical interactions involving binary stars, since these close binaries will produce blue stragglers via gradual coalescence or mass transfer. Given the fact that a surprisingly large number of primordial binaries have been discovered in NGC 5053 by Pryor and his collaborators, it is difficult to reject the possibility that at least some of the cluster's blue stragglers have a collisional origin. The characteristics (e.g., degree of central concentration, mass function, binary properties, and rotational rates) of the merged stars are described.

1. INTRODUCTION

If primordial binary stars exist in globular clusters, then there are (at least) three different ways in which blue stragglers can be produced: (1) by the building up of low-mass companions in binary systems via mass transfer (McCrea 1964), (2) by the merger of short-period binaries after a gradual loss of orbital energy (Zinn & Searle 1976), and (3) by the merger of stars during dynamical interactions involving binary stars (Hoffer 1983). The third mechanism includes both binary-single and binary-binary collisions (especially the latter), and the present paper will improve upon the predictions for this scenario presented in Leonard (1989) in an attempt to reject the hypothesis. There are additional mechanisms which may produce blue stragglers in star clusters [see Sec. I of Leonard (1989) for a summary], but this paper will consider only those mechanisms which involve binary stars.

Hills & Day (1976) were the first to demonstrate that the merger of stars following physical collisions between single stars (i.e., single-single collisions) can produce many blue stragglers in dense globular clusters. The hydrodynamical simulations of Benz & Hills (1987) show convincingly that a collision between two main-sequence stars results in a massive well-mixed star. However, a serious problem with the collisional hypothesis is that Nemec & Harris (1987) and Nemec & Cohen (1989) discovered many blue stragglers in the low-density globular clusters NGC 5466 and NGC 5053, contrary to the Hills and Day prediction that not even one collisionally merged star should be present in either system due to the very low frequency of single-single collisions. This observation was viewed as evidence in favor of the merging of short-period-binaries hypothesis since Renzini *et al.* (1977) predicted that the low-density clusters should contain many such blue stragglers. The basis for the Renzini *et al.* prediction is that the close primordial binaries in the low-density clusters should be essentially unaltered by dynamical interactions, and the very closest of these binaries should coalesce into single objects after $\simeq 10^{10}$ yr. On the other hand, Leonard (1989) pointed out that a collisional origin for the blue stragglers cannot be so easily dismissed if significant numbers of primordial binaries exist in the low-density clusters.

It was Hoffer (1983) who first noted that very close encounters between stars are likely to occur during binary-binary interactions, and such encounters can result in physical stellar collisions even if the binaries involved are relatively wide. Now since the cross section for a binary-binary collision is roughly a/R times larger than that for a single-single collision [compare Eqs. (7) and (8) of Leonard 1989], where a is the binary semimajor axis and R is the stellar radius, and since a/R is typically much larger than unity, it is clear that binary-binary collisions can produce merged stars much more efficiently than can single-single collisions. In fact, Leonard (1989) argued that a primordial binary frequency of $\simeq 10\%$ might be enough for binary-binary collisions to account for the majority of the blue stragglers observed in NGC 5053 and NGC 5466.

The calculations leading to the suggestion that the collisions-of-binaries mechanism might work were fairly simple ones, and it was immediately clear that a much more detailed investigation was required in order to seriously test the hypothesis. In particular, better predictions were needed regarding the rate at which binary-binary collisions produce blue stragglers in low-density clusters such as NGC 5053 and NGC 5466. Hence, this paper extends the earlier study of Leonard (1989) by considering a large number of binary-binary scattering experiments in which the stars are allowed

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to instantly merge if they touch. Note that this criterion has been previously applied to binary-single scattering by Hut & Inagaki (1985) and McMillan (1986). In addition, the new simulations allow us to determine certain characteristics of the merged stars (e.g., their degree of central concentration, mass function, binary properties, and rotational rates) which provide observational tests of the hypothesis.

2. THE EXPERIMENTS

The cluster chosen to test the collisions-of-binaries hypothesis is the low-density globular cluster NGC 5053. This cluster has been the subject of deep CCD studies by Nemec & Cohen (1989) and Fahlman *et al.* (1991). The cluster has also been the subject of a survey by Pryor *et al.* (1991) to search for binary stars among the cluster giants. NGC 5053 represents an extreme test of the collisional hypothesis because of its small central density and core mass. The number of blue stragglers produced by binary-binary collisions in the core of a star cluster (where most of such interactions in a cluster will occur) should correlate with cluster properties according to

$$N_{bs} \propto f_b^2 r_c^3 n_0^2 / \sigma_0, \tag{1}$$

where f_b is the fraction of binaries in the core of the cluster, r_c is the core radius, n_0 is the central number density, and σ_0 is the central velocity dispersion [see Eq. (14) of Leonard 1989]. NGC 5053 has one of the lowest values of $r_c^3 n_0^2/\sigma_0$ of all the globular clusters that have been studied in detail in recent years. Thus, if the collisions-of-binaries mechanism works in NGC 5053, then it should work better in most other clusters.

Nemec & Cohen (1989) found 24 blue stragglers in NGC 5053 brighter than the main-sequence turnoff. From Fig. 4 of their paper, we estimate the mean main-sequence lifetime of these stars to be $\simeq 6\times 10^9$ yr. Thus, if a given mechanism is to account for all of the blue stragglers in the cluster, then it must produce one blue straggler every $\simeq 6\times 10^9/24 = 2.5\times 10^8$ yr. We also note here that the metal-poor isochrones presented in Table V of McClure *et al.* (1987) indicate that the blue stragglers in NGC 5053 studied by Nemec and Cohen have masses exceeding $\simeq 0.9\,\mathcal{M}_\odot$.

The structural parameters and mass function adopted for NGC 5053 are those of the best-fit multimass King model presented in Fahlman *et al.* (1991). For the mass range 0.35–0.78 $\mathcal{M}_\odot$, the central number density is $n_0 = 7.9$ pc^{-3}, and the core radius is $r_c = 8.0$ pc. If we describe the mass function of single stars and binary primaries by a power law of the form

$$\frac{dN}{d\mathcal{M}} \propto \mathcal{M}^{-(1+x_c)}, \tag{2}$$

then $x_c = 0.5$ in the core of NGC 5053.

A central one-dimensional velocity dispersion for NGC 5053 of $\sigma_0 = 1.5$ km s^{-1} will be adopted. This figure is consistent with the radial-velocity observations of cluster giants described by Pryor *et al.* (1991). Fortunately, the exact value of σ_0 assumed in this investigation is not very critical to the results.

The escape velocity from the center of the cluster corresponding to $\sigma_0 = 1.5$ km s^{-1} is $V_{esc,0} \simeq 2\times\sqrt{3}\sigma_0 = 5.2$ km s^{-1}. Here we have made use of the relation $V_{esc} \simeq 2V_{rms}$, where V_{rms} is the root-mean-square velocity in the cluster. We have also made use of the relation $V_{rms} \simeq \sqrt{3}\sigma_0$. Thus, if a binary-binary collision involving main-sequence stars in the core of NGC 5053 produces a merged star of mass $\mathcal{M} \geqslant 0.9\,\mathcal{M}_\odot$ and velocity $V < 5.2$ km s^{-1}, then such a star is *instantly* a blue straggler.

For now we will make the liberal assumption that the core of NGC 5053 consists entirely of binaries. We have chosen to do this because to conclusively reject the hypothesis that the collisions-of-binaries mechanism is the dominant source of blue stragglers in the cluster, we must show that the mechanism does not work even for the most generous of assumptions. Section *3.1* of this paper will consider binary frequencies less than 100%.

There is increasing observational evidence for a population of primordial binaries in globular clusters. Pryor *et al.* (1989) found six binary red giants in six high-density globular clusters, and extrapolated that result (to include undetected binaries with both shorter and longer periods) to arrive at a binary frequency of $\simeq 10\%$ for the main-sequence stars. More recently, Pryor *et al.* (1991) found several binaries among the red giants surveyed in NGC 5053 and NGC 5466, and estimated that the binary frequency in these two low-density clusters is two to three times greater than in the high-density clusters. A binary frequency of 20%–30% is surprisingly high, considering that for years it was believed that globular clusters contained very few binaries. Note that even a low binary frequency for the entire cluster can imply a very high binary frequency in the core because of mass segregation. For example, the two-component model of Spitzer & Mathieu (1980) had an overall binary frequency of 10% and yet had a core binary frequency approaching 100%. However, in a cluster with a more realistic mass function, the segregation of binaries is not expected to be as extreme.

The mean rate of collisions between binaries in the core of a star cluster can be estimated from

$$\Gamma_{bb} \simeq \tfrac{1}{2} N_c n_c \langle \sigma_{bb} V_{rel} \rangle, \tag{3}$$

where N_c and n_c are the total number and mean density of binary stars in the core, respectively, σ_{bb} is the cross section for a collision, and V_{rel} is the asymptotic relative velocity of approach of two binaries. In a King model, $N_c = (2/3)\pi n_0 r_c^3$ and $n_c = n_0/2$ (King 1966). Here we have assumed that the binary frequency in the core is 100%. For σ_{bb} we will simply use the geometrical cross section πb^2_{max}, where b_{max} is the maximum impact parameter. Gravitational focusing will be taken into account during the numerical simulations of the collisions. For V_{rel} we will use $\sqrt{2}\times\sqrt{3}\sigma_0$, since for a Maxwellian velocity distribution the root-mean-square relative velocity $V_{rel,rms}$ is equal to $\sqrt{2}$ times the root-mean-square velocity V_{rms}, and the latter is roughly equal to $\sqrt{3}\sigma_0$. Thus, the mean time between binary-binary collisions is

$$t_{bb} = \Gamma_{bb}^{-1} \simeq \sqrt{6}/\pi^2 n_0^2 r_c^3 b^2_{max} \sigma_0. \tag{4}$$

Assigning the parameters the observed values for NGC 5053 (which are summarized in Table 1) and adopting a maximum impact parameter of 100 AU, we find a mean time between collisions of $t_{bb} = 2.2\times 10^7$ yr. Thus, in order to produce one blue straggler every 2.5×10^8 yr, one such star must be produced in every 11 binary-binary collisions. Altering the maximum impact parameter by a factor λ changes the mean time between collisions by a factor λ^{-2} and the required attempt to success ratio by a factor λ^2. For example, most of the collisions presented in this paper have $b_{max} = 10\,000$ AU, and thus the critical attempt to success

TABLE 1. The adopted parameters for NGC 5053.

Parameter	Adopted Value
range in mass	0.35 to 0.78 $\mathcal{M}_\odot$
n_o	7.9 pc^{-3}
r_c	8.0 pc
x_c	0.5
σ_o	1.5 km s^{-1}
$V_{esc,o}$	5.2 km s^{-1}

ratio is $11\times(10\,000/100)^2 = 110\,000$. However, in order to avoid confusion, we will scale our results to the $b_{\rm max} = 100$ AU case regardless of the actual maximum impact parameter used, so that the critical attempt to success ratio will be 11 in all cases.

The scattering experiments were carried out using Murray Alexander's FOURBL code [see Alexander (1986) for a description], which was modified to allow stars to instantly merge if they touched. For each experiment, the user selects the masses of the four stars involved, the semimajor axes of the binaries, the eccentricities of the binaries, the asymptotic relative velocity of approach of the binaries, and the impact parameter of the collision. The code randomly generates the initial orientations and orbital phases of the binaries from the appropriate distributions.

The mass of the primary of each binary $\mathcal{M}_1$ was generated using Eq. (2) with $x_c = 0.5$, and the binary mass ratios $q = \mathcal{M}_2/\mathcal{M}_1$, where $\mathcal{M}_2$ is the secondary mass, were generated using

$$\frac{dN}{dq} \propto q^{\beta}, \tag{5}$$

where β is a constant. It is not clear what value of β and range in q are appropriate for NGC 5053. The value of β in a star cluster probably increases with time as binaries suffer exchange interactions with each other and with single stars. The minimum value of q may also evolve. The "neutral" choice of $\beta = 0$ has been adopted in the simulations presented in this paper (unless otherwise noted), and a range in q from 0.1 to 1.0 has been used in all simulations.

Figures 1 and 2 are artificial color–magnitude diagrams of NGC 5053 generated using the adopted primary mass function and binary mass ratio distribution, and assuming a binary frequency of 100%. The colors and magnitudes are based on the 16 Gyr isochrone presented in Table V of McClure *et al.* (1987). A distance modulus of 16.0 has been adopted (Fahlman *et al.* 1991). No observational errors have been included in Fig. 1, while Fig. 2 includes Gaussian errors with a magnitude-dependent dispersion given by

$$\sigma_m = 0.02\exp(m - 22), \tag{6}$$

where m is either B or V. (Figure 2 also contains some artifi-

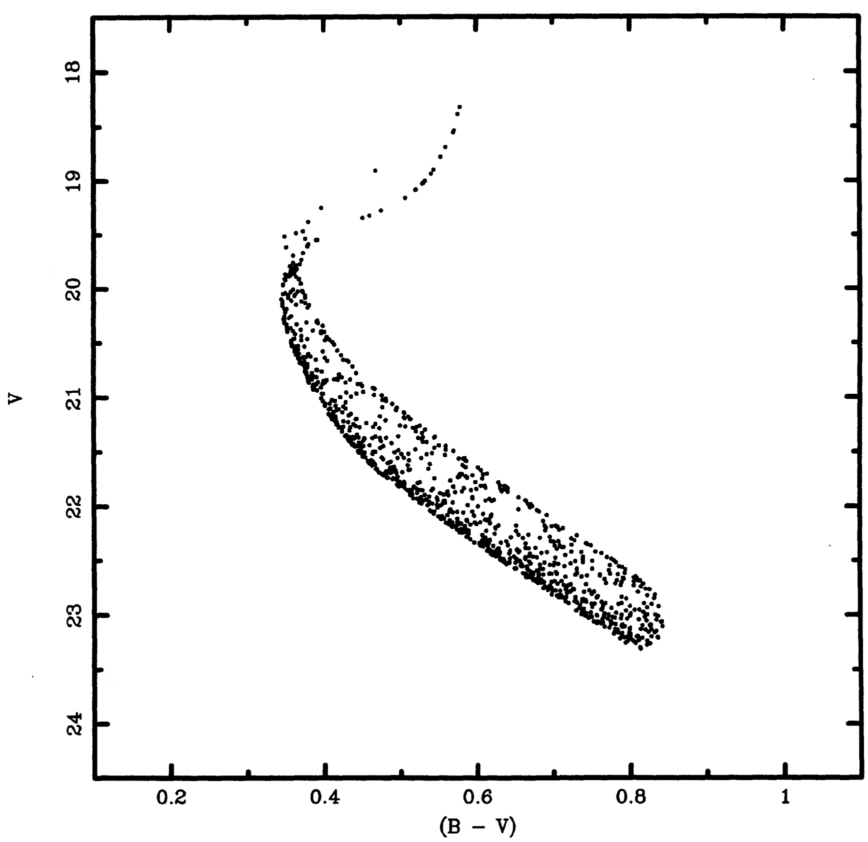

FIG. 1. Artificial color–magnitude diagram for NGC 5053 generated from the mass-function and mass-ratio distribution discussed in Sec. 2. The binary frequency is 100%, and no observational errors have been included.

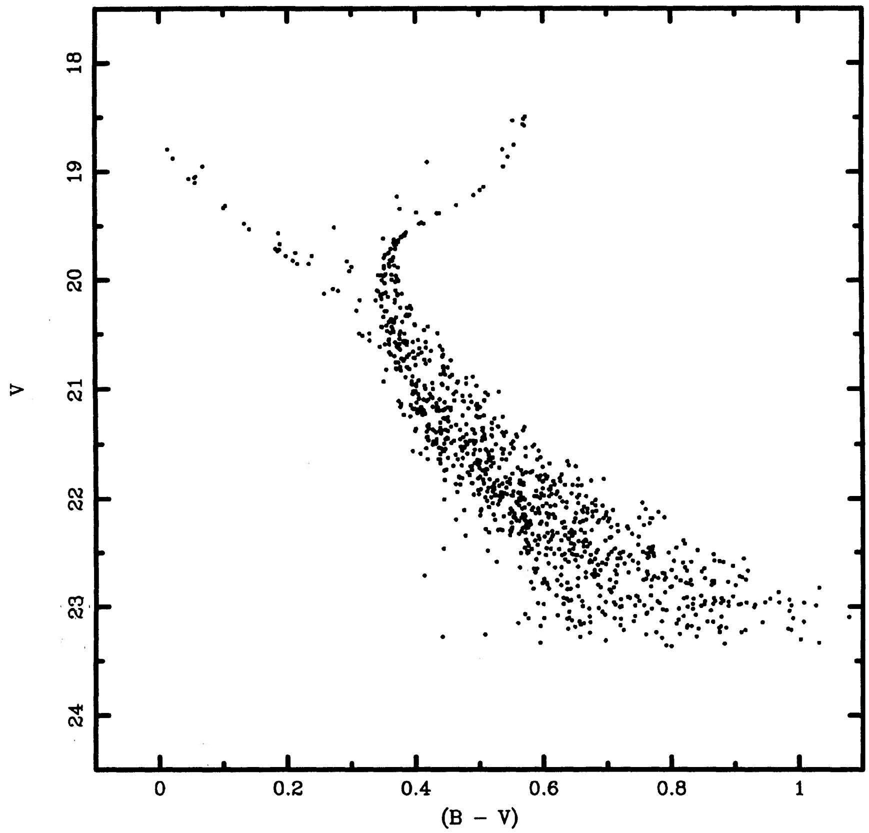

FIG. 2. Same as Fig. 1 but with photometric errors in B and V added to the points. In addition, artificial blue stragglers, generated in the manner described in Sec. *3.2*, have been included.

cial blue stragglers; these will be discussed in Sec. *3.2.*) Although Fig. 1 shows that binary companions widen the main sequence considerably, Fig. 2 reveals that observational error makes this widening inconspicuous, and only a detailed statistical analysis using accurate error estimates would reveal an abundance of binaries. We note that Fig. 2 is qualitatively similar in appearance to the color–magnitude diagram of the core of NGC 5053 to which Eq. (6) empirically applies (Fahlman & Richer 1990).

The binary periods P were generated using

$$\frac{dN}{d\log P} = \text{constant}, \tag{7}$$

which is a good approximation for the period distribution of the binaries in the Galactic disk (Abt 1983). The range in periods considered was from 1 to 10^5 days. The dividing line between hard and soft binaries (as defined by Hut 1983) in the core of NGC 5053 lies in the period range from 10^4 to 10^5 days, and depends on the masses of the stars involved. Thus, cluster binaries with periods exceeding 10^5 days have likely disrupted by now, and need not be considered further.

Two distributions of binary eccentricities were considered. The first was simply $e = 0$ for all binaries, since primordial binaries in the Galactic disk are generally observed to have circular orbits (e.g., examine the catalog of Batten 1967). However, considering the short timescale for interactions involving binary stars in the core of NGC 5053, the eccentricities have likely evolved away from their initial distribution towards a distribution of the form

$$\frac{dN}{de} \propto e. \tag{8}$$

Equation (8) was derived for binaries formed via the three-body mechanism (Heggie 1975), but also applies to binaries which survive binary-single and binary-binary interactions (see Sec. *3.2*). The Heggie distribution was used for most of the experiments reported in this paper.

Note that when generating binary eccentricities from any distribution in which eccentricities approaching unity are possible, it is important to ensure that each binary has a pericenter larger than the sum of the radii of the component stars. Otherwise the binary will merge at the next pericenter. Also, even a pericenter twice the sum of the radii can result in enough dissipation of orbital energy to quickly circularize the orbit. The criterion used in this study for accepting an eccentricity from the Heggie distribution was

$$a(1-e) \geqslant 4(R_1 + R_2), \tag{9}$$

where a is the binary semimajor axis, and R_1 and R_2 are the radii of the primary and secondary. Binaries satisfying Eq. (9) should be able to maintain their eccentricities. Note that it is sometimes impossible to satisfy Eq. (9) with a positive eccentricity for binaries with periods less than a few days. Thus, we have adopted $e = 0$ (i.e., we have assumed circularization) for all binaries with $P < 10$ days.

Initial random velocities were generated for both binaries in each experiment in order to calculate the asymptotic relative velocity of approach and the center-of-mass velocity of the four-body system. Each of three orthogonal velocity components for each binary was randomly generated from a Gaussian distribution with a dispersion of $\sigma_0 = 1.5$ km s^{-1}. We will assume that the velocity dispersion varies little within the core of the cluster, which is a good approximation. The total velocity of each binary was limited to be less than the escape velocity from the cluster center, which also varies by only a small amount within the cluster core. The difference of the two velocities yields the asymptotic relative velocity of approach, and the mass-weighted sum of the velocities divided by the total mass of the four-body system yields the center-of-mass velocity. The latter is added on to the kick that a merged star produced in the interaction receives.

The impact parameter for each experiment was generated from

$$\frac{dN}{db} \propto b. \tag{10}$$

The maximum impact parameter $b_{\max}$ should be chosen to be large enough to include essentially all of the interactions which produce merged stars. For the experiments presented in this paper involving binaries with periods less than 10^3 days, $b_{\max} = 1000$ AU was used. For the interactions involving binaries with periods up to 10^4 and 10^5 days, $b_{\max} = 3000$ and 10 000 AU were used, respectively. Note that 10 000 AU is small compared with the mean separation between stars at the center of NGC 5053, which is $\simeq (3/4\pi n_0)^{1/3} = 64\,000$ AU.

To save quite a bit of computer time, it was decided to only integrate those binary-binary collisions which had a pericenter (of the orbit that the binaries would follow if the binaries were combined into single stars located at their centers of mass) satisfying

$$p < 4 \max(a_1, a_2), \tag{11}$$

where a_1 and a_2 are the semimajor axes of the two binaries involved. It is difficult to imagine that mergers can be anything but rare during interactions with larger pericenters. The pericenter can be found using the gravitationally focused cross section

$$\sigma_{gf} = \pi b^2 = \pi p^2 \left(1 + \frac{2G(\mathcal{M}_1 + \mathcal{M}_2)}{pV_{\rm rel}^2}\right), \tag{12}$$

where b is the impact parameter, G is the gravitational constant, $\mathcal{M}_1$ and $\mathcal{M}_2$ are the binary masses, and $V_{\rm rel}$ is the asymptotic relative velocity of approach. Equation (12) can be derived by assuming the conservation of energy and momentum during a two-body encounter.

The stars were allowed to instantly merge if they touched. The adopted radius-mass relation was

$$R = (\mathcal{M}/\mathcal{M}_\odot) R_\odot. \tag{13}$$

According to the information given in Table V of McClure *et al.* (1987), Eq. (13) is accurate for masses ranging from 0.4 $\mathcal{M}_\odot$ up to the mass where the main sequence begins to turn up. The relation continues to be a good approximation down to 0.1 $\mathcal{M}_\odot$, according to Table 5 of D'Antona (1987). Subgiants have much larger radii than is given by Eq. (13), but the effect is small on the overall merger rate since there are relatively few subgiants. Once two stars merge, the new star is assigned a radius appropriate for its mass according to Eq. (13). It is possible for all four stars in an interaction to successively merge into one massive star, but in the vast majority of cases only two stars merge.

Instant merging may be an unrealistic approximation, but it is the best one can do until hydrodynamical binary-binary collisions can be performed in quantity. It is likely that including hydrodynamical effects in the models will increase the number of mergers because tidal interactions will result in a loss of orbital energy which in turn will result in closer encounters. Also, the stars will not settle down to the size given by Eq. (13) on a timescale which is short compared with the interaction timescale, and thus the merged stars will be larger targets for subsequent merges in hydrodynamical simulations than they were in the current series of experiments.

3. RESULTS AND DISCUSSION

3.1 The Production Rate of Blue Stragglers

The number of binary-binary collisions with $\beta = 0$ required to produce one blue straggler is presented in Tables 2 and 3 for binaries with circular orbits and a Heggie distribution of eccentricities, respectively. We have considered several arbitrary period ranges denoted by the lower and upper period limits $P_{\rm low}$ and $P_{\rm upp}$, respectively. Two numbers, scaled to $b_{\max} = 100$ AU, are given for each range in period considered. The first is the number of collisions required to produce one "instant" blue straggler, which is defined to be a merged star of mass $\mathcal{M} \geqslant 0.9\ \mathcal{M}_\odot$ and velocity $V < 5.2$ km s^{-1}. The second is the number of collisions required to produce one bound merged star of any mass, which includes both instant stragglers and stars which will someday be observable as blue stragglers. The bound merged stars with $\mathcal{M} < 0.9\ \mathcal{M}_\odot$ will be called "latent" stragglers. The ratio of the two numbers is slightly less than a factor of 2 in most of the cases presented in the two tables. The number of experiments carried out for each combination of parameters was large enough to produce at least 25 instant stragglers. Thus, the Poisson uncertainties of the attempt to success ratios presented in the two tables are 20% and 15% or better for instant stragglers and bound merged stars, respectively. This is true for all of the simulations presented in this paper.

A comparison of Tables 2 and 3 reveals that collisions between eccentric binaries result in more mergers than do collisions between binaries with circular orbits. The ratio of rates is slightly less than a factor of 2 for most of the period ranges in the tables. It is not at all surprising that eccentric binaries produce more mergers than do binaries with circular orbits, since the components of very eccentric binaries suffer close encounters even before the collision begins, and thus only a small external perturbation is required to precipitate a collision.

TABLE 2. The number of collisions, scaled to $b_{\max} = 100$ AU, between binaries with circular orbits required to produce one blue straggler/one bound merged star.

log P_{low} \ log P_{upp}	1	2	3	4	5
0	688/489	328/212	194/132	167/89	89/46
1	—	211/137	146/90	97/59	78/46
2	—	—	107/53	59/31	80/44
3	—	—	—	71/38	55/30
4	—	—	—	—	85/40

TABLE 3. The number of collisions, scaled to $b_{max} = 100$ AU, between binaries with eccentric orbits required to produce one blue straggler/one bound merged star.

log P_{low} \ log P_{upp}	1	2	3	4	5
0	688/489	273/168	131/69	65/36	52/31
1	—	135/73	78/42	56/31	50/25
2	—	—	70/36	38/23	26/19
3	—	—	—	37/23	30/17
4	—	—	—	—	33/23

Tables 2 and 3 are quite similar aside from the difference in scaling. Collisions between short-period binaries ($P < 10^2$ days) are not very effective at producing a large number of blue stragglers for two reasons. First, even though mergers are likely to occur during strong interactions involving short-period binaries, the strong interactions themselves are quite rare. This is because $b_{max} = 1000$ AU (the value used for $P_{upp} \leqslant 10^3$ days) usually results in a pericenter between binaries [given by Eq. (12)] which is much larger than the sum of the semimajor axes of two short-period binaries. Second, most of the merged stars produced by collisions involving short-period binaries are ejected from the cluster. The kick that a star receives during a binary-binary interaction is, on average, proportional to the precollision orbital velocity of the binaries, and the latter can be large for short-period binaries. Some of the merged stars receive kicks of several tens of km s^{-1}, and thus they become Population II runaways, analogous to the classical Population I runaways which can be produced by strong interactions involving short-period binaries in young open star clusters (Leonard & Duncan 1988, 1990). Note that merged runaways can be distinguished from unmerged runaways since the former are likely to be rapid rotators (see Sec. *3.2*).

Collisions involving binaries with periods exceeding 10^2 days are the most effective at producing bound merged stars. These long-period binaries work better than do the short-period binaries because they are much larger targets, and thus they are involved in strong interactions much more frequently than are short-period binaries. The increase in the probability of a strong interaction more than makes up for the decrease in the probability that a physical stellar collision will occur during interactions involving these relatively wide systems. Also, most of the merged stars produced during interactions involving long-period binaries remain bound to the cluster. For collisions between binaries with periods approaching 10^5 days (e.g., consider the 10^4–10^5 day range), the probability that an encounter close enough to result in a physical stellar collision has become so small that the number of binary-binary collisions required to produce one blue straggler begins to increase. This is the case in spite of the huge cross section for a strong interaction.

The most favorable period range in Table 3 is from 10^2 to 10^5 days, and thus we will examine this case in more detail. Also, we will henceforth consider collisions between eccentric binaries, since it is likely that each binary has participated in several weak interactions before participating in one strong enough to produce a merger, and these precursor interactions will set up a Heggie distribution of eccentricities. Figure 3 shows the number of collisions required to produce one instant straggler and one bound merged star for the 10^2–10^5 day range, $\beta = 0$, a Heggie distribution of eccentricities, and $b_{max} = 100$, 300, 1000, 3000, and 10 000 AU. The results have been scaled by a factor of $(100/b_{max})^2$ so that the critical attempt to success ratio is 11 in all cases. Figure 3 reveals that at least $b_{max} = 3000$ AU is required to arrive at the result that one out of 27 collisions is likely to produce an instant straggler, and one in 17 a bound merged star.

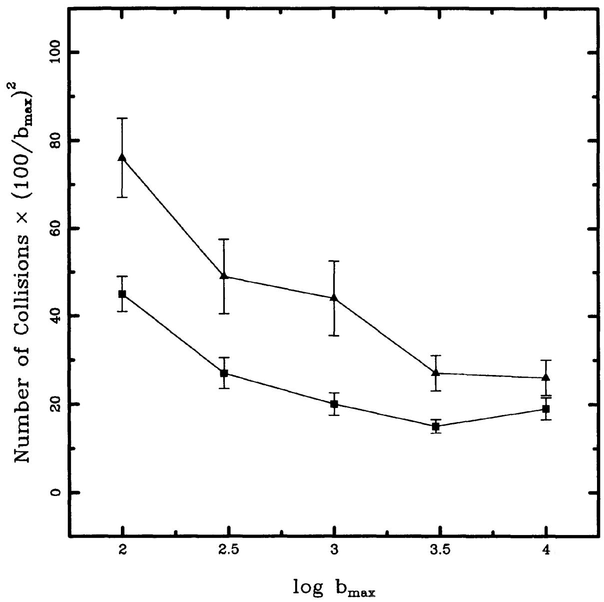

FIG. 3. Triangles and squares represent the number of binary-binary collisions required to produce one instant blue straggler (i.e., a bound merged star of mass $\mathscr{M} \geqslant 0.9 \mathscr{M}_{\odot}$) and one bound merged star of any mass, respectively, as a function of the maximum impact parameter b_{max}. The colliding binaries have a uniform distribution of mass ratios and a range in periods from 10^2 to 10^5 days.

Considering a spectrum of binary mass ratios which is peaked at $q = 1$ results in a slightly better production rate. Figure 4 shows the number of collisions required to produce one instant straggler and one bound merged star for a period range from 10^2 to 10^5 days, a Heggie distribution of eccentricities, $b_{max} = 10\,000$ AU, and $\beta = -1$, 0, 1, and 2. The results have been scaled by a factor of $(100/10\,000)$.^{2} A value of β greater than zero improves the production rate slightly, which is reasonable since the larger β is, the larger the secondaries are, which means a larger cross section for a physical stellar collision. A β value of 2 yields one instant straggler out of every 21 collisions, and one bound merged star out of every 15 collisions. Note that the improvement in the production rate of bound merged stars as β goes from -1 to 2 is not a great as the improvement in the production rate of instant stragglers; the largest effect with the increase of β over this range is a shift to a higher average mass of the merged stars.

Let us now compare the attempt to success ratios for the most favorable case of periods in the range from 10^2 to 10^5 days with the critical attempt to success ratio of 11. For the $\beta = 0$ case, we see from Fig. 3 that one in 27 collisions produces an instant straggler, which is off from the critical attempt to success ratio by a factor of 2.5. However, since the

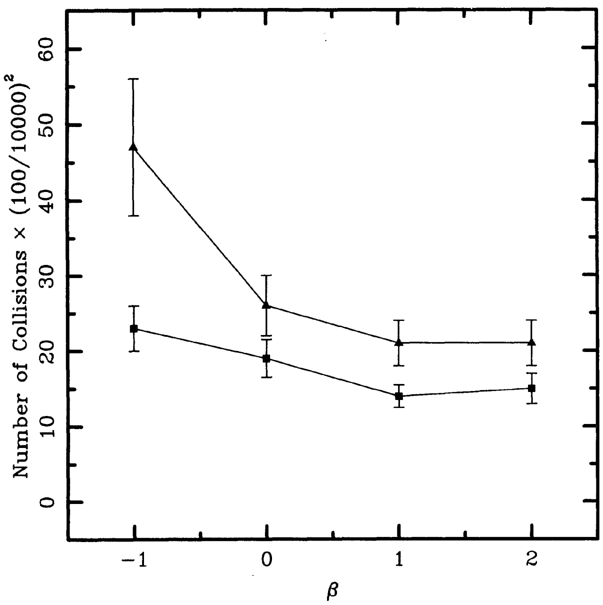

FIG. 4. Number of binary-binary collisions required to produce one instant blue straggler (triangles) and one bound merged star (squares) as a function of β. The latter parameterizes the mass-ratio distribution, and is defined in Eq. (5). The range in periods of the colliding binaries is from 10^2 to 10^5 days.

latent stragglers will eventually become blue stragglers (albeit with reduced lifetimes), then perhaps it would be more accurate to consider the mean of the attempt to success ratios for instant stragglers and bound merged stars. The mean attempt to success ratio is $(27 + 17)/2 = 22$, which is a factor of 2 off from the critical ratio. For $\beta = 2$ the mean ratio is $(21 + 15)/2 = 18$, which is less than a factor of 2 off. We will henceforth adopt an attempt to success ratio of $(22 + 18)/2 = 20$ for the case of binaries with periods in the range from 10^2 to 10^5 days and a Heggie distribution of eccentricities. Thus, if all of the main-sequence stars in the core of NGC 5053 are such binaries, then $11/20 \times 100\% = 55\%$ of the cluster's blue stragglers can be accounted for by mergers following physical stellar collisions.

Although the most favorable case of binaries with periods in the range from 10^2 to 10^5 days cannot be ruled out by the observations, it is probably more realistic to consider a period distribution which includes some short-period binaries, which results in a reduced production rate. From Table 3 we see that the attempt to success ratios for the cases of periods is the range from 1 to 10^5 days, 10 to 10^4 days, and 10 to 10^5 days are all quite similar, and the mean of the attempt to success ratios for instant stragglers and bound merged stars is roughly 40. This is a factor of 2 worse than the case of periods in the range from 10^2 to 10^5 days. Thus, if all of the main-sequence stars in the core of NGC 5053 are binaries with a realistic range in periods, then $11/40 \times 100\% = 28\%$ of the blue stragglers can be accounted for by mergers following physical stellar collisions.

The requirement that one out of 11 binary-binary collisions produces a blue straggler in order to account for 100% of the blue stragglers in NGC 5053 assumes that the fraction of binaries f_b in the core of that cluster is unity, and a smaller value of f_b means that a much smaller (i.e., a much more stringent) attempt to success ratio is necessary in order to meet the same requirement. The total production rate of collisionally merged blue stragglers is given by

$$r_{\rm tot} = r_{\rm bb} f_b^2 + 2 r_{\rm bs} f_b (1 - f_b) + r_{\rm ss} (1 - f_b)^2, \qquad (14)$$

where $r_{\rm bb}$, $r_{\rm bs}$, and $r_{\rm ss}$ are those rates at which binary-binary, binary-single, and single-single collisions produce blue stragglers, respectively. Note that $r_{\rm ss} \ll r_{\rm bb}$, and single-single collisions can be safely ignored if $f_b > 0.1$. For $f_b = 0.5$, the fraction of the blue stragglers which can be accounted for by collisions between binaries with periods in the range from 10^2 to 10^5 days and a Heggie distribution of eccentricities is $(11/20) \times (0.5)^2 = 0.14$ for the most favorable period range, and $(11/40) \times (0.5)^2 = 0.07$ for the realistic period ranges. Binary-single scattering experiments performed using Alexander's code reveal that $r_{\rm bs}/r_{\rm bb} \simeq 1/7$ for binaries with periods in the ranges under consideration and a Heggie distribution of eccentricities. Binary-single collisions are much less effective than are binary-binary collisions at producing close encounters between stars because in the latter case one of the four stars is typically ejected very early in the interaction which allows the remaining three stars to become much more tightly bound than is possible during a binary-single collision. Thus, for $f_b = 0.5$ we find that binary-single collisions can account for additional blue straggler fractions of $2 \times (1/7) \times (11/20) \times (0.5) \times (1.0\text{–}0.5) = 0.04$ and $2 \times (1/7) \times (11/40) \times (0.5) \times (1.0\text{–}0.5) = 0.02$ for the most favorable and realistic period ranges, respectively. Thus, both kinds of collisions combined can account for 18% and 9% of the NGC 5053 blue stragglers, respectively, if $f_b = 0.5$.

The above rates are increased when tidal capture is considered. Thus far we have assumed that two stars merge, with little mass loss, when they touch. This is consistent with the hydrodynamical simulations of Benz & Hills (1987). However, tidal capture binaries are created without the stars actually touching, and these close binaries can eventually produce a blue straggler either by merging into one object due to a gradual loss of orbital energy, or when the more massive star evolves off the main sequence and dumps material onto its companion. Since tidal capture can occur in globular clusters if the centers of two stars pass within a distance of twice the sum of the radii of the two stars [see Fig. 4 of Lee & Ostriker (1986)], and since, in the gravitationally focused regime, the cross section for an encounter with a pericenter of a given size is proportional to the pericenter [see Eq. (12)], then tidal capture can result in roughly a factor of two increase in the production rate of blue stragglers. This conclusion is consistent with Table 1 of Krolik (1983). Thus, a core binary frequency of 50% may be enough for binary-binary collisions to account for 18%–36% of the blue stragglers in NGC 5053.

The collision of binaries mechanism works better in NGC 5466 than it does in NGC 5053 because the former has a much larger central density than does the latter. Using Eq. (1) and the data for the two clusters supplied in Webbink (1985), the ratio of the expected number of blue stragglers in NGC 5466 to the expected number in NGC 5053 is 4, assuming that the binary fractions and binary populations are similar in both clusters. Since only twice as many blue stragglers are observed in NGC 5466 than in NGC 5053 [compare the results of Nemec & Harris (1987) with those of Nemec & Cohen (1989)], then the critical attempt to success ratio is twice as large in NGC 5466 than it is in NGC 5053. Therefore, it is more difficult to reject the collisions-of-binaries hypothesis for the NGC 5466 blue stragglers than it is for the NGC 5053 stragglers.

3.2 The Properties of the Blue Stragglers

Let us now turn to the properties of the merged stars. We will concentrate on the merged stars produced by collisions between binaries with periods in the range from 10^2 to 10^5 days, $\beta = 0$, and a Heggie distribution of eccentricities. The results from the five values of b_{max} presented in Fig. 3 have been combined into one sample in order to improve the statistics.

Figure 5 displays the velocity of each merged star produced versus the star's mass. The velocity corresponds to a time well after the binary-binary interaction is over. For merged stars in multiple star systems, the velocity corresponds to the center of mass of the system. As can be plainly seen from the figure, most of the merged stars are bound to the cluster. The corresponding diagram for merged stars produced by collision between short-period binaries is stretched in velocity such that most of the merged stars are unbound. The corresponding diagram for a larger value of β is shifted to higher masses, with little change in the width of the mass distribution. A smaller value of β yields a shift in the opposite direction.

Figure 6 shows the velocity distribution of the stars shown in Fig. 5. The distribution differs from the velocity distribution of the parent binaries (i.e., the binaries which collided to produce the merged stars) in that it has a smaller mean velocity and an excess of both low and high-velocity stars.

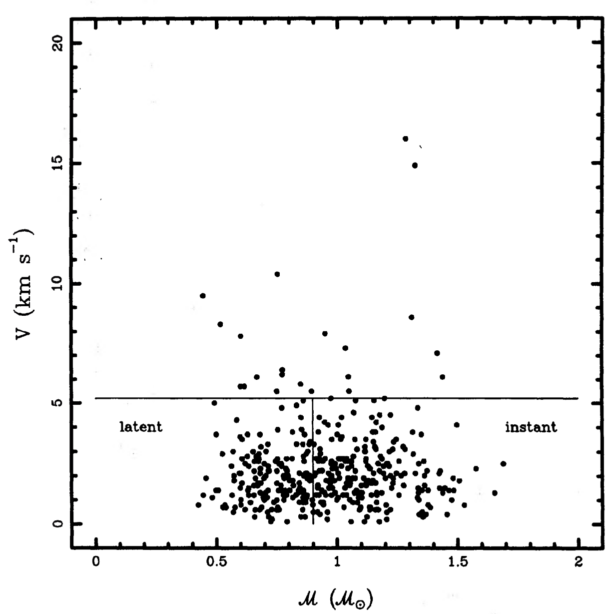

FIG. 5. Plot of mass vs velocity for each merged star produced by collisions between binaries with a uniform distribution of mass ratios and a range in periods from 10^2 to 10^5 days. Only those merged stars with a velocity $V < 5.2$ km s^{-1} (indicated by the horizontal line) are bound to the cluster. Bound merged stars of mass $\mathcal{M} \geqslant 0.9\,\mathcal{M}_\odot$, i.e., to the right of the vertical line, are instant blue stragglers; those to the left are latent blue stragglers.

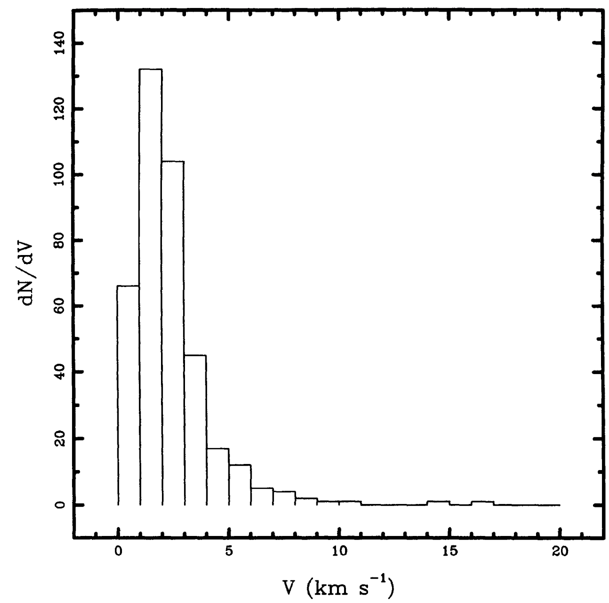

FIG. 6. Velocity distribution of the merged stars shown in Fig. 5.

Thus, for periods in the range from 10^2 to 10^5 days, the decrease in velocity which occurs when a pair of binaries form a temporary four-body system exceeds, on average, the kick given to the merged star as a result of the binary-binary interaction. Thus, the resulting blue stragglers are expected to be more centrally concentrated than the binaries from which they formed. This is consistent with the observation of Nemec & Cohen (1989) that the blue stragglers in NGC 5053 are significantly more centrally concentrated than the red giants. Note that collisions between short-period binaries produce blue stragglers which are not very centrally concentrated, and thus such stragglers can only achieve a high degree of central concentration after suffering two-body interactions with other stars in the cluster.

Since the blue stragglers produced by the collisions-of-binaries mechanism are likely created with a velocity distribution which is different from that of the parent binaries, such blue stragglers are not likely to be (at least initially) in energy equipartition with the other stars in the cluster. Therefore, it would be difficult to estimate the mean mass of the blue stragglers by comparing their radial distribution with that of another class of stars in the cluster since the assumption of energy equipartition may not be valid.

Equation (1) does not take into account the ejection of merged stars from the cluster, and thus the equation must be modified. This can be done by multiplying the right-hand side of Eq. (1) by

$$F = \frac{\int_0^{V_{esc}} N(V)dV}{\int_0^{\infty} N(V)dV}, \tag{15}$$

where V_{esc} is the escape velocity from the core of the cluster and $N(V)$ is the velocity distribution of newly formed merged stars. The latter depends on the precollision velocity distribution of the binaries and on the characteristics of the binary population. The velocity distribution shown in Fig. 6 has $F = 0.94$.

Figure 7 shows the mass function of the bound merged

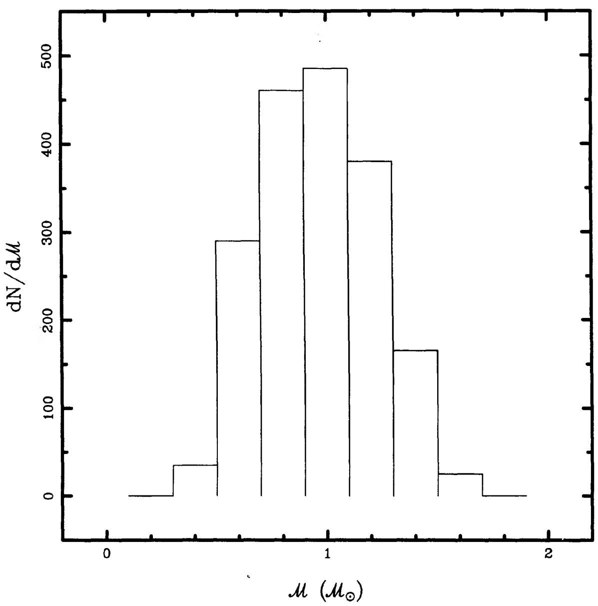

FIG. 7. Mass function of the bound merged stars shown in Fig. 5.

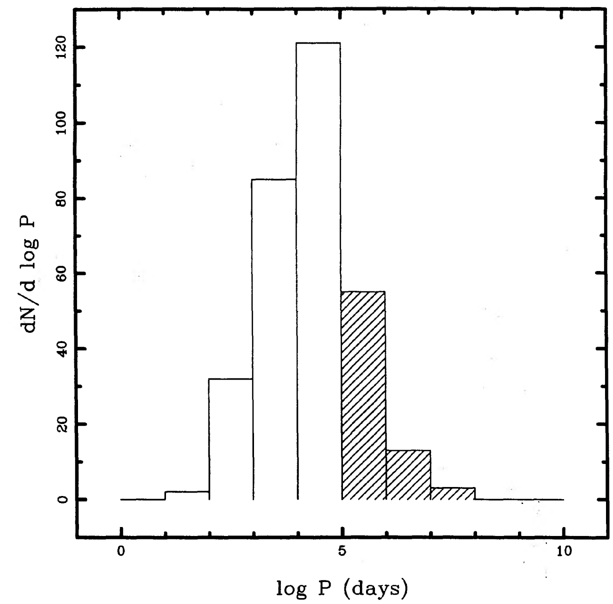

FIG. 8. Period distribution of the bound merged stars shown in Fig. 5 in binary systems. Binaries with periods exceeding 10^5 days (shaded) are soft in NGC 5053.

stars shown in Fig. 5. The distribution can be crudely represented by a Gaussian with a mean mass of $0.95\mathcal{M}_\odot$ and a dispersion of $0.30\mathcal{M}_\odot$, although the core of the mass function is too broad to be perfectly represented by a Gaussian. The spread in mass in Fig. 7 of 0.42–1.69 $\mathcal{M}_\odot$ is similar to the spread in the mass of the parent binaries, which is $1.1\times0.35 = 0.39\mathcal{M}_\odot$ to $2.0\times0.78 = 1.56\mathcal{M}_\odot$. However, the agreement is to a large extent coincidental, and a larger set of experiments would result in a greater range in mass. For example, the total range in mass for all of the merged stars produced in all of the experiments with $\beta = 0$ reported in this paper is 0.22–2.78 $\mathcal{M}_\odot$.

Most of the merged stars reside in binary systems. Of the bound merged stars presented in Fig. 5, 78% and 7% are members of binary and triple systems, respectively. Note that some of the latter systems may eventually break up into a binary and a single star. What typically happens during a binary-binary interaction which produces a merged star is that one star is shot off quickly, and carries away enough energy to allow two of the remaining three stars to merge. The third star typically goes into a loose orbit around the merged pair and becomes the binary companion. Note that these binary blue stragglers are able to participate in another binary-binary collision which may result in additional physical stellar collisions. Also, mass may be transferred onto the secondary when the blue straggler evolves off the main sequence, which may allow the secondary to become a blue straggler.

Figure 8 shows the period distribution of the bound binary systems that contain at least one merged star. Triples are also included in Fig. 8, and for these systems the period used in the figure corresponds to that of the companion closest to the merged star, or to that of the hierarchical orbit of the merged star about two unmerged stars if the latter make up a tightly bound pair. As a general rule of thumb, the binary periods are comparable with or longer than the periods of the parent binaries. Note that the binaries with periods exceeding 10^5 days are soft in NGC 5053, and thus they will quickly disrupt. Excluding such binaries results in binary and triple frequencies of 60% and 5%, respectively.

The distribution of mass ratios of the bound binaries and triples that contain at least one merged star is presented in Fig. 9. The merged star has been assumed to be the primary, and this results in a value of $q \equiv \mathcal{M}_2/\mathcal{M}_1$ greater than unity if the unmerged star is the more massive of the two, or, in the

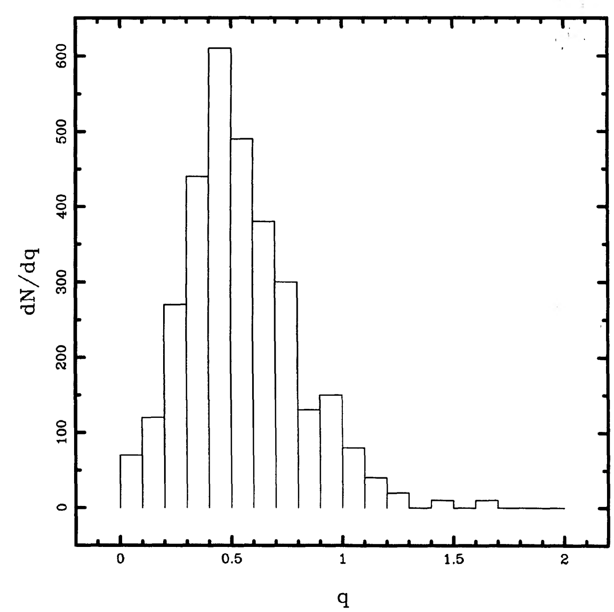

FIG. 9. Distribution of mass ratios of the bound merged stars shown in Fig. 5 in binary systems.

case of triple systems, if the merged star is in a hierarchical orbit around a tight pair of unmerged stars which has a total mass exceeding that of the merged star. The distribution in Fig. 9 differs from the parent distribution [i.e., Eq. (5) with $q = 0.1$–1] in that it is peaked at $q \simeq 0.45$ with most binaries having $q > 0.45$. Note that there is no correlation between the mass of the merged star and its companion. Also, the distributions of mass ratios for hard and soft binaries are quite similar.

Figure 10 displays the distribution of binary eccentricities of the bound binaries and triples that contain at least one merged star. The distribution is similar to a Heggie distribution (the dashed line), except for a possible flattening of the distribution at large eccentricities. The flattening is more noticeable for hard binaries. The collisions involving binaries with circular orbits resulted in a similar distribution, which suggests that a Heggie distribution is a good approximation for the eccentricities of the binaries which emerge from strong dynamical interactions. This is reasonable because two strongly interacting binaries can be considered to be a small star cluster that is capable of producing binaries via the three-body process.

The artificial blue stragglers in Fig. 2 were generated using the mass function in Fig. 7 and the binary mass ratio distribution in Fig. 9. The brightnesses and colors of massive metal-poor main-sequence stars with a helium fraction $Y = 0.25$ were derived using the luminosities and temperatures given in Mengel *et al.* (1979), and the bolometric corrections and colors given in VandenBerg (1985) and Bell & VandenBerg (1987). Fifty artificial merged stars have been added to Fig. 2, and 26 are brighter than the main-sequence turnoff. This is similar to the number observed in NGC 5053 by Nemec & Cohen (1989). The scatter in the brightness and color of the blue stragglers is mostly due to the presence of the binary companions. Note that only one epoch of blue straggler formation is represented in Fig. 2. It would be more realistic to display several epochs of merged star formation using a mass spectrum with a mean mass which decreases with time (this should flatten the mass spectrum), and to include stellar evolution (this will steepen the spectrum).

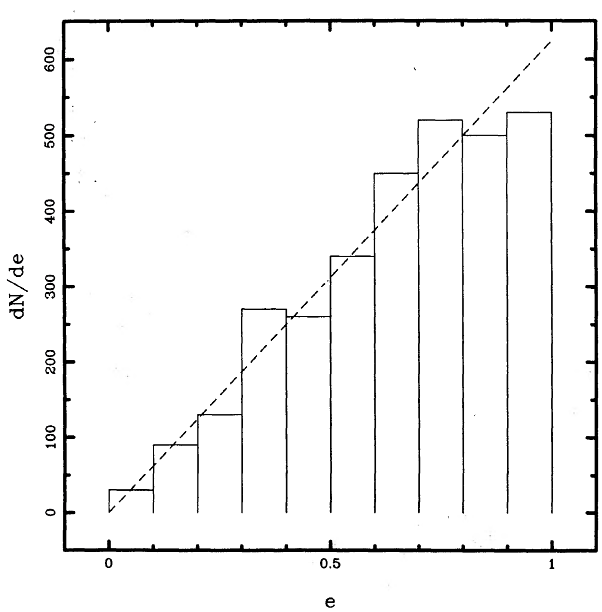

FIG. 10. Distribution of eccentricities of the bound merged stars shown in Fig. 5 in binary systems. The dashed line is the Heggie distribution normalized to the sample size.

The blue stragglers in Fig. 4 of Nemec & Cohen (1989) have a roughly uniform distribution in color with a fairly sharp cutoff at the blue end. This is similar to the distribution of blue stragglers shown in Fig. 2. A more detailed comparison is difficult because the blue stragglers in Fig. 2 represent only one epoch of blue straggler formation, and are unevolved.

The rotational periods of the merged stars can be crudely estimated as follows. Let L be the angular momentum of two stars which have touched and are about to merge. We will assume the merged star rotates as a rigid body, and thus $L = I\omega$, where I is the moment of inertia of the merged star and ω is its angular velocity. The latter can be converted to a rotational period via $P_{\rm rot} = 2\pi/\omega$. According to the data given in Sec. 76 of Allen (1973), $I \simeq 0.06 \mathscr{M} R^2$ for the Sun, which we will adopt for the merged stars.

The resulting rotational periods are quite short, being less than 0.1 day in most cases. This rapid rotation may mean that the two stars cannot merge immediately, but rather there may be an intermediate contact binary stage. A disk may also form. Note that Mateo *et al.* (1990) have discovered some contact or near contact binaries among the blue stragglers in NGC 5466.

The merged blue stragglers may not have rotational periods as short as predicted by the simple calculation for two reasons. First, the assumption of rigid body rotation is probably a poor one, and the interior of the merged star will likely rotate much faster than the surface, which will result in a longer surface rotational period. Second, a small amount of mass loss can carry away quite a bit of angular momentum, which can significantly reduce the rotational rate. Note that the merging of short-period binaries and binary-mass-transfer mechanisms also predict rapidly rotating blue stragglers.

If the merged stragglers are cool enough to have convective envelopes, then magnetic braking will slow their rotational rates considerably. However, the brightest blue stragglers in Fig. 2 are too hot to have convective envelopes, and thus they are expected to be rapid rotators.

Wheeler (1979) and Maeder (1987) have argued that blue stragglers may be stars with extended lifetimes due to rotationally induced mixing. If the hydrogen in the envelope of a main-sequence star can be slowly mixed into the hydrogen-burning core, then the star's lifetime can be increased by as much as an order of magnitude. This follows from the fact that only $\simeq 10\%$ of the hydrogen is converted into helium during the main-sequence evolutionary stage under normal circumstances. In our picture, the extended lifetime induced by rapid rotation would reduce the rate at which binary-binary collisions must produce merged stars in order to account for the majority of the blue stragglers observed in NGC 5053.

4. SUMMARY

A large number of binary-binary and binary-single scattering experiments have been carried out in an attempt to test the hypothesis that the blue stragglers in the low-density globular clusters are the result of physical stellar collisions during strong interactions involving primordial binary stars. The simulations presented in this study differ from those presented in the earlier study by Leonard (1989) in that stars were allowed to instantly merge if they touched. The study has been tailored to NGC 5053, since this cluster represents an extreme test of the collisions-of-binaries hypothesis due to its low density.

It has been found that the collisions which are the best at producing merged stars that would remain bound to NGC 5053 are those between binaries with periods exceeding 10^2 days. Collisions between binaries with a Heggie distribution of eccentricities produce roughly twice as many blue stragglers as do collisions between binaries having the same period distribution but with circular orbits. A spectrum of binary mass ratios peaked at equal masses results in only a slight improvement in the production rate compared with a uniform distribution of mass ratios. Binary-single collisions are much less effective than binary-binary collisions at producing merged stars.

If the binary frequency in the core of NGC 5053 is 100%, then 28%–55% of the blue stragglers observed in the cluster can be accounted for by physical stellar collisions during binary-binary interactions. A core binary frequency of 50% is still enough for interactions involving binary stars to account for 9%–18% of the observed blue stragglers. Including tidal capture binaries (which can eventually produce blue stragglers) formed during interactions involving binary stars roughly doubles the number of blue stragglers that can be accounted for. The collisions-of-binaries mechanism has less difficulty accounting for blue stragglers in NGC 5466 than it does in NGC 5053 due to the much higher central density of the former cluster.

Given that Pryor *et al.* (1991) have discovered a surprisingly high frequency of binaries in NGC 5053 and NGC 5466, it is difficult to reject the possibility that at least some of the blue stragglers observed in these clusters were formed directly or indirectly as a result of strong interactions involving binary stars. However, it is likely that the merging of short-period binaries and the binary-mass-transfer mechanisms also contribute some blue stragglers. Indeed, two mechanisms can even work together in some cases to produce blue stragglers (e.g., tidal capture binaries can be produced via binary-binary collisions, and then one of the other two mechanisms can turn these close binaries into blue stragglers).

The collisions-of-binaries mechanism can produce blue stragglers which are quite centrally concentrated, and which have a distribution in the color–magnitude diagram similar to what is observed. Most of the merged blue stragglers are in binary systems with periods exceeding those of the binaries which collided to produce them. These binary systems have a distribution of mass ratios peaked at $q \simeq 0.45$ with most having $q > 0.45$, and have a Heggie distribution of eccentricities. The blue stragglers produced by the collisions-of-binaries mechanism are (at least initially) rapid rotators, which may extend their lifetimes.

As usual, more observational and theoretical work needs to be done on the problem. From an observational standpoint, better estimates of the frequency of binaries in the low-density clusters are needed, and the physical properties of the blue stragglers should be determined. On the theoretical side, hydrodynamical simulations [similar to those of Cleary & Monaghan (1990) and Goodman & Hernquist (1991)] of binary-binary collisions between long-period binaries would help ascertain the predicted properties of the blue stragglers produced by the collisions-of-binaries mechanism.

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