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ASTRONOMY AND ASTROPHYSICS

Synchrotron radiation treated by the Weizsäcker-Williams method of virtual quanta

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Summary. The method of virtual quanta is employed in a treatment of synchrotron radiation. It is shown that classical radiation is adequately described as Thomson scattering of magnetic field virtual photons by the electron. However, when $\gamma B \simeq 10^{14}$ Gauss, where γ is the electron Lorentz factor, Compton scattering becomes important. The result is a reduction in the synchrotron loss rate which agrees closely with quantum electrodynamics. Classical theory fails to produce a cutoff in the emission spectrum at the electron energy, because it ignores the uncertainty principle, and assumes that the electron motion is well-defined at very short times. As a corollary, it is also suggested that the influence of virtual quanta on protons and nuclei sets upper limits on their energy in a given magnetic field.

Key words: radiation mechanism – synchrotron radiation

1. Introduction

It has recently been shown (Lieu et al., 1987) that the influence of radiation reaction on the orbit of a synchrotron electron leads to a significant classical correction to the characteristics of the emission when μ, the fractional energy radiated during the crossing period of a synchrotron pulse, approaches unity (or, equivalently, when $\gamma B \simeq 10^{16}$ Gauss, where γ is the electron Lorentz factor). Among the modifications is an enhancement in the synchrotron loss rate, although severe difficulties with "pre-acceleration" and "Schott Energy" render the validity of the entire classical treatment questionable. Furthermore, before the regime $\mu \simeq 1$ is reached, the synchrotron spectrum has a peak at $\hbar\omega > \gamma m_e c^2$, where $\gamma m_e c^2$ is the energy of the emitting electron (in fact, this occurs at $\mu \sim 0.01$). The conclusion is that, in the limit $\mu \simeq 0.01$ ($\gamma B \simeq 10^{14}$ Gauss), synchrotron radiation can only be described with the help of quantum mechanics (see Ginzburg, 1979; and references cited).

In the case $\gamma \gg 1$ and $\gamma B \simeq 10^{14}$ Gauss Berestetskii et al. (1980) have pointed out that it is not the quantum mechanics of electron motion (i.e. Landau levels) but the second quantisation of the radiation field which defines the nature of the problem. Sokolov and Ternov (1968) used perturbation theory to obtain the necessary corrections to the radiation characteristics and found that the synchrotron spectrum always exhibits a cutoff at the electron total energy. A detailed analysis by Ritus (1972) justified the validity of the perturbation approach in this limit. More recently, Harding and Preece (1987) published calculations in the limit $B \sim 4.4\ 10^{13}$ Gauss, where both the motion and the radiation field are quantized.

The present work provides a simple, although approximate theory of synchrotron radiation based on the Fermi-Weizsäcker-Williams method of virtual quanta (Fermi, 1924; Weizsäcker, 1934; Williams, 1935). As long as numerical precision is not strictly required, the method can reproduce standard properties of classical radiation and leads to a reduction in the energy loss rate due to quantum recoil which agrees closely with that of Sokolov and Ternov (1968). Moreover, it explains why classical electrodynamics fails to predict a cutoff in the synchrotron spectrum and thus violates energy conservation.

We have been motivated to study synchrotron radiation from this point of view for the following reasons:

(i) In cyclotron radiation by non-relativistic electrons the angular dependence of the differential emission rate is $(1+\cos^2\theta)$, which is exactly the pattern produced by the Thomson scattering of incident radiation propagating in the $\pm z$-directions (assuming that $\boldsymbol{B} = B\hat{\boldsymbol{k}}$).

(ii) The total synchrotron loss rate is identical to that of an electron undergoing inverse Compton scattering in the Thomson limit, the energy density of the ambient field being $B^2/8\pi$. Indeed, the total loss can be obtained as the product of the Thomson scattering cross section $(8\pi/3)(e^2/mc^2)$ and the apparent electromagnetic energy flux seen in the frame of the rotating electron, viz. $c\gamma^2 B^2/4\pi$ (e.g. Baylis et al., 1967).

2. The method of virtual quanta

As in the case of electron bremsstrahlung, laboratory synchrotron radiation can be approximately described by the Thomson scattering of virtual photons in the rest frame of the electron. Consider an electron moving with speed $v \simeq c$ in a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{k}}$ having a trajectory given by

$$x = R\sin\omega_0 t, \quad y = R\cos\omega_0 t.$$

With reference to a coordinate system in which the particle appears instantaneously at rest at time $t=0$, its trajectory is found by Lorentz transformation to be

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$$x' = \gamma (R \sin \omega_0 t - v t) \tag{1a}$$

$$y' = R \cos \omega_0 t \tag{1b}$$

$$t' = \gamma \left(t - \frac{v}{c^2} R \sin \omega_0 t \right). \tag{1c}$$

Since it is impossible to express t as an analytical function of t', the time dependence of the particle position (x', y') in the rest frame can only be written implicitly in terms of t. The velocity of the particle in this frame is given by

$$v'_x = \frac{dx'}{dt'} = \frac{dt}{dt'} \frac{dx'}{dt} = -\frac{v(1 - \cos \omega_0 t)}{1 - \frac{v^2}{c^2} \cos \omega_0 t},$$

$$v'_y = \frac{dy'}{dt'} = \frac{dt}{dt'} \frac{dy'}{dt} = -\frac{v \sin \omega_0 t}{\gamma \left(1 - \frac{v^2}{c^2} \cos \omega_0 t\right)}. \tag{2}$$

In the dipole approximation we assume that the electron motion and subsequent radiation is caused by an incoming electromagnetic wave with net polarization vector along the $x' y'$ plane. In general, the virtual photons can propagate in two directions. As long as the electron remains non-relativistic in its instantaneous rest frame, these should be $\pm z'$. However, such an assumption is approximate, because (2) indicates that the electron accelerates rapidly from $v' = 0$ to $v' \simeq c$. At present only the $+z'$ direction is considered, as results on the emission properties remain unaffected, provided that they are averaged over all outgoing solid angles.

The spectral flux (energy per unit area per unit frequency) of this virtual incident field is given by the expression

$$\left[\frac{d^2 E'}{d\omega' \, dA'}\right]_{\text{incident}} = \frac{\omega'^2 m^2 c}{4\pi^2 e^2} \left| \int_{-T}^{T} v'(t') \, e^{i\omega' t'} \, dt' \right|^2 \tag{3}$$

where $(-T, T)$ is the time of interaction, which is the duration of the synchrotron pulse as measured in the rest frame. Since, in such a frame, the motion of the electron on either side of the pulse is assumed to be non-relativistic, effects of retarded time are ignored. The errors introduced by the method are small even for ultra-relativistic laboratory motion, and tend to zero for non-relativistic laboratory motion (i.e., cyclotron radiation).

Consider an incident electromagnetic wave with a spectral flux density given by (3), which Thomson scatters an electron initially at rest. By means of the Thomson cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}_0|^2 \tag{4}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_0$ are respectively the polarization of the incoming and outgoing wave, the differential emission rate (i.e. energy radiated per unit solid angle per unit frequency) for outgoing radiation of polarization $\boldsymbol{\varepsilon}'$ is easily seen to be

$$\frac{d^2 I'}{d\omega' d\Omega'} = \frac{e^2 \omega'^2}{4\pi^2 c^3} \left| \int_{-T}^{T} \boldsymbol{\varepsilon}' \cdot \boldsymbol{v}'(t') \, e^{i\omega' t'} \, dt' \right|^2.$$

In the t' integration we now make a change of variable $t' \to t$ via (1c). As a result of this transformation, the limits of integration are $-1/\gamma\omega_0 \leqq t \leqq 1/\gamma\omega_0$, i.e., the synchrotron pulse duration in laboratory frame. Substituting (2) for the velocity and ignoring terms of order $< 1/\gamma^2$ we obtain

$$\frac{d^2 I'}{d\omega' d\Omega'} = \frac{e^2 \omega'^2}{4\pi^2 c} [|\varepsilon'_x|^2 |I'_1|^2 + |\varepsilon'_y|^2 |I'_2|^2] \tag{5}$$

where

$$I'_1 = \frac{1}{\gamma} \int_{-\infty}^{\infty} e^{i\gamma\omega'\left(\frac{1}{\gamma^2} + \frac{\omega_0^2 t^3}{6}\right)} dt$$

$$I'_2 = -\int_{-\infty}^{\infty} \omega_0 t \, e^{i\gamma\omega'\left(\frac{1}{\gamma^2} + \frac{\omega_0^2 t^3}{6}\right)} dt \tag{5}$$

and the reason for replacing $(-1/\gamma\omega_0, 1/\gamma\omega_0)$ by $(-\infty, \infty)$ is explained in Jackson (1975).

Since the synchrotron beam is directed along the $+x'$-axis in the laboratory frame, it is appropriate to assign the direction $\boldsymbol{n}'$ and polarization $\boldsymbol{\varepsilon}'$ of the outgoing radiation as follows:

$$\boldsymbol{n}' = (\cos\theta', 0, \sin\theta'), \; \boldsymbol{\varepsilon}'_1 = (\sin\theta', 0, -\cos\theta'), \; \boldsymbol{\varepsilon}'_2 = (0, 1, 0),$$

where θ' is the angle between the radiation propagation vector and the electron velocity (x' axis, see Fig. 1), and 1, 2 denote polarization states.

Then the differential emission rate in the electron rest frame has the form

$$\frac{d^2 I'}{d\omega' \, d\Omega'} = \frac{e^2 \omega'^2}{4\pi^2 c} [|I'_2|^2 + \sin^2\theta' \, |I'_1|^2]. \tag{6}$$

Transforming now to the laboratory frame by means of the Doppler shift formulae valid for relativistic velocities in the laboratory frame

$$\cos\theta' = \frac{1 - \gamma^2\theta^2}{1 + \gamma^2\theta^2}; \quad \omega' = \frac{1 + \gamma^2\theta^2}{2\gamma} \omega \tag{7}$$

to obtain

$$\frac{d^2 I}{d\omega \, d\Omega} = \frac{4e^2}{3\pi^2 c} \left(\frac{\omega}{\omega_0}\right)^2 \frac{1}{\gamma^4} [K^2_{2/3}(\zeta) + 2\psi^2 (1 + \psi^2)^{-2} K^2_{1/3}(\zeta)], \tag{8}$$

where $\psi = \gamma\theta$, $\zeta = \sqrt{2}\omega(1 + \psi^2)/3\gamma^3\omega_0$ and the Airy integrals $K(x)$ are defined in the Appendix. This resembles closely the exact differential emission rate of classical synchrotron radiation (Jackson, 1975). The angular distribution of total intensity, $dI/d\Omega$, cuts-off at $\theta \sim 1/\gamma$ in both cases. A comparison of $dI/d\omega$, the spectra summed over all directions, is given in Fig. 2 (the relative

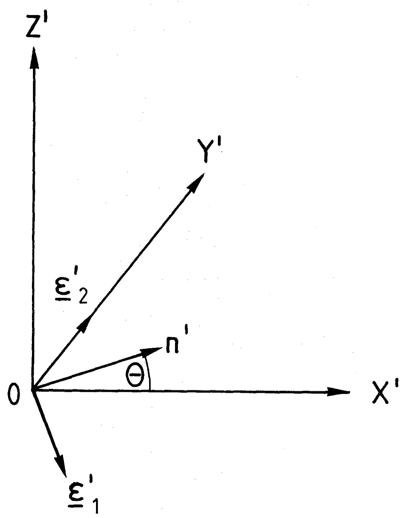

Fig. 1. Direction $\boldsymbol{n}'$ and polarization $\boldsymbol{\varepsilon}'$ of outgoing (scattered) electromagnetic waves in the electron instantaneous rest frame. The magnetic field and the propagation direction of incoming (virtual) quanta are both along $+z'$

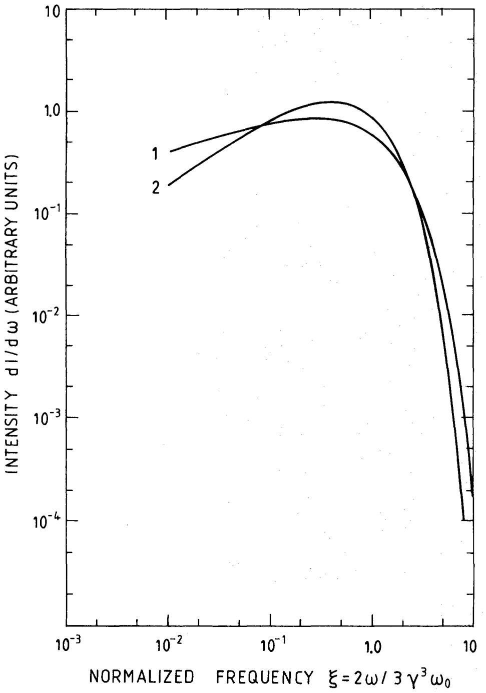

Fig. 2. Angle-integrated spectra of conventional synchrotron radiation (curve 1) and radiation due to Thomson scattering of virtual quanta in the electron rest frame (curve 2)

shape of the two spectra is independent of γ and ω_0). The total emission rate (energy per unit time) as obtained from (8) is

$$\frac{dE}{dt} = 0.84 \frac{e^2 \omega_c^2 \gamma^2}{c} \tag{9}$$

whereas the exact formula

$$\frac{dE}{dt} = 0.667 \frac{e^2 \omega_c^2 \gamma^2}{c} \tag{10}$$

is only smaller by 26% in this asymptotic relativistic limit. The conclusion is that synchrotron radiation can reasonably be considered as Thomson scattering of virtual photons in the rest frame of the electron. Discrepancies from the exact theory are surmised as being due to the neglect of relativistic effects in the electron rest frame.

3. Compton scattering of virtual quanta: the quantum correction of synchrotron radiation

Within the frame work of the virtual quanta method it is a straightforward matter to introduce the necessary quantum corrections on synchrotron radiation. As noted earlier, if $\gamma B \geqq 10^{14}$ Gauss, peak photon energies in the emission spectrum approach the electron energy. In our instantaneous rest frame this means the virtual quanta which influence the electron have energies $\sim m c^2$. Thomson scattering is therefore modified by the Compton effect.

The spectral flux of incident virtual photons is still given by (3) [1], but only up to frequency components $\omega' < m c^2/\hbar$. It is no longer possible to Fourier analyze a classical motion beyond this frequency limit because the characteristics of the motion at frequencies $> m c^2/\hbar$ are established by measurements of the electron at times $\Delta t \lesssim \hbar/m c^2$, measurements which inherently create an uncertainty in the electron energy $\Delta E \geqq m c^2$. For an electron which accelerates from rest at $t = 0$ and remains non-relativistic at small times, such an energy uncertainty is sufficient to render the orbit completely indeterminate. In fact, very similar arguments have been used in the problem of bremsstrahlung to secure a lower limit on the impact parameter of the electron (see e.g. Heitler, 1965).

For Compton scattering each outgoing photon of frequency ω' and direction θ' corresponds to an incoming photon of frequency ω'/η, where

$$\eta = 1 - \frac{\hbar \omega'}{m c^2} (1 - \sin \theta') \tag{11}$$

is the Compton redshift. In addition the Thomson cross-section (4) is modified by the Klein-Nishina formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{m c^2}\right)^2 \eta^2 \left[|\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}_0|^2 + \frac{1}{4} (\eta + \eta' - 2) \right].$$

The differential emission rate (6) becomes

$$\frac{d^2 I'}{d\omega' \, d\Omega'} = \frac{e^2 \omega'^2}{4\pi^2 c} \eta^2 \left[|\varepsilon_x'|^2 + \frac{1}{4} (\eta + \eta^{-1} - 2) \right] \left| I_1' \left(\frac{\omega'}{\eta}\right) \right|^2 + \left[|\varepsilon_y'|^2 + \frac{1}{4} (\eta + \eta^{-1} - 2) \right] \left| I_2' \left(\frac{\omega'}{\eta}\right) \right|^2, \tag{12}$$

where the functions I_1', I_2' are given by (5).

The results are now transformed to the laboratory frame. Applying (7), the redshift coefficient (11) becomes

$$\eta = 1 - \frac{\hbar \omega}{2\gamma m c^2} (1 - \psi)^2,$$

where $\psi = \gamma \theta$. The rest of (12) is dealt with in exactly the same way as outlined in the derivation of Eqs. (5)–(8). The final expression is

$$\frac{d^2 I}{d\omega \, d\Omega} = \frac{4 e^2}{3\pi^2 c} \eta^2 \left(\frac{\omega}{\omega_0}\right)^2 \frac{1}{\gamma^4} \left\{ \left[1 + \frac{1}{4} (\eta + \eta^{-1} - 2) \right] K_{2/3}^2 (\xi) \right\} + \left\{ \frac{1}{2} \left[4\psi^2 (1 + \psi^2)^{-2} + \frac{1}{4} (\eta + \eta^{-1} - 2) \right] K_{1/3}^2 (\xi) \right\}, \tag{13}$$

where

$$\xi = \frac{\sqrt{2}\omega}{3\gamma^3 \omega_0 \eta} (1 + \psi^2).$$

4. Interpretation of results and comparison with QED theory

Integration of (11) over frequencies and solid angles (see Appendix) yields the total synchrotron emission rate:

$$\frac{dE}{dt} = 0.84 \frac{e^2 \omega_c^2 \gamma^2}{c} f_1 (\varepsilon); \tag{14}$$

where

$$f_1 (\varepsilon) \begin{Bmatrix} = 1, \varepsilon = 0 \\ < 1, \varepsilon > 0 \end{Bmatrix}$$

and $\varepsilon = 3\gamma^2 \hbar \omega_0 / 2 \sqrt{2} m c^2 \simeq 2.4 \; 10^{-14} \gamma B$ (Gauss). $f_1 (\varepsilon)$ measures the suppression of synchrotron loss rate due to quantum effects.

[1] The reason is, when $\gamma B \sim 10^{14}$ Gauss, the deviation of electron orbit from circularity is not large enough to affect its radiation; for such effects to be significant, we need $\mu \simeq 1$, or $\gamma B \simeq 10^{16}$ Gauss. In this limit the radiation field can no longer be treated as perturbation (Lieu, 1987; Ritus, 1972).

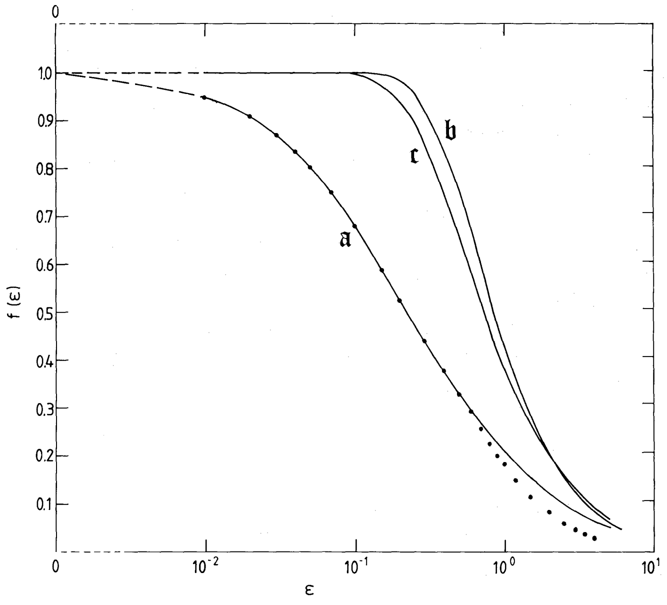

Fig. 3. Plots of various estimates of f, the factor which measures suppression of total synchrotron loss rate due to quantum effets (for more precise definition, see text). The dots (f_1) are obtained by the method of virtual quanta, curve a (f_2) is the exact QED result (Sokolov and Ternov, 1968). Curves b and c (f_3 and f_4) are crude estimates of f obtained by integrating the classical differential emissivity of this work and of the exact synchrotron theory, over all angles and all frequencies up to the absolute cutoff at $\hbar\omega = \gamma m c^2$

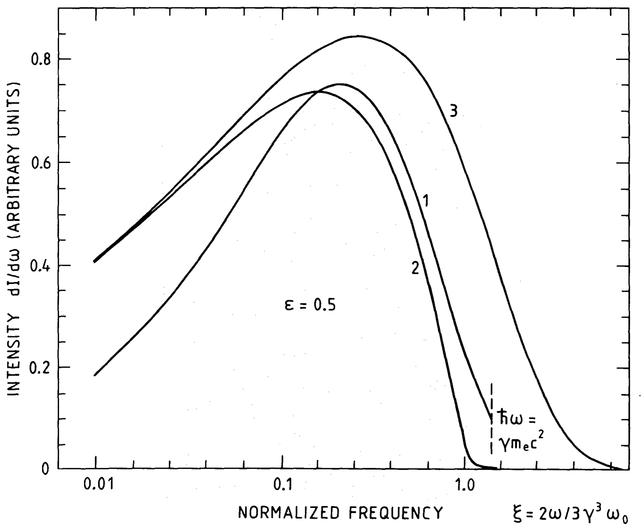

Fig. 4. Spectra of synchrotron radiation in the quantum limit $\varepsilon = 0.5$, as given by: curve 1, the method of virtual quanta; curve 2, exact QED calculations; curve 3, classical theory

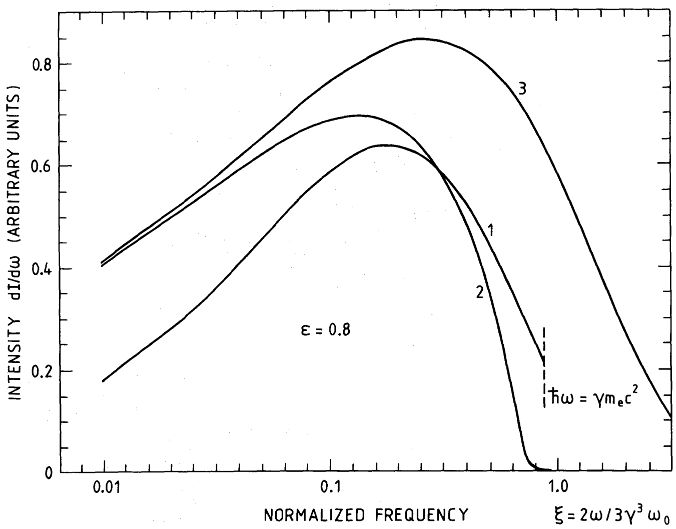

Fig. 5. Same as Fig. 4, except $\varepsilon = 0.8$

QED perturbation calculations (Berestetskii et al., 1980) give

$$\frac{dE}{dt} = 0.667 \, \frac{e^2 \omega_c^2 \gamma^2}{c} f_2(\varepsilon), \tag{15}$$

where $f_2(\varepsilon)$ has also the property of $f_1(\varepsilon)$ in (14b). Note that in the limit $\varepsilon \simeq 0$, (14), (15) become (9), (10). Figure 3 shows how $f_1(\varepsilon)$ and $f_2(\varepsilon)$ decrease with ε. In view of the simplicity of the virtual quanta method when compared with perturbation calculations, the resemblance between the results is striking. Also plotted in Fig. 3 are similar reduction factors $f_3(\varepsilon)$, $f_4(\varepsilon)$ estimated respectively by introduction of artificial cutoffs at $\hbar\omega = \gamma m c^2$ in the spectra of Thomson scattering (i.e. (8) integrated over all angles) and of classical synchrotron theory. Evidently, both latter approaches grossly over-estimated f, because they ignored the reduction in cross-section due to quantum recoil.

Figures 4 and 5 show representative spectra of synchrotron radiation at $\varepsilon \simeq 1$, as given by the virtual quanta method [i.e. (13) integrated over all angles] and by the perturbation method. The abrupt cutoff at $\hbar\omega = \gamma m c^2$ in the spectra of the former result is because the Heisenberg uncertainty principle enforces an absolute upper limit of $\hbar\omega' \sim m c^2$ on the energy of virtual quanta in the electron rest frame. Perturbation theory on the other hand, provides a more accurate spectral profile prior to the absolute cut-off energy. For this reason, the present method over-estimates the radiation at high energy.

5. The behaviour of high energy protons and nuclei in a magnetic field

Quantum limited synchrotron radiation can hardly be a dominant source of energy loss for protons, because of the much larger rest mass. When a peak synchrotron photon satisfies $\hbar\omega \sim \gamma m_p c^2$ (i.e. the equivalent value of ε for protons ≈ 1) we have

$$\gamma B \geqq 2.81 \; 10^{20}\,\text{Gauss} \tag{16}$$

which describes an unattainable situation in astrophysics. Let us however consider the possibility of the virtual photons from the static magnetic field interacting with a non-elementary particle. Before the limit in (16) is reached, the proton undergoes pion production. This is allowed in terms of energetics when the virtual quantum in the proton rest frame has an energy $\hbar\omega' > 140$ MeV, or when

$$\gamma B \geqq 4.29 \; 10^{19}\,\text{Gauss}. \tag{17}$$

For stable nuclei, photodisintegration is allowed when $\hbar\omega' \sim 8\,\mathrm{MeV}$ (binding energy per nucleon), or

$$\gamma B \geqq 2.45 \frac{A^2}{Z}\, 10^{18}\,\mathrm{Gauss}\,, \tag{18}$$

where Z and A are respectively the atomic and mass (nucleon) number of the nucleus. The magnetic virtual photons could also scatter off protons and nuclei to continually produce positron-electron pairs (see e.g. Allkofer, 1975; Hillas, 1983). The threshold is $\hbar\omega' > 2m_e c^2 \simeq 1\,\mathrm{MeV}$, which implies

$$\gamma B \geqq 3.06\; 10^{17}\,\mathrm{Gauss} \tag{19}$$

for protons and

$$\gamma B \geqq 3.06 \frac{A^2}{Z}\, 10^{17}\,\mathrm{Gauss} \tag{20}$$

for nuclei. The criteria for particle production by virtual quanta (17)–(20) may set stringent limits on the acceleration of cosmic ray protons and nuclei in a given magnetic field. Note that these processes take place well in advance of the synchrotron limit (16). In general they must be unimportant in intergalactic space where the energy density of the (real) cosmic background radiation far exceeds that of the virtual photons of the magnetic field. However, it would appear that they may be important in the polar regions of neutron stars ($B \approx 10^{12}$ Gauss) for mirroring particles.

6. Conclusions

The Weizsäcker-Williams method of virtual quanta reproduces, with remarkable simplicity, all qualitative features of classical and quantum synchrotron radiation. More specifically, the following points are noteworthy:

(i) Equation (8) describes the essential behaviour of the spectrum, polarization, and angular distribution of classical radiation.

(ii) The Compton effect offers an explanation as to why the total synchrotron emission rate is lower than that given by the classical formula when $\gamma B \geqq 10^{14}$ Gauss. The manner in which this quantum suppression effect increases in severity with increasing γB is in close agreement with exact calculations (Fig. 3).

(iii) The method also provides a reason for the breakdown of classical theory at $\hbar\omega \sim \gamma m_e c^2$. Heisenberg's uncertainty principle is responsible for the absence of virtual quanta with energies $\hbar\omega' \geqq m_e c^2$ and hence the absence of real quanta with energies $\hbar\omega > \gamma m_e c^2$.

Nevertheless, the results are not completely satisfactory: in particular, the spectra deviate from those obtained by more exact calculations (Figs. 2 and 4) and the total energy loss rate does not agree exactly with the classical result [Eqs. (9) and (10)]. At least some of the inaccuracy must be associated with the neglect of the effects of retarded time in the calculations.

Appendix

Characteristics of outgoing radiation for electron scattering with magnetic virtual quanta

Equation (6) gives the differential rate of outgoing radiation when magnetic virtual quanta scatters an electron in its rest frame, viz.

$$\frac{d^2 I'}{d\omega\, d\Omega} = \frac{e^2 \omega'^2}{4\pi^2 c} \left[|I_2'|^2 + \sin^2\theta' \,|I_1'|^2\right].$$

For an equivalent expression in the Laboratory frame, we apply the Lorentz invariance of $\omega^{-2}(d^2 I/d\omega\, d\Omega)$ and the Doppler shift formulae (7). The result is

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 \frac{4}{\gamma^4}\left[|I_2|^2 + 2\psi^2(1+\psi^2)^{-2}|I_1|^2\right],$$

where $\psi = \gamma\theta$, and

$$I_1 = \int_{-\infty}^{\infty} \exp\left[i\frac{3}{2}\zeta\left(x + \frac{1}{3}x^3\right)\right] dx$$

$$I_2 = \int_{-\infty}^{\infty} x \exp\left[i\frac{3}{2}\zeta\left(x + \frac{1}{3}x^3\right)\right] dx$$

and $\zeta = \sqrt{2}\,\omega(1+\psi^2)/3\gamma^3\omega_0$. Next, we note that $|I_1|$ and $|I_2|$ are respectively equal $2/\sqrt{3}\,K_{1/3}(\zeta)$ and $2/\sqrt{3}\,K_{2/3}(\zeta)$, where $K_{1/3}$ and $K_{2/3}$ are Airy Integrals (Modified Bessel Functions).

Hence we have

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 \frac{4}{\gamma^4}\left[K_{2/3}^2(\zeta) + 2\psi^2(1+\psi^2)^{-2} K_{1/3}^2(\zeta)\right]$$

which is (8).

The total emission rate dE/dt (energy s^{-1}) in laboratory frame is obtained by integrating over all angles and frequencies. Note that

$$\omega^2 d\omega = \frac{27\gamma^9\omega_0^3}{2\sqrt{2}}(1+\psi^2)^{-3}\zeta^2 d\zeta$$

and

$$d\Omega = \cos\theta\, d\theta\, d\Phi = 2\pi\, d\theta = \frac{2\pi}{\gamma} d\psi\,.$$

We now have

$$I = \frac{18\sqrt{2}e^2}{\pi c}\gamma^4\omega_0 \int_{-\infty}^{\infty}(1+\psi^2)^{-3} d\psi \int_0^{\infty} \left[K_{2/3}^2(\zeta) + 2\psi^2(1+\psi^2)^{-2}K_{1/3}^2(\zeta)\right]\zeta^2 d\zeta\,.$$

To get dE/dt, divide I by the gyro-period $2\pi/\omega_0$. This, together with numerical integration of I, gives

$$\frac{dE}{dt} = 0.840\,\frac{e^2\omega_c^2\gamma^2}{c}\,,\quad \omega_c = \gamma\omega_0$$

which is (9).

Exactly the same procedures can be applied to the Compton corrected differential emission rate (13), to obtain the total loss rate (14) via the redshift ratio η. The Compton suppression factor $f(\varepsilon)$ emerges as a complicated double integral for I, similar to the above; it can only be evaluated numerically.

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