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Trapped One-Armed Corrugation Waves and QPO's

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Abstract

One-armed corrugation waves (one-armed warps) are very low-frequency modes of oscillations in relativistic, geometrically thin disks. It is shown that some modes of such one-armed corrugation waves can be trapped in the innermost region of accretion disks. The necessary conditions for trapping to occur are that (i) a pressure maximum exists in the inner region of the disks, as in geometrically thick disks, and (ii) the rotation of the central object is slow. In the case of low-mass X-ray binaries (LMXB's), the frequencies of these trapped oscillations can become comparable with those of observed quasi-periodic oscillations (QPO's) for reasonable values of parameters. The trapped oscillations are, however, generally leaked toward the central object, because the trapped region is too close to the inner edge of the disks. This leakage makes the oscillations quasi-periodic, and also will induce time variations in X-rays from the neutron star surface.

Key words: Low mass X-ray binaries; One-armed corrugation waves; QPO's; Trapping of waves.

1. Introduction

Quasi-periodic oscillations (QPO's) are frequently observed in X-rays from low-mass X-ray binaries (LMXB's). Roughly speaking, they are divided into two classes. One class comprises low-frequency QPO's the frequencies of which are around 6 Hz (which occur in a normal branch of the spectral hardness-intensity diagram); the other comprises high-frequency QPO's whose frequencies are 20 Hz–50 Hz (which occur in a horizontal branch) (e.g., Hasinger 1988).

The X-ray spectra from LMXB's consist of two components. One is a hard black-body component of ~2 keV and the other is a softer component of ~1 keV. The black-body component of ~2 keV is usually thought to represent thermal radiation from the gas hitting the neutron star surface, while the soft component of ~1 keV represents radiation from the accretion disk.

The time variations of QPO's are known to occur in the hard black body component of ~2 keV (Mitsuda et al. 1984). The thermal time scale in the neutron star atmosphere is, however, much shorter than the period of QPO's. Hence, it is difficult to consider that the origin of time variability of QPO's is intrinsic on the neutron star surface. This suggests that the origin of QPO's is in the inner part of accretion disks: the accretion rate of the gas hitting the neutron star surface changes quasi-periodically as the result of some instability in the inner part of the accretion disks (Inoue 1988). The problem is the origin of such quasi-periodic variations in the inner part of accretion disks. If we focus our attention on low-frequency QPO's, the required time scale of the variations is 0.16 s, much longer than the dynamical time scale of disks, the latter being about 1 ms in the case of accretion disks around neutron stars.

In relation to this problem we emphasize here that relativistic accretion disks can oscillate at very low frequencies. Such low-frequency oscillations are one-armed corrugation waves (Kato 1989). In some disks such one-armed corrugation waves can be trapped in the innermost region of the accretion disks when the central object is not rotating much. This trapping is, however, generally incomplete, because the trapped region is so close to the inner edge of the disk that the oscillations are leaked toward the central object. Because of this leakage the oscillations are quasi-periodic; further, the gas accretion onto the central object becomes non-axisymmetric. The latter means that the surface region of the neutron stars that is heavily struck by gas rotates very slowly when observed from some outer fixed system. This indicates that the X-ray luminosity from the neutron star surface varies with a long period and with small amplitude, when observed from an outer fixed system. This might be QPO's observed in LMXB's.

Considering the above possible application, we examine in this paper the conditions of trapping and the conditions for the frequencies of trapped oscillations to become of the order of those of observed QPO's.

2. One-Armed Corrugation Waves

Corrugation waves are wavy oscillations of disks (a kind of warp) by which the disk plane deviates from the original equatorial plane in the vertical direction with some wavelength in the radial direction.

We first emphasize that one-armed corrugation waves have a particular position among various disk oscillation modes in relativistic disks, in the sense that they are extremely low-frequency modes of oscillations. The cause of this low frequency is related to the fact that the frequency of vertical oscillations of disks is roughly equal to the angular frequency of disk rotation. How is the closeness of these two frequencies related to the slowness of one-armed corrugation waves? This comes from the following situations (Kato 1989).

Let us tentatively consider a zero-temperature disk rotating with the relativistic Keplerian angular velocity, $\Omega_{\mathrm{K}}(r)$, around a non-rotating central object, where r is the radial distance from the rotation axis of the central object. (We employ cylindrical coordinates (r, ϕ, z), where the origin is at the center of the central object and $z = 0$ is the unperturbed disk plane(the equatorial plane)). Let us consider the displacement of a particle from the equatorial plane in the vertical direction (z-direction). The particle feels a gravitational restoring force toward the equator. Since this force is proportional to the vertical displacement when the displacement is small, the particle harmonically oscillates around the equator with a certain frequency. This frequency is equal to the relativistic Keplerian angular velocity, $\Omega_{\rm k}$, of the disk when the central object is not rotating. That is, the vertical oscillation period of the particle coincides with the rotation period of the particle around the central object.

We now consider a bunch of particles with the same angular momentum. They are assumed to have been at the same position A (not on the equatorial plane) at an initial time and to have started to turn around the disk center with circular orbits at the same time; however, the speed is different in the vertical direction. After one turn around the disk center, they again gather at the initial point A at the same time. This implies that the one-armed pattern is maintained without any time change ($\omega = 0$) in a zero-temperature disk rotating around a non-rotating central object.

Mathematically speaking, this represents the following situation. Let us consider a small-amplitude perturbation of m-arms in the azimuthal direction, i.e., the perturbation is assumed to be proportional to $\exp[i(\omega t - m\phi)]$. Then, the frequency of warps of zero-temperature disks in the vertical direction can be described by

$$(\omega - m\Omega_{\rm k})^2 = \Omega_\perp^2, \tag{2.1}$$

where $\Omega_\perp$ is the frequency of vertical oscillations of particles around the equatorial plane, and is equal to $\Omega_{\rm k}$ in the present problem. This dispersion relation shows that $\omega = 0$ for perturbations of $m = 1$.

In the limit of zero temperature, the oscillations at various places on the equator are independent. If pressure exists, however, they become coherent modes of oscillations with some wavelengths in the radial direction. In other words, the one-armed perturbations of $\omega = 0$ considered above become coherent oscillations when the disk has temperature. The frequency of the oscillations is, however, very low in thin disks for the following reason. If the restoring force of disk rotation contributes to make the motion oscillatory, the frequency of the oscillatory motion is generally of the order of the angular velocity of disk rotation (or epicyclic frequency). In the present problem, however, the restoring force making the motion oscillatory is the gas pressure alone: The rotation has no primary contribution. This implies that the frequency of the one-armed oscillations is very low.

To estimate the frequency of the one-armed corrugation waves somewhat generally, the rotation of the central object is hereafter taken into account. That is, the gravitational potential due to the central object is not always assumed to be spherically symmetric. If no pressure exists, vertical oscillations of disks and disk oscillations in the equatorial plane are independent as

$$(\omega - m\Omega)^2 - \Omega_\perp^2 = 0 \tag{2.2}$$

for vertical oscillations and

$$(\omega - m\Omega)^2 - \kappa^2 = 0 \tag{2.3}$$

for radial oscillations, where $\Omega(r)$ is the angular velocity of disk rotation and κ is the epicyclic frequency, defined by

$$\kappa^2 = 2\Omega\left(2\Omega + r\frac{d\Omega}{dr}\right). \quad (2.4)$$

If pressure exists, however, these two oscillation modes are coupled to give the following dispersion relation (cf. Okazaki et al. 1987 and Kato 1989):

$$[(\omega - m\Omega)^2 - \kappa^2][(\omega - m\Omega)^2 - \Omega_\perp^2] = k^2 c_s^2 (\omega - m\Omega)^2. \quad (2.5)$$

This relation has been obtained by assuming that the disks are isothermal in the vertical direction and that the perturbations are local in the radial direction. Here, c_s is the sound speed and k is the radial wavenumber of perturbations. That is, perturbations are approximated to be proportional to $\exp[i(\omega t - m\phi)]\exp(ikr)$.

Dispersion relation (2.5) is a simple generalization of that obtained by Kato (1989) for the case where the central object is not rotating to the case where it is rotating. That is, $(\omega - m\Omega)^2 - \Omega_k^2$ in equation (3.13) in Kato (1989) is replaced in equation (2.5) by $(\omega - m\Omega)^2 - \Omega_\perp^2$.

It becomes obvious from physical argument and from the procedure of mathematical derivation of the dispersion relation in Kato (1989) and in Okazaki et al. (1987) that the essential part of the effects of rotation of the central object on the dispersion relation is taken into account by this replacement. It should also be noted here that in deriving dispersion relation (2.5) the effects of accretion flow were neglected. Because of this, the dispersion relation can not be applied to a region too close to the sonic point, where accretion flow transits the sound velocity.

To have a rough image on the order of frequencies of one-armed oscillations, let us take $m = 1$ and consider the case where the difference between Ω^2 and $\Omega_\perp^2$ is so small that we have $|2\omega\Omega| > |\Omega^2 - \Omega_\perp^2|$. Then, since $\kappa \sim 0$ near the inner edge of the accretion disks, we have from equation (2.5) (e.g., Kato 1989)

$$|\omega| \sim \frac{k^2 c_s^2}{2\Omega^2}\Omega. \quad (2.6)$$

This shows that the frequency of one-armed oscillations is smaller than the dynamical one ($\sim \Omega$) by a factor of $(kc_s/\Omega)^2$. Here, if we take $k = 2\pi/r_g, c_s/c = 10^{-3}$ and as Ω the Keplerian velocity at $3r_g$ with 1 $M_\odot$, equation (2.6) gives $\omega \sim 14\ s^{-1}$ or 2 Hz, which is comparable with the frequency of low-frequency QPO's.

3. Trapping of One-armed Corrugation Waves

Although we have known a rough order of frequencies of one-armed corrugation waves, an important point to be emphasized here is that some modes of the oscillations can be trapped within the inner region of disks. This is interesting in relation to quasi-periodic oscillations (QPO's) observed in X-ray sources.

The right-hand side of equation (2.5) is always positive and tends to zero when the wavenumber k vanishes. This means that the spatial region where waves of a given frequency can propagate on the disk is restricted to the region specified by

$$[(\omega - m\Omega)^2 - \kappa^2][(\omega - m\Omega)^2 - \Omega_\perp^2] > 0. \quad (3.1)$$

In the inner region of disks we have $\Omega_\perp > \kappa$. Hence, when $m = 1$, inequality (3.1) is realized in the regions of $(\omega - \Omega)^2 - \Omega_\perp^2 > 0$ or of $(\omega - \Omega)^2 - \kappa^2 < 0$. The former inequality is satisfied by $\omega > \Omega + \Omega_\perp$ or $\omega < \Omega - \Omega_\perp$. We consider hereafter the case of $\omega < \Omega - \Omega_\perp$, because this case is interesting from the point of wave trapping.

We first consider geometrically thin disks surrounding nonrotating central objects. Since the difference between Ω and $\Omega_\perp$ is small in such disks (it is noted that $\Omega_\perp = \Omega_{\rm k}$ in the case of non-rotating central object), the qualitative form of the radial distribution of $\Omega - \Omega_\perp$ can change much by the form of the pressure distribution in the radial direction. As will be discussed in the next paragraph, in some disks the radial distribution of $\Omega - \Omega_\perp$ can become like that shown schematically in figure 1. That is, the value of $\Omega - \Omega_\perp$ is positive and has a maximum at a radius close to the inner edge of the disks. In such disks, waves with frequency ω somewhat smaller than the maximum value of $\Omega - \Omega_\perp$ can propagate only in a limited region near the maximum of $\Omega - \Omega_\perp$, as is demonstrated in figure 1. The region outside is an evanescent region of the waves. In other words, some wave modes are trapped in the region around the maximum of $\Omega - \Omega_\perp$. (Detailed conditions for the trapping are discussed in subsequent sections.)

Such a distribution of $\Omega - \Omega_\perp$ as is shown in figure 1 can really occur. We continue to assume that the disks are geometrically thin and that the central objects have no rotation. We additionally assume here that the specific angular momentum $l(r)$ of gas consisting of the innermost region of the disks is somewhat larger than the marginal angular momentum $l_{\rm ms}$ (the angular momentum of the particles rotating just along the marginally stable circular orbit). In such disks a steady accretion state is realized by gas being pushed to the sonic point ($r = r_{\rm s}$) (the point beyond which the infalling speed of gas becomes supersonic) by pressure rather than falling as the result of losing angular momentum by viscosity. That is, the pressure maximum will occur at a radius (say, $r_{\rm m}$) outside $r_{\rm s}(r_{\rm s} < r_{\rm m})$. We assume here that $r_{\rm m}$ is not so close to r that the inertial force is negligible in the radial force balance in the region around $r_{\rm m}$. Then, in the region $r > r_{\rm m}$ we have $\Omega < \Omega_{\rm k}$, because the pressure force is outwards; thus, $\Omega - \Omega_\perp > 0$. On the other hand, in the region of $r < r_{\rm m}$ we have $\Omega - \Omega_\perp > 0$; approaching the sonic point, however, it decreases to zero, since around the sonic point gas can fall spontaneously without being pushed from the outside. In this sense we expect a maximum of $\Omega - \Omega_\perp$ at a radius between the sonic point ($r = r_{\rm s}$) and the pressure maximum point ($r = r_{\rm m}$). This kind of distribution of $\Omega - \Omega_\perp$ (actually $\Omega - \Omega_{\rm k}$) is the same as that known in thick accretion disks (Abramowicz et al. 1978), and actually occurs in conventional α-type accretion disks when α is small and the accretion rate is high (e.g., Matsumoto et al. 1984; Abramowicz et al. 1988)

Discussions in the above paragraph are made only for the case where the central object is non-rotating. If it rotates rapidly, $\Omega_\perp$ deviates much from $\Omega_{\rm k}$, and $\Omega - \Omega_\perp$ can not have a maximum in region $r > r_{\rm s}$, when $l(r)$ is close to $l_{\rm ms}$, as is discussed in subsequent sections. Because of this, wave trapping can not be expected when the rotation of the central object is high.

The next problem to be examined here is to quantitatively estimate the frequency of trapped oscillations. Similar problems have been examined in problems of non-radial oscillation of stars (e.g., Unno et al. 1989). That is, some non-radial oscillation modes are trapped in particular regions within stars. The frequencies of such trapped oscil-

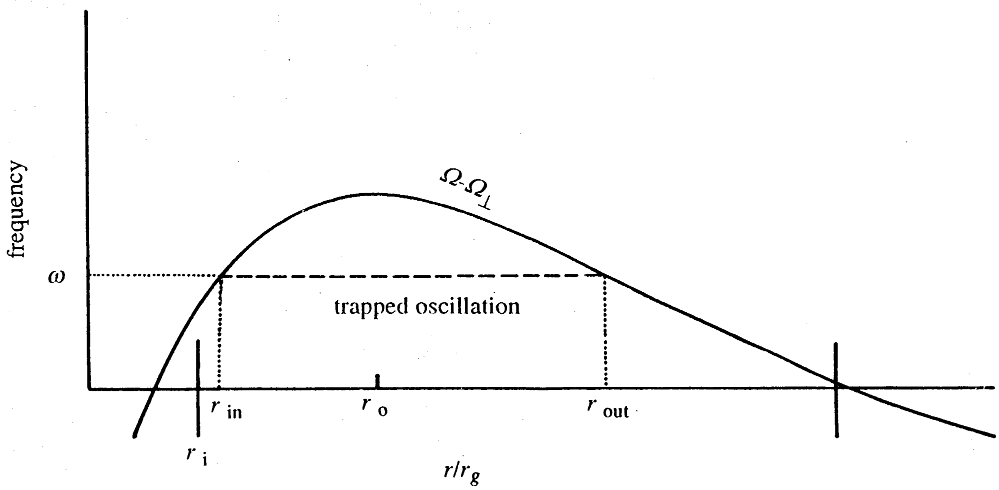


Fig. 1. Schematic diagram showing the radial distribution of $\Omega - \Omega_\perp$ in the case where trapping of waves occurs. Two vertical lines crossing the abscissa show the boundary radii of the region of $\Omega - \Omega_{\rm k} \geq 0$; the inner boundary being denoted by $r_{\rm i}$. Roughly speaking, $r_{\rm i}$ can be regarded as the inner edge of the disk. If the central object has rotation, $r_{\rm i}$ can become larger than the radius, $r_{\rm o}$, of the maximum of $\Omega - \Omega_\perp \geq 0$. In such cases there is no trapped oscillation mode. In the special case of non-rotating central object,. $r_{\rm i}$ coincides with the inner boundary of $\Omega - \Omega_\perp \geq 0$.

lation modes can be estimated with good accuracy by the WKBJ method (Shibahashi 1979; Unno et al. 1989). Adopting the same procedure, we can write the eigen-value condition in our present problem as (cf., Okazaki et al. 1987)

$$\int_{r_{\rm in}}^{r_{\rm out}} k(r)dr = \left(n + \frac{1}{2}\right)\pi, \quad (n = 0, 1, 2, 3, \ldots) \tag{3.2}$$

where $r_{\rm in}$ and $r_{\rm out}$ are the inner and the outer radius where

$\omega = \Omega - \Omega_\perp$ is realized (see figure 1); $k(r)$ is the wavenumber determined by dispersion relation (2.5).

The actual procedures to determine the eigen-frequency are as follow: Equation (3.2) is written explicitely in the form

$$\int_{r_{\rm in}}^{r_{\rm out}} \left(1 - \frac{\kappa^2}{\Omega^2}\right)^{1/2} [(\omega - \Omega)^2 - \Omega_\perp^2]^{1/2} \frac{dr}{c_{\rm s}} = \left(n + \frac{1}{2}\right)\pi. \tag{3.3}$$

The left-hand side is calculated for a given ω. In general, the calculated value is not equal to the right-hand side for arbitrary ω's. Thus, the value of ω is adjusted so that both sides coincide.

4. Various Oscillation Frequencies in Simplified Disks

To perform the integration in equation (3.3), we need to know the radial distributions of $\kappa(r), \Omega_\perp(r)$ and $\Omega(r)$ in the disks. Though the quantity $\Omega_\perp(r)$ is obtained if the metrics are specified, to know $\Omega(r)$ and $\kappa(r)$ we must specify the disk structures.

We first summarize the expressions for $\Omega_{\mathrm{k}}(r), \kappa_{\mathrm{k}}(r)$ (hereafter the subscribt k is attached to κ in order to emphasize that it is the epicyclic frequency in Keplerian disks) and $\Omega_{\perp}(r)$. We adopt the Kerr metric expressed in terms of Boyer-Lindquist coordinates, which is given in equation (A1) in the Appendix. In this case the Keplerian angular velocity Ω_{k} observed at infinity,

$$\Omega_{\mathrm{k}} = \frac{d\phi}{dt} = \frac{d\phi/ds}{dt/ds} = \frac{U^{\phi}}{U^{t}} \tag{4.1}$$

(see the Appendix for notations), is given as (e.g., Abramowicz et al. 1978; see also the Appendix)

$$\Omega_{\mathrm{k}}(r) = \pm \frac{(GM)^{1/2}}{r^{3/2} \pm a(r_{\mathrm{g}}/2)^{3/2}}, \tag{4.2}$$

where r_{g} is the Schwarzschild radius defined by $r_{\mathrm{g}} = 2GM/c^2$. Here, M is the mass of the central object, and a $(0 \le a \le 1)$ is a dimensionless parameter specifying the amount of angular momentum of the central object [$a = 0$ is the case of non-rotating central object (the case of the Schwarzschild metric), and $a = 1$ is the case where the central object has the maximum rotation (the case of the extreme Kerr)]. Both here and hereafter the upper and the lower signs are for cases where particle revolutions are prograde and retrograde, respectively.

The radial distribution of the epicyclic frequency, $\kappa_{\mathrm{k}}(r)$, in the case of the Kerr metric has been obtained by Okazaki et al. (1987). The result is

$$\kappa_{\mathrm{k}}^2(r) = \frac{GM}{r^3} \frac{1 - 6(r_{\mathrm{g}}/2r) - 3a^2(r_{\mathrm{g}}/2r)^2 \pm 8a(r_{\mathrm{g}}/2r)^{3/2}}{[1 \pm a(r_{\mathrm{g}}/2r)^{3/2}]^2}. \tag{4.3}$$

In the case of the Schwarzschild metric, κ_{k} vanishes at $r = 3r_{\mathrm{g}}$, as is well-known. In the case of the extreme Kerr ($a = 1$), equation (4.3) gives $\kappa_{\mathrm{k}} = 0$ at $r = r_{\mathrm{g}}/2$, as is expected.

To derive an expression for $\Omega_{\perp}(r)$, some calculations are necessary. The procedure is given in the Appendix. Rearranging the expression given by equation (A11), we have

$$\Omega_{\perp}^2(r) = \Omega_{\mathrm{k}}^2 \left[1 \mp 4\left(\frac{r_{\mathrm{g}}}{2r}\right)^{3/2} a + 3\left(\frac{r_{\mathrm{g}}}{2r}\right)^2 a^2\right]. \tag{4.4}$$

The next problem is to determine the angular velocity of disk rotation, $\Omega(r)$; To obtain it we must specify the disk structure. Here, as the simplest case, we adopt disks with constant specific angular momentum (i.e., $l \equiv -c^2 U_{\phi}/U_{\mathrm{t}} = \mathrm{const.}$). In general, the definitions of l and Ω give the relation

$$\Omega(r) \equiv U^{\phi}/U^{t} = -\frac{g_{\mathrm{t}\phi} + l g_{\mathrm{tt}}}{g_{\phi\phi} + l g_{\phi\mathrm{t}}}. \tag{4.5}$$

In the case of the Kerr metric, it can be written explicitly as

$$\Omega(r) = \frac{4c}{r_{\mathrm{g}}} \frac{2(l/cr_{\mathrm{g}})(r/r_{\mathrm{g}} - 1) + a}{(2r/r_{\mathrm{g}})^3 + 2(1 + r/r_{\mathrm{g}})a^2 - 4al/cr_{\mathrm{g}}}. \tag{4.6}$$

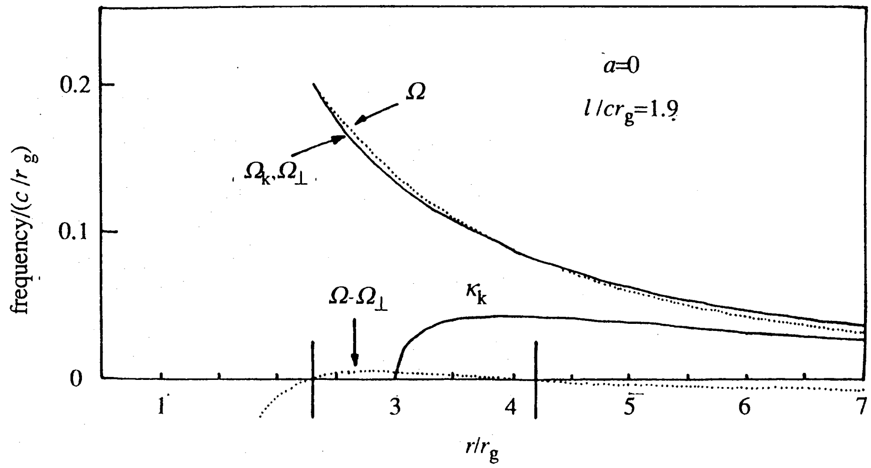


Fig. 2. Radial distributions of Ω, $\Omega_\perp(=\Omega_{\rm k})$, $\kappa_{\rm k}$, and $\Omega-\Omega_\perp$ in the case of Schwarzschild metric ($a=0$). The angular velocity of disk rotation, Ω, is obtained by assuming that specific angular momentum of gas, l, is constant with $l/cr_{\rm g}=1.9$. The marginal specific angular momentum $l_{\rm ms}$ in the case $a=0$ is $(3\sqrt{6}/4)cr_{\rm g}=1.837\ldots cr_{\rm g}$. The inner and outer boundaries of the region of $\Omega-\Omega_{\rm k}>0$ are shown by vertical lines crossing the abscissa.

Hence, the assumption which we adopt here is to take $l=$ const. in equation (4.6).

5. Frequencies of Trapped Oscillations

Let us assume for simplicity that the disk has a constant temperature in the region where the wave trapping occurs. Then, the dimensionless parameters specifying the disk structure are $l/cr_{\rm g}$ and $c_{\rm s}/c$. Another parameter is a, specifying the degree of rotation of the central object.

5.1. The Case of Non-Rotating Central Object

We first consider the case $a=0$. As shown below, this is the case where trapping most easily occurs.

For trapping to occur, the presence of a region where $\Omega\geq\Omega_\perp$ is necessary, as discussed in section 3. The region is nothing but the region of $\Omega\geq\Omega_{\rm k}(r)$ or $l\geq l_{\rm k}(r)$ in the present case. The Keplerian angular momentum $l_{\rm k}(r)$ has a minimum $l_{\rm ms}$ at $r=r_{\rm ms}$($r_{\rm ms}$ being the radius of the marginally stable circular orbit and $3r_{\rm g}$ in the present case of $a=0$), and increases in the inward and outward directions of r. Hence, if we consider a gas with $l>l_{\rm ms}$, the region of $\Omega>\Omega_\perp$ appears around $r=r_{\rm ms}$ (see figure 2 of Abramowicz et al. 1978). To have a rough image concerning the distribution and magnitude of $\Omega-\Omega_\perp$, the radial distributions of Ω, $\Omega_\perp(=\Omega_{\rm k})$, $\kappa_{\rm k}$ and $\Omega-\Omega_\perp$ are shown in figure 2 for $l/cr_{\rm g}=1.9$. This value of l is somewhat larger than $l_{\rm ms}$, which is $0.75\sqrt{6}cr_{\rm g}$ $(=1.837117\ldots cr_{\rm g})$ in the case of the Schwarzschild metric.

On the other hand, the inner edge of the disk exists near radius $r_{\rm i}$, where $\Omega=\Omega_{\rm k}$ is satisfied. [Radius $r_{\rm i}$ is the cusp of the equipotential surface, if the effects of accretion flow are neglected (Abramowicz et al. 1978).] The inner and the outer boundaries of the region of $\Omega>\Omega_\perp$ are shown in figure 2 by two vertical lines crossing the abscissa. One important point to be noted here is that the inner edge, $r_{\rm i}$, of the disk is the same

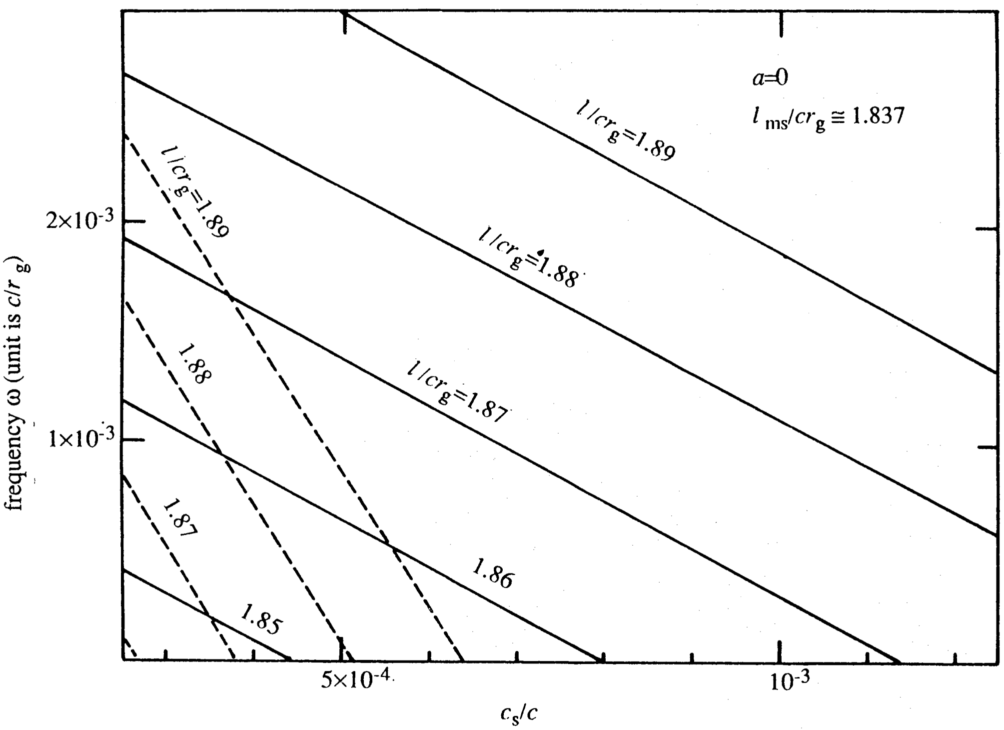


Fig. 3. Frequency-temperature relation for trapped oscillations for various values of l when $a = 0$. Solid curves are for the fundamental modes and broken curves are for the first overtone.

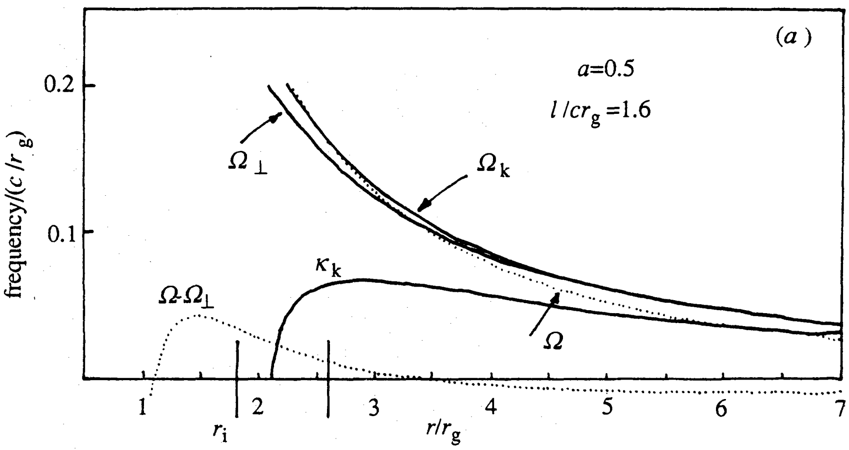
as the inner boundary of region $\Omega \geq \Omega_{\perp}$ in the present case of a non-rotating central object. Hence, region $\Omega - \Omega_{\perp} \geq 0$ is certainly in the disk.

Since the maximum of $\Omega - \Omega_{\perp}$ occurs at a radius larger than that of the inner edge, trapping occurs for some discrete modes of oscillations whose frequencies ω's are in the range $0 < \omega < (\Omega - \Omega_{\rm k})_{\rm max}$. To determine what oscillations are actually trapped, equation (3.3) should be solved. The frequencies of the resulting trapped oscillations are shown in figure 3 as a function of disk temperature for some values of l slightly larger than $l_{\rm ms}$. The solid lines are for the fundamental mode and the broken lines are for the first overtone.

5.2. Effects of Rotation of the Central Object

The next problem to be examined is the effects of the rotation of the central object on wave trapping. Equation (4.4) shows that the frequency of vertical oscillation of particles around the equatorial plane, $\Omega_{\perp}(r)$, is generally smaller than the Keplerian frequency $\Omega_{\rm k}(r)$, except for the case $a = 0$. This trend becomes stronger as r decreases or a increases. We shall consider how this affects wave trapping.

Let us first consider the case of a non-zero a with l close to $l_{\rm ms}$. Since l is close to $l_{\rm ms}$, the pressure is not so strong to change $\Omega(r)$ much from $\Omega_{\rm k}(r)$ that $\Omega - \Omega_{\perp}$ (which is $\sim \Omega_{\rm k} - \Omega_{\perp}$) increases with decreasing r by the effect of a. In other words, $\Omega - \Omega_{\perp}$ has no maximum in the region of $r > r_{\rm i}$. Such an example is shown in figure 4a with $l = 1.6cr_{\rm g}$ and $a = 0.5$. (In this case of $a = 0.5$, $l_{\rm ms}/cr_{\rm g} = 1.58\ldots$) Around $r_{\rm i}$ there is a sonic point and in the region of $r \leq r_{\rm i}$ the gas falls toward the central object



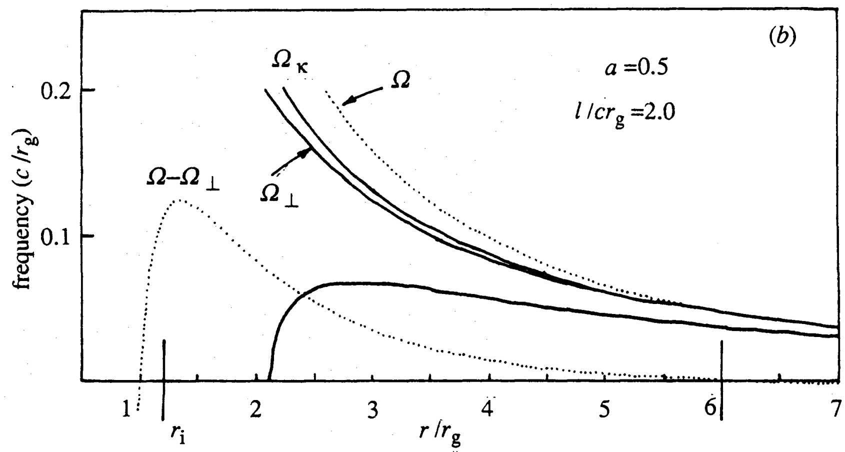


Fig. 4. The same with figure 2, except for the values of a and l. This figure is for the Kerr metric with $a = 0.5$. Figure 4a is for $l/cr_{\rm g} = 1.6$ and figure 4b for $l/cr_{\rm g} = 2.0$. The marginal angular momentum $l_{\rm ms}$ in the case of $a = 0.5$ is $1.58\ldots cr_{\rm g}$. In the case of figure 4a there exists no trapped oscillation, while in the case of figure 4b the trapping will exist.

with supersonic speed. Hence, if there is no maximum of $\Omega > \Omega_\perp$ in region $r > r_{\rm i}$, there is no trapped oscillation.

On the other hand, if we consider a disk with large l, the pressure force is not negligible in comparison with the centrifugal force. In this case, $\Omega - \Omega_\perp$ can have a maximum in the region $r > r_{\rm i}$, as is shown in figure 4b. Such a case of large l, however, means that the disks are rather thick and are outside of our present problem. Furthermore, the value of $(\Omega - \Omega_{\rm k})_{\rm max}$ is so large that the frequency of trapped oscillations becomes too high to explain observations.

To show quantitatively the situations mentioned above, the trapped frequency-disk temperature relation is shown in figure 5 for various values of l, with $a = 0.2$.

6. Discussions

We have shown the possibility that some discrete oscillation modes of one-armed corrugation waves are trapped in the innermost region of accretion disks. The presence

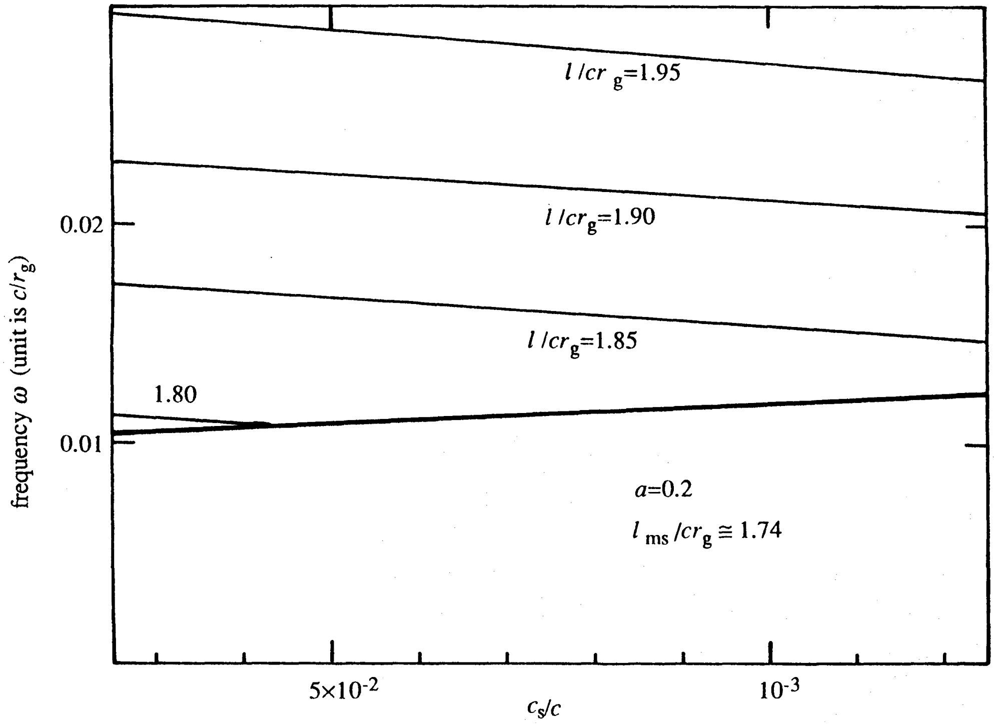


Fig. 5. The same with figure 3 except for $a = 0.2$. In this case the value of $l_{\rm ms}$ is $1.74 \ldots cr_{\rm g}$. Below the thick line there is no fundamental mode trapped.

of these trapped modes are of interest in relation to the QPO's observed in LMXB's for the following reason. As shown in previous sections, the trapped region is just outside the inner edge of disks. Hence, in actual situations the eigen-functions of trapped oscillations penetrate until the inner edge of disks. This means that the trapped oscillations are leaked toward the central object.

This leakage makes the oscillations quasi-periodic. Furthermore, this makes the accretion onto the surface of the central object non-axisymmetric. This implies that the surface region struck by much gas rotates slowly, when the phenomena are seen from an outer fixed system. Though the greatly struck region radiates stronger than do the other regions, this bright region flows away by the rotation of the central object. However, since the thermal time scale on the star surface is very short, the strongly struck region remains as the brightest region. Since the region rotates slowly, this induces the QPO's phenomena.

The necessary conditions for such trapped modes to exist are that (i) the disks must have specific angular momentum l larger than $l_{\rm ms}$ in the innermost region of disks (in other words there must be the pressure maximum in the innermost region) and (ii) the rotation of the central object must be slow.

We first discuss the first point. In accretion disks the gas falls forward the central object with supersonic velocity after passing through a sonic point, $r_{\rm s}$ (critical point). Disk models with such transonic flow have been examined in detail in cases of conventional α-disks (e.g., Muchotrzeb 1983; Matsumoto et al. 1984; Abramowicz et al. 1988). These studies show that the disk structures in the innermost region can be classified into two types. The first type is the case of large α. In this case the angular momentum distribution $l(r)$ in the innermost region is below the Keplerian one $l_{\rm k}(r)$, i.e., $l(r) < l_{\rm k}(r)$. This is because the gas falls toward the sonic point by losing angular momentum by viscosity, not by being pushed by pressure. The other type occurs when the value of α is small. In this case we have $l(r) > l_{\rm k}(r)$ (i.e., $l(r) > l_{\rm ms}$) in the innermost region, since the gas falls toward the sonic point, being pushed by pressure. The critical α separating these two types of disks exists around $\alpha \sim 0.05$ (Matsumoto et al. 1984; see also Kato et al. 1988). Results in the text show that for QPO's to occur the disks must be of the latter type.

The presence of a maximum of $\Omega - \Omega_{\perp}$ is only a necessary condition for trapping. That is, if the maximum is too small, there is either no ω which satisfies the trapped condition (3.3), or the value of ω of trapped oscillations becomes too small to explain the QPO's . Figure 3 shows that for trapped oscillations of 6 Hz (which correspond to $\omega = 2.7 \times 10^{-4}(c/r_{\rm g})$ if the mass of the central object is 1.4 $M_{\odot}$) to occur in disks with $c_{\rm s}/c = 10^{-3}$, the specific angular momentum of the gas, l, must be $l/cr_{\rm g} \cong 1.86$ at $r = r_{\rm ms}$. This value of l has been obtained by assuming that l is constant. In actual disks with viscosity, however, the value of l increases outwards. Hence, the difference between Ω and $\Omega_{\perp}$ remains positive until a larger radius in actual disks. Since the eigen-frequency, ω, is given by equation (3.3), this implies that in actual disks with a viscosity the observed frequencies of QPO's can be explained by an $l(r_{\rm ms})$ smaller than 1.86 $cr_{\rm g}$. This effect of spatial variation of l should be studied in detail in realistic disks in the future, since the quantitative results presented here will be modified. Qualitatively, however, there will be no essential change. In any case, if QPO's really occur due to the trapped oscillations discussed here, a comparison between the calculated frequencies of trapped oscillations and those observed provides a good tool to determine the disk structure in the innermost region, and will contribute much to the understanding of disk structures in real systems.

We next comment on the second point: For QPO's of the observed order of frequencies to occur, the rotation of the central object must be slow. This is consistent with the conventional image that LMXB's are aged system.

A point which should be emphasized here is that QPO's are not always observed in LMXB's: Though they are observed in some objects they are not in some others. Even in the same object they are observed during some epoch but not during some other epoch. This is not surprising from the point of view of our model, since trapping is not complete, as mentioned before. That is, the oscillations disappear when excitation processes are ceased. Second, the presence of QPO's depends sensitively on the disk structure. In relation to the last point, the question exists why low-frequency QPO's have almost the same frequency of 6 Hz, since this requires that the innermost regions of disks have a universal structure.

We have so far discussed the parameter dependences of ω by taking l and $c_{\rm s}/c$ as independent parameters. The force balance in the radial direction is $-\partial p/\rho\partial r + (\Omega^2 - \Omega_{\rm k}^2)r = 0$. Furthermore, we treat the innermost region of disks. Hence, to change l under a constant $c_{\rm s}/c$ is to consider a model sequence of disks in which the scale length (in units of $r_{\rm g}$) of the pressure variation in the radial direction changes during the sequence. This model sequence may not be suitable to evaluate parameter dependences in real systems. We therefore discuss here the parameter dependences of ω from a somewhat different point of view. That is, the radial scale of pressure change in the innermost region is assumed to be constant in units of r_g, independent of disks. Then, from the force balance in the radial direction, we have

$$\Omega - \Omega_k \propto \frac{c_s^2}{\Omega_k r_g^2}. \tag{6.1}$$

In a rough sense the frequency of trapped oscillations in disks with non-rotating central objects is of the order of $\Omega - \Omega_k$, except when ω becomes too small. Hence, in a rough sense, equation (6.1) implies that

$$\omega \propto M^{-1}\left(\frac{c_s}{c}\right)^2, \tag{6.2}$$

since $r_g \propto M$, M being the mass of the central object. This result is equivalent with what will be obtained from equation (2.6) for non-trapped one-armed corrugation waves.

We shall apply here equation (6.2) to black hole candidates. In our present paper we have assumed that a non-axisymmetric collision of gas to the neutron star surface is the direct cause of X-ray variations. In some black hole candidates, however, QPO's are observed (e.g., Ebisawa et al. 1988). This may suggest the presence of some mechanism which changes the flow energy before the flow hits the surface of the central object. Bypassing this problem, however, we shall estimate the frequency of trapped oscillations expected in black hole systems. In the case of black hole systems, the central mass is larger than that of neutron stars. Furthermore, in conventional α-models, the disk temperature decreases with an increase in the mass of the central object. Hence, if equation (6.2) is applied to black hole candidates, it is not difficult to obtain the frequencies of trapped oscillations as low as 0.08 Hz, which are observed in a black hole candidate (LMC X-1) (Ebisawa et al. 1988). The higher harmonics observed in LMC X-1 may represent non-linearity of oscillations.

Appendix. Frequency of Vertical Oscillations of Disks

We start from the Kerr metric expressed in terms of Boyer-Lindquist coordinates,

$$ds^2 = \frac{\Delta}{\rho^2}[cdt - b\sin^2\theta d\phi]^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+b^2)d\phi - bcdt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2. \tag{A1}$$

Here, functions Δ and ρ are defined by

$$\Delta = r^2 - rr_g + b^2 \tag{A2}$$

and

$$\rho^2 = r^2 + b^2\cos\theta, \tag{A3}$$

where b is a parameter representing the angular momentum per unit mass of the Kerr black hole and can change in the range $0 \le b \le r_g/2$.

The Euler equation of a free particle is given by

$$u^\mu_{;\alpha}u^\alpha = 0, \tag{A4}$$

where u^μ is the particle's 4-velocity dx^μ/ds.

We first consider a circular motion in the equatorial plane ($\theta = \pi/2$), i.e., $u^\mu = (U^t, 0, U^\phi, 0)$. It then follows from equation (A4) that the angular velocity, $\Omega_{\rm k}$, of a circular motion observed at infinity is

$$\Omega_{\rm k} = \frac{d\phi}{dt} = \frac{U^\phi}{U^t} = \pm\frac{(GM)^{1/2}}{r^{3/2} \pm b(r_{\rm g}/2)^{1/2}}. \tag{A5}$$

Here and hereafter, the upper sign refers to the prograde orbit, while the lower sign to retrograde orbit. The redshift factor U^t is also found from equation (A4) to be

$$cU^t = \frac{r^{3/2} \pm b(r_{\rm g}/2)^{1/2}}{r^{3/4}(r^{3/2} - 3r^{1/2}r_{\rm g}/2 \pm 2b(r_{\rm g}/2)^{1/2})^{1/2}}. \tag{A6}$$

We next consider a motion slightly perturbed from a circular orbit. The coordinate velocity is written as

$$\frac{dx^\mu}{dt} = (1, v^r, v^\theta, \Omega_{\rm k} + v^\phi), \tag{A7}$$

where v^r, v^θ and v^ϕ are the velocity components associated with the infinitesimal perturbations. To derive linearized equations for v^r, v^θ, and v^ϕ, it is convenient to rewrite the Euler equation (A4) in the form

$$\left[\left(\frac{dx^\mu}{dt}\right)_{;\nu} + \frac{d\ln u^t}{dx^\nu}\frac{dx^\mu}{dt}\right]\frac{dx^\nu}{dt} = 0. \tag{A8}$$

Here, u^t is

$$u^t = U^t\left\{1 + U^t\left[\left(r^2 + b^2 + \frac{r_{\rm g}}{r}b^2\right)\Omega - \frac{r_{\rm g}}{r}cb\right]v^\phi\right\}, \tag{A9}$$

which is obtained from a linearized form of $u_\mu u^\mu = 1$.

Substituting equations (A7) and (A9) into equation (A8), we obtain linearized equations for v^r, v^θ, and v^ϕ. Among them, the equations for v^r and v^ϕ have already been derived by Okazaki et al. (1987) in order to calculate the epicyclic frequency $\kappa_{\rm k}$. Hence, we restrict our attention here to the equation for v^θ. After lengthy, but straightfoward, calculations, we have

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)v^\theta = -\Omega_\perp^2\delta\theta, \tag{A10}$$

where

$$\Omega_\perp^2(r) = \Omega_{\rm k}^2\left(1 + 3\frac{b^2}{r^2} + \frac{2r_{\rm g}b^2}{r^3}\right) \mp \frac{2cr_{\rm g}b}{r^3}\Omega \tag{A11}$$

and $\delta\theta$ is the displacement of the particle in the θ-direction. Equation (A10) shows that the frequency of small amplitude oscillations around the equatorial plane is $\Omega_\perp$.

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