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THE THERMODYNAMICS OF ANISOTROPIC RADIATING SYSTEMS[1]

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ABSTRACT

A formalism is presented which employs the principles of statistical mechanics to approximate the state of a radiation field that is out of equilibrium with the material medium through which it passes. Central to the theory is the statistically constrained angular distribution of the photons and the *statistically independent* energy distribution. We are addressing here the problem of finding a statistical distribution for the *photons*, not that of finding the occupation numbers for the matter, as in the problem of statistical equilibrium in non–LTE stellar atmospheres. We argue that the physics underlying the derived specific intensity of the radiation field is valid for gray transport, and we discuss possibilities of adapting the theory to nongray transport. We show that the functional form of the distribution function leads to a natural closure of the hierarchy of radiative moment equations at the third angular order. In suitable limiting cases, the method reduces to a second-order truncation of the moments, namely a variable Eddington factor method. Thus the techniques discussed may be applied to models of radiation hydrodynamics in a wide variety of astrophysical problems. A few examples of currently interesting applications in high-energy astrophysics are briefly discussed. The consequences of this theory for some basic thermodynamic relations, and a similar treatment for neutrino transport, are also discussed. In subsequent work, the formalism shall be used to solve the classical gray atmosphere as a means of testing the theory.

Subject headings: hydrodynamics — plasmas — radiative transfer — stars: atmospheres

I. INTRODUCTION

There have been two basic obstacles to achieving a realistic treatment of radiative energy and momentum transport through an astrophysical plasma: (1) a self-consistency problem, in which the properties of the radiation field at a point in space are determined by those of the gas *and* vice versa; (2) a generally large discrepancy betweeen the mean free paths of the material particles and of the radiation. Experience has shown that the first difficulty is more easily mastered than the second. Self-consistent numerical models of radiation hydrodynamics have been worked out to calculate the spectral fluxes in the atmospheres of X-ray bursting neutron stars (London, Taam, and Howard 1986) to generate reasonable estimates of light curves from the time-dependent evolution of Type II supernova explosions (e.g., Grasberg, Imshennik, and Nadezhin 1971; Falk and Arnett 1977) and to investigate the importance of neutrino transport during gravitational collapse of stars (e.g., Wilson 1971, 1974; Arnett 1977), just to name a few examples. However, with a few exceptions, these treatments have succumbed to the second obstacle either by ignoring it or by making an ad hoc approximation to acknowledge its reality.

In this manner, variable Eddington factor parametrizations, of which flux-limited diffusion theory is a special instance (Levermore 1984), have been the most common ways of obtaining a closed pair of radiative angular moment equations (i.e., energy and momentum conservation). Truncation of the moment equations by means of a prescribed relationship between the moments requires some *a priori* knowledge of the angular dependence of the radiation field as a function of optical depth τ. In flow calculations, variable Eddington factors usually furnish the necessary information at all τ through an interpolation between the optically thick limit, for which the Eddington factor $f_E \equiv \langle \mu^2 \rangle = \frac{1}{3}$ (where $\mu \equiv \cos \theta$ is the cosine of the zenith angle θ), and the streaming limit, for which $f_E = 1$. In most cases where variable optical depth transport is important, however, such information does not seem available from the physical input of the model. Within the context of a *local* theory of thermodynamics and hydrodynamic flow of the gas, a satisfactory model of a *nonlocal* radiation field has not yet been constructed.

Of course, the most accurate treatments are those which attack the full angle and frequency-dependent transfer equation. For this reason, and because of the physical inadequacies of the above methods, much effort has been expended toward finding the specific intensity $I(\boldsymbol{x}, t, \nu, \theta)$ directly. Well-established calculational techniques exist for planar geometry (Chandrasekhar 1960; Kourganoff 1963; Feautrier 1964; Rybicki 1971) and for spherical geometry (Chapman 1966; Larson 1969; Hummer and Rybicki 1971). However, these methods have nearly always been confined to time-independent, static situations. Complications inevitably arise, and costs are prohibitive, when these techniques are adapted for use in hydrodynamic calculations. This is often the case even for variable Eddington factor techniques in which f_E is determined by *direct iteration*, since then the full transfer equation must be employed.

Therefore, a new moment approximation is useful in flow calculations only if it provides a coherent physical scenario for variable optical depth transport and does not add significantly to computing costs. Toward these ends, we choose not to address the latter concern until further progress is made with the former. Moment closure methods are often chosen for their computational efficiency, and it is clear that they represent a viable "class" of solutions. Hence, within this framework it seems crucial to build a more solid physical foundation upon which cost-cutting techniques at the computational level can be developed.

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In this paper, we present a new formalism for the physics of radiative transport. We exploit the fact that the photon distribution appearing in the transfer equation is an ensemble-averaged function (Tubbs 1978) and we adopt the techniques of statistical mechanics to derive an approximate functional form of the (ν, θ) dependence of the specific intensity. In § II, we derive the distribution function, emphasizing the approximations made and the physical conditions under which they are valid. In § III, we discuss the closure of the radiative moment equations. In § IV, we discuss possible implications of the theory for more generalized forms of the thermodynamic Gibbs relation and of the nonequilibrium entropy balance equation. A nongray generalization of the theory is presented in § V. A special discussion of contributions of local emission to the radiation field is given in § VI. The relevance of the methods to the study of high-energy astrophysical systems is discussed in § VII. A similar treatment for neutrino transfer (as in the gravitational collapse of stars), is discussed in § VIII. Conclusions are presented in § IX.

II. PHYSICS OF THE PHOTON DISTRIBUTION

a) Basic Assumptions

A radiation field will be out of equilibrium with the fluid medium in which it is embedded if the photons interact weakly with the matter on the time scale over which the flow variables change or if there are few interactions on the length scale over which gradients in the material occur. In a rough sense, the first situation causes departures of the radiative energy distribution from a Planck function at the local matter temperature, whereas the second causes anisotropies in the angular distribution of the radiation. Thus in the *nonequilibrium diffusion* limit, the fast hydrodynamic time scale does not allow the radiation field to relax to $B_\nu(T_m)$, but photon mean free paths are short enough for the diffusion approximation to be valid (see Mihalas and Mihalas 1984). In this case, the energy density is usually described as aT_R^4, , although, in general, the radiation field is not Planckian, and $T_R \neq T_m$. Alternatively, we can consider the fast expansion of a sphere of Planckian radiation of temperature T_R into a surrounding spherically symmetric medium of tenuous matter. The expanding radiation decouples from the matter, retaining the same Planckian frequency profile at temperature T_R (while T_m drops off as r increases), but with a *diluted* total energy and number density (Fu and Arnett 1985), and with a forward peaking of the angular distribution.

In many situations, all of the above effects may be relevant. For the moment, let us focus on the total radiative contributions to the energy and momentum transport in the fluid, not on the details of the spectrum. We seek to solve the frequency-integrated moment equations, where the opacities may be treated as gray, if necessary. Hence, we are motivated to make the following assumptions:

1. The energy distribution can be described by a function of two variables, a temperature, T_R, and another that allows the total number density to vary independently of the energy profile. Two-parameter transport was suggested by Imshennik and Nadezhin (1973) in the diffusion limit and was used by Mazurek (1975) and Sato (1975) in calculations of gravitational collapse. Within the context of equilibrium diffusion methods, the assumption of a Fermi-Dirac distribution with nonzero chemical potential for neutrinos was an obvious conceptual improvement over one-parameter blackbody diffusion used earlier in collapse simulations. With the introduction of an additional thermodynamic variable, the equation of lepton number conservation took on an increased significance. Fu and Arnett (1985) suggested the idea of two-parameter *photon* transport, employing a Bose-Einstein function with nonzero chemical potential for nonquilibrium diffusion calculations. Since the number conserving process of Compton scattering can thermalize photons to Bose-Einstein (Kompaneets 1957), this approximation may be appropriate for many optically thick, high-temperature environments in astrophysics. Moreover, in the optically thin regime, *dilution* and *number conservation* call for an additional degree of freedom to describe the energy and number densities of the propagating photon gas. For these reasons, we shall approximate the energy distribution at all τ as Bose-Einstein with nonzero chemical potential.

2. The specific intensity is separable in frequency and angular dependences, $I(\nu, \theta) = f(\nu)g(\theta)$. This is a natural assumption to make if the opacity is ideally *gray*, for if there is a single, frequency-independent optical depth scale, there can be only one angular distribution (as a function of τ). Thus in the general nongray case, $g(\theta)$ becomes a *mean* angular distribution obtained from a suitable mean optical depth scale. To find $g(\theta)$ in the frequency-integrated moment equations, we must then take into account the opacity averages $\kappa_E = \int E_\nu k_\nu d\nu/E$, the energy mean, and κ_F, the flux mean, analogously defined. Under the separability assumption, $\kappa_F = \kappa_E$, defines an appropriate mean optical depth scale. It can be seen that separability may be tacitly assumed whenever a gray Eddington factor method is invoked to close the moment equations. Also, because optical depth is frequency dependent, the temperature T_R and chemical potential $\mu_R = -\alpha k T_R$, which characterize the radiation at a single point in space, must also be frequency-averaged quantities representing the mean nonlocal energy distribution. This shall be discussed in more detail in § V.

3. The specific intensity at all τ is the *most probable distribution subject to the appropriate set of local thermodynamic constraints.* Hence the photon distribution function represents the state of statistical equilibrium of the photon gas. Two of these constraints are the total energy density E and number density N, in accordance with the first assumption of this section. We assert that a third independent constraint, which regards the *anisotropy* of the radiation field as a *thermodynamic condition*, is the number flux F_N. In devising a frequency specific variable Eddington factor scheme, Minerbo (1978) was the first to use the flux F_ν [equivalent to $(F_N)_\nu$ for monochromatic quantities] as a statistical constraint. His main result was a two-parameter angular distribution (for which he subsequently considered a one-parameter limit). In what follows, we generalize this kind of analysis to include the energy distribution as well, obtaining a more complete and consistent physical picture.

It is apparent that, in some sense, the distribution function must be the statistically most probable one. The key problem is finding the *accessible possibilities* and assessing the *probability of occurrence* of each possibility. To see why classical ensemble theory, which hinges on the assumption of equal probabilities of all accessible states, is appropriate for our treatment of the radiation, consider the special case of local thermal equilibrium of the radiation field with the matter. Here the photon mean free path is the same order of magnitude as that of the matter with itself. The multitude of photon/matter interactions which thermalize the radiation field to the local temperature of the matter T_m takes place entirely within a specific volume of photon gas, which is the "local thermodynamic system" under consideration. These same interactions allow us to construct a statistical ensemble for which the sole constrained quantity is the photon energy E. The statistical basis for the thermodynamics is clear in this case, since the macroscopic quantities (expectation values) of the system arise from microscopic processes occurring entirely *within the system.*

The situation is qualitatively different, of course, when the photon mean free path is long and the radiation field is strongly anisotropic. Here the set of macroscopic quantities which describe the local state of the photon gas arise from a multitude of interactions occurring over regions much bigger than the specific volume. This set of quantities is generally much larger than that for isotropic equilibrium, since it must include information about the angular distribution of the radiation (i.e., the higher order angular moments). It is still reasonable, however, to assume that a correspondingly larger set of constrained quantities defines a set of accessible states of the system just as the single quantity E defines a more unrestricted set of accessible states in the case of thermal equilibrium. The main difference is that, in the former case, the microprocesses which establish the state of statistical equilibrium described by *local* variables (including the higher order moments) occur mostly *outside* of the specific volume. Because these constraints alone determine which states are accessible, each microstate is as likely to occur as any other. In our approximation, therefore, we step away from isotropic equilibrium by including the higher order moments as independent constraints on the local state of the photon gas. Specifically, in this initial investigation, we consider the lowest order angular information available, namely that contained in the radiative *flux*. We stress that in this approximation we include only the first-order anisotropy, namely $\langle \cos \theta \rangle$. Therefore, harmonic information contained in the higher order moments $\langle \cos^3 \theta \rangle$, $\langle \cos^4 \theta \rangle, \ldots,$ does not furnish new information about the angular spectrum but provides greater *resolution* of the angular distribution function (Lamb 1986). Hence, we would expect that the inclusion of higher order moments (by including higher order moment equations and eliminating lower order equations) in this scheme should result in a more accurate solution. Olson and Weaver (1982) conducted a study in which several variable Eddington factors were used to solve a time-dependent linear Marshak wave problem in planar geometry. They indeed found that all the variable Eddington factors tested showed a significant increase in accuracy of roughly a factor of 10 from 10%–20% (when compared to the known analytic solution) to 1%–2% when the moment equations were closed at the fourth angular order rather than the conventional second order. Another difficulty mentioned by Olson and Weaver that seems to be solved by higher order closures is the presence of unphysical discontinuities in the radiation field when the equations are closed at the second order. These "shocks" in the radiation field disappear when the equations are closed at the fourth order in terms of only the even moments. This suggests that the problem of introducing a nonlinear closure relation into a set of linear equations is not an insurmountable one, but can be remedied by including higher order angular moments as speculated by the above authors.

b) *Derivation of the Statistical Distribution Function*

In microcanonical ensemble theory, the goal is to construct the entropy function for the system in terms of the occupation numbers of the particles, and to maximize it subject to the appropriate set of integral constraints (see, e.g., Pathria 1972; Landau and Lifshitz 1938). In order to do this, consider the separability assumption outlined previously. We see that it is equivalent to an assumption of statistical independence of the energy and angular distributions, whereby the joint probability $P(\epsilon, \theta)$ that a particle has energy ϵ and is oriented at an angle θ is the product of two independent probabilities, $P(\epsilon, \theta) = P(\epsilon)P(\theta)$. Therefore, to find the entropy, we have to solve two separate statistical problems, one pertaining to the occupation of the energy levels of the system, the other pertaining to the occupation of the "angular levels"; it is the combination of these independent contributions which gives the total entropy of the photon gas.

Let us first consider, then, the problem of finding the energy distribution. This is just the classical problem of finding the mean occupation numbers for a Bose gas. The relevant constraint relations are

$$\sum_j n_j^\epsilon = N , \qquad \sum_j n_j^\epsilon \epsilon_j = E ,$$

where n_j^ϵ is the number of particles in energy state j with energy ϵ_j. For a certain distribution of occupation numbers or *distribution set* $\{n_j^\epsilon\}$ of the N particles over all the ϵ_j, it can be shown that the total number of microstates or "complexions" of the system is

$$W\{n_j^\epsilon\} = \prod_j \frac{(n_j^\epsilon + g_j^\epsilon - 1)!}{n_j^\epsilon!(g_j^\epsilon - 1)!} ,$$

where g_j is the degeneracy of the level ϵ_j. This result follows from a standard combinatorial analysis for a gas of identical, indistinguishable particles with no other restrictions (see, e.g., Pathria 1972). Suppose that $\{n_j^\epsilon\}$ is the most probable distribution set, and let Ω be the *total number* of accessible microstates of the gas. It can then be shown for a large system of N particles that $W\{n_j^\epsilon\} \approx \Omega$, that is, $\{n_j^\epsilon\}$ is overwhelmingly probable. The entropy of the gas is, by definition,

$$S_\epsilon \equiv k \ln \Omega \approx k \ln W\{n_j^\epsilon\} .$$

Under the assumption of statistical independence, we can now consider separately the problem of finding the angular occupation numbers of the gas. We regard the gas as N bosons distributed over "angular states" θ_i, each with a degeneracy g_i^θ, so that the constraint relations are

$$\sum_i n_i^\theta = N , \qquad \sum_i n_i^\theta \cos \theta_i = \frac{F_N}{c} ,$$

and the total number of microstates for a distribution set $\{n_i^\theta\}$ is

$$W\{n_i^\theta\} = \prod_i \frac{(n_i^\theta + g_i^\theta - 1)!}{n_i^\theta!(g_i^\theta - 1)!} ,$$

which is completely similar to $W\{n_j^\theta\}$, since the same condition of indistinguishability applies. The entropy of the system would be

$$S_\theta \approx k \ln W\{n_i^\theta\} .$$

It is easy to combine these results. The total number of complexions of the photon gas with respect to energy and angular orientation is just the product

$$W\{n_i^\theta, n_i^\epsilon\} = W\{n_i^\theta\}W\{n_j^\epsilon\} ,$$

and the total entropy is

$$S = k \ln W = k \ln W\{n_i^\theta\} + k \ln W\{n_j^\epsilon\} = S_\theta + S_\epsilon .$$

To find the occupation numbers n_i^θ and n_j^ϵ, we must maximize S subject to the constraints given above. When S is a maximum, any arbitrary variation of S must vanish, so

$$\delta S_\theta + \delta S_\epsilon = 0 .$$

However, δS_θ and δS_ϵ are independent since δS_θ involves changing the numbers n_i^θ, and δS_ϵ involves changing the numbers n_j^ϵ. Therefore, we require that the two variations vanish separately:

$$\delta S_\theta = 0 , \qquad \delta S_\epsilon = 0 .$$

For $\delta S_\theta = 0$ we have the two constraints:

$$\sum_i n_i^\theta = N \text{ and } \sum_i n_i^\theta \cos\theta_i = F_N/c .$$

Therefore, we can use the method of Lagrange multipliers to obtain the variational equation

$$\delta S_\theta - \left(\eta \sum_i \delta n_i^\theta + \lambda \sum_i \delta n_i^\theta \cos\theta_i\right) = 0 ,$$

where η is the multiplier associated with the constraint $\sum_i n_i^\theta = N$, and λ is the multiplier associated with the constraint $\sum_i n_i^\theta \cos\theta_i = F_N/c$. By substituting the expression for S_θ in this equation and taking the variation, the result

$$n_i^\theta = \frac{1}{\exp(\lambda \cos\theta_i + \eta) - 1} \tag{1}$$

is obtained. Likewise, for $\delta S_\epsilon = 0$, we have the constraints $\sum_j n_j^\epsilon = N$ and $\sum_j n_j^\epsilon \epsilon_j = E$, and the corresponding variational equation is

$$\delta S_\epsilon - \left(\alpha \sum_j \delta n_j^\epsilon + \beta \sum_j \delta n_j^\epsilon \epsilon_j\right) = 0 ,$$

where $\alpha \equiv -\mu_\epsilon/kT_R$ and $\beta \equiv 1/kT_R$ are the more familiar Lagrange multipliers related to the chemical potential and temperature of the gas. This, of course, leads to the Bose-Einstein occupation number:

$$n_j^\epsilon = \frac{1}{\exp(\beta\epsilon_j + \alpha) - 1} . \tag{2}$$

Since the occupation numbers are unnormalized probability densities, it follows from $P(\epsilon, \theta) = P(\epsilon)P(\theta)$ that the joint occupation number $n(\epsilon, \theta)$ must be proportional to the product of the two individual functions, equations (1) and (2):

$$n(\epsilon, \theta) \sim \left(\frac{1}{e^{\alpha+\beta\epsilon} - 1}\right)\left(\frac{1}{e^{\eta+\lambda\cos\theta} - 1}\right) . \tag{3}$$

In order to obtain a continuum distribution function, consider how momentum space would now be filled for a distribution function with the energy *and* angular dependences as given in equation (3). Since the energy distribution in this approximation is derived in exactly the same way as for an isotropic Bose-Einstein distribution, it is reasonable to require that, for a given α, β, each spherical shell of phase space contain the same total number of particles as does an isotropic radiation field:

$$\frac{2}{h^3} p^2\left[\int n(\epsilon, \theta) d\Omega\right] dp = \frac{2}{h^3} 4\pi p^2 dp\left[\frac{1}{\exp(\beta\epsilon + \alpha) - 1}\right] .$$

where $p = \epsilon/c$. Therefore, the total energy and number are given in terms of α and β alone:

$$N = \frac{8\pi}{h^3c^3}\int_0^\infty \frac{\epsilon^2 d\epsilon}{\exp(\alpha + \beta\epsilon) - 1} , \qquad E = \frac{8\pi}{h^3c^3}\int_0^\infty \frac{\epsilon^3 d\epsilon}{\exp(\alpha + \beta\epsilon) - 1} .$$

It follows that the angular distribution should be normalized so that it is a *phase function*, i.e., a function which merely redistributes the particles about the polar axis in each shell of phase space, giving the result

$$n(\epsilon, \theta) = \left[\frac{4\pi}{\exp(\alpha + \beta\epsilon) - 1}\right]\left[\frac{1}{\exp(\eta + \lambda\cos\theta) - 1} \Big/ \oint \frac{d\Omega}{\exp(\eta + \lambda\cos\theta) - 1}\right] . \tag{4}$$

This can be compared with Minerbo's result (his eq. [2.7]). His distribution contains no separate energy distribution and no normalization of the angular distribution. Therefore, in his expression, λ and η contain an implicit frequency dependence. For a gray system, his specific intensity integrated over all frequencies would diverge. For a nongray system, the shortcomings of this distribution is discussed below in § IIIc. Furthermore, failure to normalize the angular distribution excludes some physically realizable cases (which are beyond the scope of this work) as discussed in the next section.

In retrospect, it might seem odd to include N as a statistical constraint *twice*, resulting in the Lagrange multipliers μ and η. Although the constraints have often been alluded to as macroscopic quantities, it is clear that they are, more precisely, relations which represent conditions on the ways in which the single particle states can be filled. It is then obvious that the single constrained quantity N gives rise to two separate conditions, one on the distribution of particles in their angular states, the other on the distribution of particles in their energy states.

III. CLOSURE OF THE MOMENT EQUATIONS

a) Properties of the Angular Distribution

Let us define the angular distribution in equation (4) as $\Phi(\theta)$. Then

$$\Phi(\theta) = \left[\frac{1}{\exp(\eta + \lambda \cos\theta) - 1}\right] \Big/ \left[\oint \frac{d\Omega}{\exp(\eta + \lambda\cos\theta) - 1}\right]. \tag{5}$$

An obvious requirement of equation (5) is that it always be positive. This can be satisfied if $\lambda \cos\theta + \eta > 0$, or if $\eta > |\lambda|$. Thus η is restricted to positive values (as is the photon chemical potential $\mu_\epsilon = -\alpha k T_R$). Also, in most cases of astronomical interest, the angular distribution must be forward-weighted. This is easily shown to be equivalent to the condition $\lambda < 0$. It can also be shown that the Eddington factor behaves correctly in the optically thick and thin limits, that is,

$$f_E \rightarrow \begin{cases} \dfrac{1}{3} & \text{as} \quad \tau \rightarrow \infty, \quad \Phi \rightarrow \dfrac{1}{4\pi}, \\ 1 & \text{as} \quad \tau \rightarrow 0, \quad \Phi \rightarrow \delta(\cos\theta - 1), \end{cases}$$

and that $\frac{1}{3} < f_E < 1$ for a smooth variations of the parameters η, λ within their allowed ranges.

Levermore (1984) showed that another requirement of the angular distribution is that it satisfy the inequality $H/J \leq K/J \leq 1$. Since this is equivalent to the condition $\langle\mu\rangle^2 \leq \langle\mu^2\rangle \leq 1$, it must hold for all variable Eddington schemes. Levermore showed that it will generally hold for the function $g(\Omega)$, where $g(\Omega) \equiv I(\nu, \Omega)/[cE(\nu)]$, if $\oint g(\Omega)d\Omega = 1$ and if $g(\Omega)$ is nonnegative. It is obvious that $\Phi(\theta) = g(\Omega)$ and that $\Phi(\theta)$ is nonnegative and normalized to unity; hence, the requirement is met.

Two other classes of solution are allowed by equation (5). Those defined by the parameter constraints $\lambda\cos\theta + \eta < 0$, $\lambda > 0$ are forward-weighted solutions; those with $\eta = 0$ are *sideward*-weighted. This latter class of solutions is relevant inside a strong, optically thin radiating shock, and is characterized by $f_E \leq \frac{1}{3}$. Such cases are not of immediate interest, so these solutions shall not be considered here. Moreover, the acceptance of both classes of solutions depends on the acceptance of the heuristically derived "phase function" normalization of the angular distribution contained in equation (5). As an example, for $\lambda\cos\theta + \eta < 0$, $n(\theta)$ in equation (1) (or the numerator in eq. [5]) is negative, but because of the denominator in equation (5), we still have $\Phi(\theta) > 0$. This normalization has two distinct advantages: (1) it gives a clear picture of how momentum space is filled by the anisotropic distribution in comparison to the isotropic Bose-Einstein distribution, and (2) it frees the definition of an "angular state" from the inevitable stigma it would have as a "quantum angular state," which it would automatically be if actual *magnitudes* of the angular moments depended on λ and η, as in Minerbo (1978; also, see discussion in§ IIIc below). However, the result (1) follows from a much more formal mathematical development than does the normalized form in equation (5). At present, it would seem premature to place demands on equation (5) rigorous enough to regard these other solutions as entirely viable ones. We feel it wiser to confine subsequent considerations here to those solutions (namely $\lambda\cos\theta + \eta > 0$) which are valid *independently* of the normalization, i.e., they give both expressions (1) and (5) positive. Indeed, the full issue of normalization and the possible inclusion of sideward-peaked solutions are the subjects of another paper.

b) Closure Method for the Equations of Radiation Hydrodynamics

In a typical variable Eddington closure scheme, f_E is expressed as a function of F_E/cE, which we shall define as the "streaming factor," f_{stream}. For our *two-parameter* angular distribution, however, f_{stream} and f_E are both independent functions of λ and η. Therefore, the hierarchy of moment equations must be closed at the *third* order rather than the second. In order to illustrate how this higher order closure may be implemented, consider the radiation hydrodynamics of spherical flows. Castor (1972) gives the fluid frame equation of radiative transfer:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + 4\pi r^2 \rho \frac{\partial I_\nu}{\partial m}\cos\theta - \sin\theta\left[\frac{\cos\theta}{c}\left(\frac{3v}{r} + \frac{\partial \ln\rho}{\partial t}\right) + \frac{1}{r}\right]\frac{\partial I_\nu}{\partial\cos\theta} + \left[\frac{\cos^2\theta}{c}\left(\frac{2v}{r} + \frac{\partial\ln\rho}{\partial t}\right) - \sin\theta\frac{v}{cr}\right]\nu\frac{\partial I_\nu}{\partial\nu}$$
$$- 3\left[\frac{\cos^2\theta}{c}\left(\frac{2v}{r} + \frac{\partial\ln\rho}{\partial t}\right) - \sin\theta\frac{v}{cr}\right]I_\nu = -\chi_\nu I_\nu + j_\nu\,, \tag{6}$$

which leads to the relevant frequency-integrated moment equations:

$$\frac{1}{c}\frac{\partial J}{\partial t} + 4\pi r^2 \rho \frac{\partial H}{\partial m} + \frac{2H}{r} - (3K - J)\frac{v}{cr} - (J + K)\frac{1}{c}\frac{\partial \ln \rho}{\partial t} = -\bar{\chi}_J J + j ,$$

$$\frac{1}{c}\frac{\partial H}{\partial t} + 4\pi r^2 \rho \frac{\partial K}{\partial m} + \frac{3K - J}{r} - \frac{2v}{cr} H - \frac{2}{c}\frac{\partial \ln \rho}{\partial t} H = -\bar{\chi}_H , \qquad (7)$$

$$\frac{1}{c}\frac{\partial K}{\partial t} + 4\pi r^2 \rho \frac{\partial L}{\partial m} + \frac{2}{r}(2L - H) + (3M - 5K)\frac{v}{cr} + (M - 3K)\frac{1}{c}\frac{\partial \ln \rho}{\partial t} = -\bar{\chi}_K K + \frac{j}{3} ,$$

where the first two expressions of equation (7) are Castor's (1972) equations (33) and (34). Notice that whereas the evolution equation for H contains moments up to K only, the equation for K contains both L and M, the third- and fourth-order moments, respectively; in general, a moment equation of order n derived from equation (6) contains the moments of order n, $n + 1$, and $n + 2$. Formally, then, we can write

$$\frac{H}{J} = f_{\rm stream}(\lambda, \eta) , \qquad \frac{K}{J} = f_{\rm E}(\lambda, \eta) .$$

plus the higher order relations

$$\frac{L}{J} = f_L(\lambda, \eta) , \qquad \frac{M}{J} = f_M(\lambda, \eta) .$$

Consequently,

$$\frac{L}{J} = f_L(f_{\rm stream}, f_{\rm E}) , \qquad \frac{M}{J} = f_M(f_{\rm stream}, f_{\rm E}) ,$$

and closure is accomplished by writing the third expression of equation (7) in terms of J, H, and K.

If we let

$$f(\epsilon) = 8\pi/(h^3c^3)\{\epsilon^2/[\exp(\alpha + \beta\epsilon) - 1]\} ,$$

then

$$E = \int_0^\infty f(\epsilon)\epsilon d\epsilon , \qquad N = \int_0^\infty f(\epsilon) d\epsilon ,$$

$$F_E = \int_0^\infty f(\epsilon)\epsilon d\epsilon \oint \cos\theta g(\theta) d\Omega , \qquad F_N = \int_0^\infty f(\epsilon) d\epsilon \oint \cos\theta g(\theta) d\Omega ,$$

$$P_E = \int_0^\infty f(\epsilon)\epsilon d\epsilon \oint \cos^2\theta g(\theta) d\Omega , \qquad P_N = \int_0^\infty f(\epsilon) d\epsilon \oint \cos^2\theta g(\theta) d\Omega ,$$

so that the moments in energy space, denoted by the subscript E, and those in number, denoted by the subscript N, are related by

$$\frac{E}{N} = \langle\epsilon\rangle = \frac{F_E}{F_N} = \frac{P_E}{P_N} = \cdots ,$$

or

$$\frac{J}{J_N} = \frac{H}{H_N} = \frac{K}{K_N} = \frac{L}{L_N} = \frac{M}{M_N} = \frac{N}{N_N} = \cdots , \qquad (8)$$

where J, H, K, ..., have their usual definitions, and the moments with subscript N are the corresponding quantities in number space. Since the scheme presented above allows us to determine all the moments in energy space from the first three moment equations, we see that knowledge of one of the moments in number space, say J_N, then allows us to determine all the corresponding number moments from equation (8). The inclusion of the *number conservation* equation is necessary for this purpose. In spherically symmetric flows, this equation would be similar in form to the first of equations (7). Hence, the scheme which allows us to determine all the moments in energy and number includes the number conservation equation as a fourth independent moment equation. This is consistent with the fact that our photon distribution is a function of four variables, the Lagrange multipliers α, β, λ, η.

c) *One-Parameter Limiting Cases*

For the Bose-Einstein distribution, large and small values of $\alpha = -\mu_\epsilon kT_R$ yield the Maxwell-Boltzmann and Planck limits, respectively. Somewhat similar limits exist for the angular distribution. For large values of $\eta(\eta > 0)$, the "angular chemical potential," we have

$$\frac{1}{\exp(\lambda\cos\theta + \eta) - 1} \approx e^{-\eta} e^{-\lambda\cos\theta} , \qquad (9)$$

and $\Phi(\theta)$ becomes independent of η. The quantities of $f_{\rm stream}$ and $f_{\rm E}$ are given by

$$f_{\rm stream} = \coth a - \frac{1}{a}, \qquad f_{\rm E} = 1 - \frac{2}{a}\left(\coth a - \frac{1}{a}\right), \tag{10}$$

where $a = |\lambda|$. It is easy to see that a ranges from zero in the diffusion regime to infinity in the streaming regime. Because of equation (10), we shall call equation (9) the "hyperbolic" limit. For small values of η, we have

$$\frac{1}{\exp(\lambda \cos\theta + \eta) - 1} \approx \frac{1}{\lambda \cos\theta + \eta}, \tag{11}$$

and $\Phi(\theta)$ depends solely on the ratio η/λ. The quantities $f_{\rm stream}$ and $f_{\rm E}$ are given by

$$f_{\rm stream} = \frac{2}{\ln (z-1)/(z+1)} + z, \qquad f_{\rm E} = z\left[\frac{2}{\ln (z-1)/(z+1)} + z\right], \tag{12}$$

where $z = -\eta/\lambda$. Here z ranges from infinity in the optically thick limit to unity in the optically thin limit. Because of equation (12), we shall call equation (11) the "logarithmic" limit.

The results (10) and (12) can easily be plotted. In Figure 1, we show the hyperbolic and logarithmic limits superposed on a plot of various other variable Eddington schemes taken from Levermore (1984) (see this paper for details on the individual schemes and references). In this space of K/J versus H/J, these parameterized curves (e.g., the hyperbolic curve parameterized by a) define certain "Eddington trajectories" which span the entire range of optical depth. Since the hyperbolic and logarithmic curves define the extreme limits of our approximation, we see that the shaded space between the curves is the allowed space in which acceptable forward-weighted Eddington trajectories must lie for the statistical approximation. Notice that only two of the other schemes lie entirely in this space: (1) Levermore's (1984) "Lorentz" approximation, for which

$$f_{\rm stream} = \frac{1 + 3\beta^2}{3 + \beta^2}, \qquad f_{\rm E} = \frac{4\beta}{3 + \beta^2},$$

where Levermore assumes that a radiation anisotropy can be transformed away by a Lorentz boost with $\beta = v/c$; (2) Kershaw's (1976) parabolic Eddington factor, $K/J = \frac{1}{3} + \frac{2}{3}(H/J)^2$. Two more or less obvious possible schemes, the Eddington limit, $K/J = \frac{1}{3}$, and a linear interpolation, $K/J = \frac{1}{3} + \frac{2}{3}H/J$, are unacceptable in this approximation. The curve labeled "Kosirev, $\eta = 2$, $R = 10$" is obtained from an accurate numerical solution to the spherical transfer equation (Rogers and Martin 1984) and is typical of the type of problem which is of primary interest (see § VII below). The Kosirev transfer problem is defined by $\rho\kappa = \alpha r^{-n}$, and the example shown has $n = 2$ and a radius equal to 10 times the scale radius, in this case $r_s = \alpha$. The Kosirev geometry for $R > 1$ is characterized

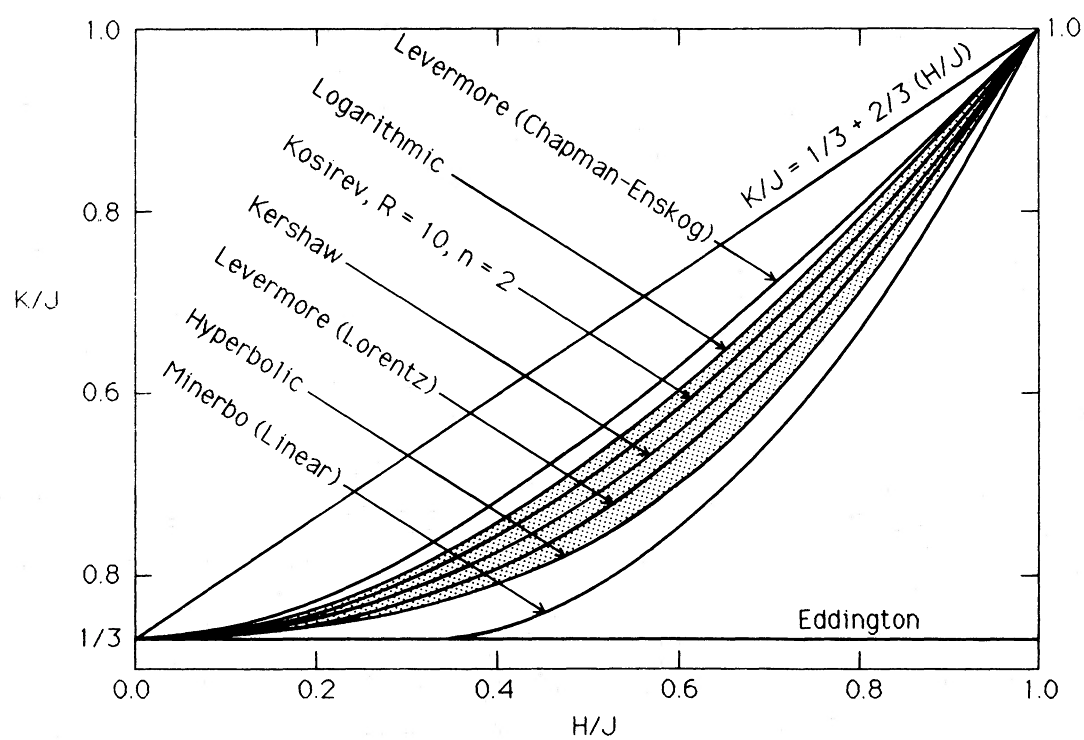

FIG. 1.—The "Eddington trajectories" for various moment closure schemes. The figure is taken from (Levermore 1984), with several other curves superposed.

by a rapid and extreme forward peaking of the radiation field as r increases. Other Kosirev solutions shall be discussed within the context of the statistical solutions in a companion paper (Fu 1987).

The hyperbolic limit was first considered by Minerbo (1978), and is labeled "Minerbo(statistical)" in Figure 1. In his maximum entropy, frequency specific formulation of the angular distribution, the parameters λ and η vary with ν and determine both the shape and *amplitude* of the energy spectrum as well. Accordingly, his specific intensity is given simply by

$$I(\nu, \theta) = \frac{2h\nu^2}{c^2} \frac{1}{\exp(\lambda \cos\theta + \eta) - 1} . \tag{13}$$

The difficulty with this formula is that the value of ν restricts the acceptable values of λ and η, hence the shape of the angular spectrum. For instance, at very high photon energies, η must be large since $I(\nu) \sim \nu^3$, and $I(\nu, \theta)$ would invariably increase rapidly with ν unless the angular contribution becomes small. In Figure 1, therefore, the only allowed Eddington trajectories would lie in the vicinity of the hyperbolic limit. This is not a serious drawback for a closure method per se, since the entire range of τ can be covered by this particular limit; however, the large η approximation in equation (13) must always give small occupation numbers. Indeed, Minerbo argued that the occupation numbers are small in astrophysical photon transport; hence the angular distribution should be restricted to the hyperbolic limit. To investigate this claim briefly, consider a simple example: a radiation field exhibiting a Planck behavior over a large portion of the continuum spectrum. Obviously, the Planck occupation number $[\exp(h\nu/kT) - 1]^{-1}$, is small only in the Wien limit. Therefore, we do not expect the claim to hold at low energies. Nevertheless, let us test its validity by evaluating *mean* intensities of the Planck distribution. Let the temperature of the radiation be $T_R \approx T_{\rm bb}$ and consider a mean intensity of $4\pi B(\nu, T_{\rm bb})$, both in the diffusion and streaming limits. The peak of the Planck energy distribution occurs at $x_{\rm peak} \equiv h\nu^{\rm peak}/kT_{\rm bb} \approx 5$. At this frequency, the mean Planck intensity is $\sim 6.8 \times 10^{-3}(8\pi h\nu^3/c^2)$. We see that the claim is valid in this region of the spectrum, for the corresponding expression for the mean intensity in terms of λ and η is

$$J_\nu = \frac{8\pi h\nu^3}{c^2} e^{-\eta} \frac{\sinh a}{a} .$$

In the isotropic limit, $a \ll 1$, $\sinh a/a \approx 1$; hence, it is required that $e^{-\eta} = 6.8 \times 10^{-3}$. This is marginally consistent with the claim that η is large. The approximation becomes better, of course, as ν increases. In the streaming limit, the requirement for the same mean intensity is

$$e^{-\eta} \sinh a/a = 6.8 \times 10^{-3} .$$

Since it always must hold that $\eta > a$, and since $a \gg 1$ in the optically thin regime, we have $e^{-\eta} \sinh a/a \approx e^{-\eta+a}/a \ll 1$, so that this last equality can be satisfied for $\eta, a \gg 1$. However, the approximation breaks down in the Rayleigh Jeans limit, since $B(\nu) \approx 2h\nu^3/c^2(1/x)$, and $x \ll 1$; it would be required that $e^{-\eta} \sinh a/a = 1/x$, and this can be satisfied in neither the thick nor the thin limit.

IV. THE ENTROPY BALANCE EQUATION FOR AN ANISOTROPIC RADIATION FIELD

Let us now investigate the consequences of the above theory for some basic thermodynamic considerations. Below we shall see that an expression for the entropy of an anisotropic radiation field as a functional of the energy and angular occupation numbers can easily be written down, and from this expression a generalized *Gibbs relation* can be derived. This will be used, along with the conservation equations for photon energy density E, number density N, and number flux F_N, to derive an equation of entropy balance for the photon gas. New terms in both the total entropy flux F_S and the entropy generation per unit volume σ appear. These terms are interpreted as contributions to the entropy dynamics resulting from the quantity of "directionality" of the radiation field, defined as F_N/c^2.

The equation of entropy balance is of fundamental importance in the theory of nonequilibrium thermodynamics. Together with a set of phenomenological laws describing irreversible fluxes and Onsager's reciprocal relations, it forms the basis of the study of evolving thermodynamic systems. The lack of uniformity in a system is taken into account in the standard entropy balance equation (see De Groot and Mazur 1984) by the presence of the "thermodynamic forces" appearing in the terms characterizing the internal generation of entropy. These "forces" or "affinities" are gradients of combinations of intensive properties of the system.

However, we note that the anisotropy of the system is not explicitly represented in the classical theory as an independent quantity. In the above theory, however, the anisotropy appears as an *independent extensive thermodynamic property*. We expect, therefore, that a new entropy balance equation can be derived which takes this effect into account.

a) The Generalized Gibbs Relation

The occupation number (1) derived from the statistical constraints of F_N and of N, the total number flux and number density of the photon gas, gives the angular distribution of the nonequilibrium radiation field; under the assumption of statistical independence of the angular and frequency dependences of the distribution function, the energy distribution is given in equation (2) and follows from the constraints of E and N. The expression for the entropy in terms of these occupation numbers follows from microcanonical ensemble theory and can be written

$$\frac{S}{k} = (\eta + \alpha)N + \lambda \frac{F_N}{c} + \beta E + \left[\sum_i g_i^\theta \ln\left(\frac{e^{\eta + \lambda \cos\theta_i}}{e^{\eta + \lambda \cos\theta_i} - 1}\right) + \sum_j g_j^\epsilon \ln\left(\frac{e^{\alpha + \beta\epsilon_j}}{e^{\alpha + \beta\epsilon_j} - 1}\right)\right] .$$

Keeping in mind the restrictions on the ranges of the parameters λ, η, β, and α, i.e., $\eta, \alpha > 0$, $\lambda < 0$, $\beta > 0$, we define

$$\frac{\eta}{\beta} \equiv -\mu_\theta , \quad \frac{\alpha}{\beta} \equiv -\mu_\epsilon , \quad \frac{\lambda}{\beta} \equiv -\gamma_F ,$$

and the expression for S becomes

$$TS = -(\mu_\theta + \mu_\epsilon)N - \gamma_F \frac{F_N}{c} + E + kT\Big[\quad\Big],$$

where the bracket denotes the bracketed term above. Notice that this term looks like the sum of the logarithms of two grand partition functions (see e.g., Pathria 1972), one of which pertains to the angular distribution.

The thermodynamic Gibbs relation describes the total infinitesimal change in the entropy in terms of changes in the *extensive parameters* of the system. For the classical, general case, there parameters are E, V, and N:

$$TdS = dE + PdV - \mu dN .$$

Here, however, we have another extensive parameter, F_N/c. Using the expression for S above, the Gibbs relation must then be

$$TdS = dE + PdV - (\mu_\theta + \mu_\epsilon)dN - \frac{\gamma_F}{c}\, dF_N ,$$

with the pressure given by

$$\frac{PV}{kT} = \sum_i g_i^\theta \ln\left(\frac{e^{\eta + \lambda \cos\theta_i}}{e^{\eta + \lambda\cos\theta_i} - 1}\right) + \sum_j g_j^\epsilon \ln\left(\frac{e^{\alpha + \beta\epsilon_j}}{e^{\alpha+\beta\epsilon_j} - 1}\right),$$

where the second term is the classical expression.

b) The Entropy Balance Equation

Writing the Gibbs relation as an evolution equation for the entropy, we have

$$T\frac{dS}{dt} = \frac{dE}{dt} + P\frac{dV}{dt} - (\mu_\theta + \mu_\epsilon)\frac{dN}{dt} - \frac{\gamma_F}{c}\frac{dF_N}{dt} .$$

The first law of thermodynamics for the radiation states that

$$\frac{dE}{dt} + P\frac{dV}{dt} = \frac{1}{\rho}(\Lambda_E - \nabla\cdot \boldsymbol{F}_E) ,$$

where Λ_E is a net source of photon energy via emission from the matter. Also, the conservation equations for photon number density and number flux are

$$\frac{dN}{dt} = \frac{1}{\rho}(\Lambda_N - \nabla\cdot\boldsymbol{F}_N) , \qquad \frac{1}{c^2}\frac{dF_N}{dt} = -\frac{\bar{\chi}_N}{c}F_N - (\nabla\cdot\boldsymbol{P}_N)_x$$

where $\bar{\chi}_N$ is the number flux mean extinction coefficient and x is the coordinate direction of the flux F_N. Substituting these conservation laws into the Gibbs relation gives

$$\rho\frac{dS}{dt} = \frac{\Lambda_E - \nabla\cdot\boldsymbol{F}_E}{T} - \frac{1}{T}(\mu_\theta + \mu_\epsilon)(\Lambda_N - \nabla\cdot\boldsymbol{F}_N) + \frac{\gamma_F}{T}[\bar{X}_N F_N + c(\nabla\cdot\boldsymbol{P}_N)_x] ,$$

Since we are interested only in one-dimensional problems (for now), we can simplify the analysis by writing

$$(\nabla\cdot\boldsymbol{P}_N)_x = \frac{\partial P_{xj}}{\partial x_j} \equiv \nabla\cdot\boldsymbol{P}_{N,x} ,$$

Using this and the identity $\nabla\cdot aA = \nabla a\cdot A + a(\nabla\cdot A)$, we can write the equation in the form of a conservation law:

$$\rho\frac{dS}{dt} = -\nabla\cdot\left(\frac{\boldsymbol{F}_E + |\mu_\theta + \mu_\epsilon|\boldsymbol{F}_N - c\gamma_F\boldsymbol{P}_{N,x}}{T}\right) - \frac{1}{T^2}(\boldsymbol{F}_E\cdot\nabla T) + \boldsymbol{F}_N\cdot\nabla\frac{|\mu_\theta + \mu_\epsilon|}{T} - c\boldsymbol{P}_{N,x}\cdot\nabla\frac{\gamma_F}{T}$$
$$+ \left(\frac{\Lambda_E + |\mu_\theta + \mu_\epsilon|\Lambda_N + \gamma_F\bar{X}_N F_N}{T}\right).$$

Therefore, the entropy generation per unit volume consists of two kinds of contributions, those arising from interactions with the matter, σ_m, and those arising from gradients in the intensive properties of the radiation, σ_g, and $\sigma_{\rm tot} = \sigma_m + \sigma_g$:

$$\sigma_m = \frac{\Lambda_E}{T} + \frac{|\mu_\theta + \mu_\epsilon|}{T}\Lambda_N + \frac{\gamma_F\bar{\chi}_N}{T}F_N , \qquad \sigma_g = -\frac{1}{T^2}(\boldsymbol{F}_E\cdot\nabla T) + \boldsymbol{F}_N\cdot\nabla\left(\frac{|\mu_\theta + \mu_\epsilon|}{T}\right) - c\boldsymbol{P}_{N,x}\cdot\nabla\frac{\gamma_F}{T} .$$

The entropy flux is

$$\boldsymbol{F}_S = \frac{\boldsymbol{F}_E + |\mu_\theta + \mu_\epsilon|\boldsymbol{F}_N - c\gamma_F\boldsymbol{P}_{N,x}}{T} .$$

and the thermodynamic forces are $\boldsymbol{\Phi}_E \equiv -\nabla T/T^2$, conjugate to the energy flux $\boldsymbol{F}_E$; $\boldsymbol{\Phi}_N \equiv \nabla(|\mu_\theta + \mu_\epsilon|/T)$, conjugate to the number flux $\boldsymbol{F}_N$, and $\boldsymbol{\Phi}_P \equiv -\nabla(\gamma_F/T)$, conjugate to the pressure flux $c\boldsymbol{P}_{N,x}$. With these definitions, the entropy balance equation becomes

$$\rho \frac{dS}{dt} = -\nabla \cdot \boldsymbol{F}_S + \sigma_{\rm tot} ,$$

with

$$\sigma_{\rm tot} = \sigma_m + \boldsymbol{F}_E \cdot \boldsymbol{\Phi}_E + \boldsymbol{F}_E \cdot \boldsymbol{\Phi}_N + c\boldsymbol{P}_{N,x} \cdot \boldsymbol{\Phi}_P .$$

Thus the inclusion of the number flux F_N as an extensive thermodynamic property of the system adds new contributions to the total entropy flux and entropy generation in the entropy balance equation. Specifically, there is a contribution to the entropy flux due to *number momentum* flow, $-c\gamma_F \boldsymbol{P}_{N,x}/T$, a thermodynamic force $-\nabla(\gamma_F/T)$ associated with the *number pressure* flux $c\boldsymbol{P}_{N,x}$, and an entropy source due to the *number momentum* deposition into the matter, $\gamma_F \bar{\chi}_N F_N/T$; in addition, there are terms containing the angular chemical potential μ_θ. The other terms are standard ones, as for diffusion flows of material particles (see De Groot and Mazur 1984).

We offer the following interpretation of the contribution of the number flux to the thermodynamics. In a sense, momentum is "directed inertia." Since F_E/c^2 is the momentum density of the radiation field, F_N/c^2 can be viewed as the "direction density" or the *directionality* of the system with respect to the direction of the net flux. Let us call F_N/c^2 the directionality, and let us consider changes in the entropy due to F_N alone. The Gibbs relation then states that

$$TdS = -\frac{\gamma_F}{c} dF_N .$$

Since $\lambda < 0$ for forward-weighted distributions, $\gamma_F > 0$, and a positive change in F_N results in a *decrease* in the entropy. This makes sense since S is a measure of the *lack of directionality* in the gas, i.e., a measure of isotropy. Of course, this is accounted for in the entropy balance equation. Whereas net outward fluxes of the *directionless* quantities of energy and photon number ($\nabla \cdot \boldsymbol{F}_E$, $\nabla \cdot \boldsymbol{F}_N > 0$) decrease S, a net outward *directionality flux* $c\boldsymbol{P}_{N,x}(\nabla \cdot c\boldsymbol{P}_{N,x} > 0)$ contributes positively to the entropy. This corresponds to an outflow or decrease of directionality in the system. There is also an entropy source within the system due to directionality deposition into the matter, $\gamma_F \bar{\chi}_N F_N/T$, and a new thermodynamic force due to the directionality gradient, $-\nabla(\gamma_F/T)$. We note that the gradients of temperature and chemical potential do not take into account the contribution to entropy changes due to directionality itself but merely provide open channels through which the directionless quantities of energy and number may pass.

Also, it may be useful to comment on the validity of this thermodynamic approach, as well as others, when applied to astrophysical models. Diffusion, in one form or another, has been used with much success in modeling time-independent, optically *thin* radiative transport. Thus the preferred solution to a problem of *space* and of *geometry* has often been the simplest one, when issues of computational effort and cost are considered. A parallel problem of *time* is encountered when dealing with dynamic systems. Quasi-static thermodynamics, along with equilibrium equations of state, may often be applied to "fit" the local behavior of rapidly evolving objects. In this work, however, we deal with the contact between statistical physics and thermodynamics at an intimate level. For instance, at this point is is not certain whether F_N is a statistically constrained quantity (following from classical arguments in ensemble theory) in a rapidly changing, optically thin environment. Tentatively, we suggest that this theory can be applied to slowly evolving systems. In other situations, the equilibrium angular distribution, approximated here to first order (since no moments of higher order than the first are considered) represents the state that the photon gas tends toward; in this case, it is a natural approximation in a "relaxation" approach to finding the angular distribution. Finally, it may be noted that the above formulation ignores some off-diagonal contributions of the pressure tensor, which in a highly anisotropic situation may be at least as important as those considered. Quantitatively, then, the above equation of entropy balance may only be regarded as a rough estimate, although the significance of the terms considered seems clear.

V. NONGRAY GENERALIZATION OF THE STATISTICAL FORMALISM

The statistical formalism above for gray radiative transport can be generalized to a *multigroup* method, in which an independent statistical problem is solved in each group. It is argued below that the treatment more accurately takes into account the "nonlocalness" of the nonequilibrium, anisotropic radiation field.

In most cases of interest, the geometry is *divergent* (e.g., spherical) and/or *directional* (having specific directions of net flux). As mentioned in § II, this encourages us to make a simple approximation of the behavior of the radiation field propagating through a material medium; : there exists a regime of large optical depth in which an equilibrium energy distribution is established, and as small optical depths are approached, the equilibrium function freezes out, with possible dilution. There are two unavoidable complications of this simple picture, however. The anisotropy of the radiation field caused by the variable optical depth scale must somehow be taken into account in the transport; this has been done in §§ II and III. Also, since radiative opacities are frequency-dependent, a major component of the "nonlocalness" of the photon gas is due to the variation of optical depth with frequency, causing the energy and angular profiles at a point in space to be a collection of partial distributions, each originating from a different depth (or range of depths, to be precise) of photon creation.

In contrast to the gray moment equations, the multigroup equations treat the radiation moments in *each group* as independent variables. This allows the energy and angular spectra to be a collection of independent profiles. Hence, to some degree of accuracy, a multigroup treatment accounts for the nonlocalness of the radiation field. With this in mind, we adapt the statistically derived distribution function which is valid for ideally gray transport for use in multigroup calculations.

Recall that the separability of angular and frequency dependences, which leads to the results (1) and (2), follows from the idealization of grayness, and is equivalent to the assumption of statistical independence of the two distributions. It seems quite natural to apply the same formalism to each group in a multigroup scheme. We shall now see that to do this in a physically consistent manner is a simple task.

Let us divide the spectrum of the radiation into G discrete frequency groups. We idealize the opacity as gray *within each group* for the sake of developing the formalism in a consistent fashion (a discussion of evaluation of the group mean opacities follows shortly). In a group denoted by k, where $k = 1, 2, \ldots, G$, we have the following occupation numbers:

$$n_{ik}^{\theta}, \text{ the occupation of angular states; } n_{jk}^{\epsilon}, \text{ the occupation of energy states in the range } (k-1)\Delta\epsilon < \epsilon \le k\Delta\epsilon \,.$$

Also we have the following constraint relations

$$\sum_i n_{ik}^{\theta} = N_k \,, \qquad \sum_i n_{ik}^{\theta} \cos\theta = \frac{(F_N)_k}{c} \,, \qquad \sum_i^{\epsilon_k} n_{jk}^{\epsilon} = N_k \,, \qquad \sum_j^{\epsilon_k} n_{jk}^{\epsilon} \epsilon_j = E_k \,.$$

Now if we allowed changes in the occupation numbers for the *kth group only*, the total number of complexions of the *entire photon gas* would be

$$W_k\{n_{ik}^{\theta}, n_{jk}^{\epsilon}\} = \left[\prod_l \frac{(n_{ik}^{\theta} + g_{ik}^{\theta} - 1)!}{n_{ik}^{\theta}!(g_{ik}^{\theta} - 1)!}\right]\left[\prod_j \frac{(n_{jk}^{\epsilon} + g_{jk}^{\epsilon} - 1)!}{n_{jk}^{\epsilon}!(g_{jk}^{\epsilon} - 1)!}\right].$$

However, we have G sets of such constraint relations, and each set is *independent*. As discussed above, this follows mathematically from the multigroup equations, and physically from the fact that the properties of each group arise from a unique range of depths of photon creation.

Therefore, the total number of complexions of the entire gas would account for all the complexions in each group, and would amount to the product of complexions in each group:

$$W\{n_i^{\theta}, n_j^{\epsilon}\} = \prod_{k=1}^{G} W_k\{n_{ik}^{\theta}, n_{jk}^{\epsilon}\} = \prod_{k=1}^{G}\left\{\left[\prod_l \frac{(n_{ik}^{\theta} + g_{ik}^{\theta} - 1)!}{n_{ik}^{\theta}!(g_{ik}^{\theta} - 1)!}\right]\left[\prod_j \frac{(n_{jk}^{\epsilon} + g_{jk}^{\epsilon} - 1)!}{n_{jk}^{\epsilon}!(g_{jk}^{\epsilon} - 1)!}\right]\right\}.$$

and the entropy of the gas is then

$$\frac{S}{k} = \ln W = \sum_{k=1}^{G} [W_k\{n_{ik}^{\theta}, n_{jk}^{\epsilon}\}] = \sum_{k=1}^{G}\left\{\sum_i \ln\left[\frac{(n_{ik}^{\theta} + g_{ik}^{\theta} - 1)!}{n_{ik}^{\theta}!(g_{ik}^{\theta} - 1)!}\right] + \sum_j \ln\left[\frac{(n_{JK}^{\epsilon} + g_{jk}^{\epsilon} - 1)!}{n_{jk}^{\epsilon}!(g_{jk}^{\epsilon} - 1)!}\right]\right\}.$$

$$S = \sum_{k=1}^{G} [(S_\theta)_k + (S_\epsilon)_k] \,,$$

In order for the entropy to be a maximum, any arbitrary variation of S must vanish; hence,

$$\sum_{k=1}^{G} [(\delta S_\theta)_k + (\delta S_\epsilon)_k] = 0$$

But, of course, each $(\delta S_\theta)_k$ and each $(\delta S_\epsilon)_k$ is independent, and so each must vanish separately:

$$(\delta S_\theta)_k = 0, \qquad (\delta S_\epsilon)_k = 0, \qquad \text{for } all\ k \,.$$

Since we have the above set of four constraint relations for each group, we have a pair of variational equations for each group:

$$(\delta S_\theta)_k - \left(\eta_k \sum_i \delta n_{ik}^{\theta} + \lambda_k \sum_i \delta n_{ik}^{\theta} \cos\theta_i\right) = 0, \qquad (\delta S_\epsilon)_k - \left(\alpha_k \sum_j^{\epsilon_k} \delta n_{jk}^{\epsilon} + \beta_k \sum_j^{\epsilon_k} \delta \eta_{jk}^{\epsilon} \epsilon_j\right) = 0 \,.$$

Therefore, the group-k distribution function is

$$n_k(\epsilon)n_k(\theta) = \left[\frac{1}{e^{\alpha_k + \beta_k \epsilon} - 1}\right]\left[\frac{1}{e^{\eta_k + \lambda_k \cos\theta} - 1}\right].$$

to within an appropriate normalization, in exact analogy to the gray case.

There are really two closure problems to solve when dealing with the radiative moment equations. In addition to finding a scheme for truncating the series of angular moments, we must find a way to evaluate the energy and flux mean opacities which appear in the source terms of the moment equations. The two-parameter energy distribution give us an improved method of approximation for both the gray and multigroup cases. These integrals would be two-temperature means with an additional variable dependence, namely the chemical potential. Also, under the assumption of separable angular and frequency dependences of the specific intensity, κ_E and κ_F become equal. Thus in the gray case, κ_E defines a mean optical depth scale which, in turn, defines the *mean angular distribution* and *mean energy distribution*, as first mentioned in § II.

VI. CONTRIBUTION OF EMISSION TO THE RADIATION FIELD IN THE OPTICALLY THIN REGIME

In the freeze-out, dilution picture mentioned at the beginning of the last section, it seems that contributions to the radiation energy density by emission from matter in the dilution regime are neglected. If LTE is assumed in the optically thin region, then the emissivity decreases proportionately to the opacity. Also, in the static limit the net emission is zero, while in the steady flow limit, it is balanced by expansion of the specific volume. Neither of these cases affects the dilution picture, since in the former instance a chemical potential gradient can account for the increasing dilution with r due to divergent geometry, and in the latter the chemical potential may change with time. Nevertheless, it is informative to obtain some estimate of how much of the radiation depleted by the dilution can be put back into the local energy density by emission.

Consider the transition from the optically thick regime, where $T_m = T_R$, to the optically thin regime, where $T_m < T_R$, to be a fairly sharp one for which we can visualize the radiation field freezing out at a point r_0, outside of which λ, the photon mean free path, is long. We wish to estimate the emission of radiation from matter in LTE in the region $r_0 < r < r_0 + \Delta r$, $\Delta r \ll r_0, \lambda$, and to compare this to the dilution of the radiation field over the same region. Figure 2 describes the situation. In spherical geometry, radiation at temperature T_R has been diluted over Δr by the amount $e^{-\alpha} = [r_0/(r_0 + \Delta r)]^3 \approx 1 - 3\Delta r/r_0$, where we crudely estimate the dilution as the ratio of volumes of everything inside the inner and outer radii. Therefore, we approximate the energy density in the region, neglecting emission, to be

$$E_i = \bar{a}e^{-\alpha}T_R^4 ,$$

where $\bar{a} = 0.92a$ is the Maxwell-Boltzmann radiation constant. This formula, valid in the Maxwell-Boltzmann limit, may be used for these purposes for *any* value of α, since it incurs a maximum error of only 8% (in the Planck limit). Now in the same time interval $\Delta t \approx \Delta r/c$, $(\lambda \gg \Delta r)$, the change in E due to net emission over Δr is

$$\frac{1}{c}\,\dot{E} \approx \rho\kappa\{B[T_m(r_0 + \Delta r)] - E[T_R = T_m(r_0), \alpha]\} . \tag{15}$$

Thus we have

$$\frac{\Delta E_{\rm emi}}{E_i} \approx \rho\kappa c\Delta t\,\frac{aT_m^4 - \bar{a}e^{-\alpha}T_R^4}{\bar{a}e^{-\alpha}T_R^4} ,$$

with $T_m = T_m(r_0 + \Delta r)$. Setting $a/\bar{a} \approx 1$, we have

$$\frac{\Delta E_{\rm emi}}{E_i} \approx \frac{\Delta r}{\lambda}\left[e^{\alpha}\left(\frac{T_m}{T_R}\right)^4 - 1\right] .$$

Now writing $T_m \approx T_R + (dT_m\, dr)\Delta r$, we have

$$\frac{\Delta E_{\rm emi}}{E_i} \approx \frac{\Delta r}{\lambda}\left(1 + \frac{3\Delta r}{r_0}\right)\left(1 + \frac{1}{T_R}\frac{\partial T_m}{\partial r}\,\Delta r\right)^4 - 1 , \tag{16}$$

which, for $\Delta T_m \ll T_m$, becomes

$$\frac{\Delta E_{\rm emi}}{E_i} \approx \frac{\Delta r}{\lambda}\left(\frac{3\Delta r}{r_0} + \frac{4\Delta T_m}{T_R}\right) . \tag{17}$$

Hence, the contribution from emission is of second order in importance, since from the above,

$$E_f \approx aT_R^4 + \Delta E_{\rm dil} + \Delta E_{\rm emi} = aT_r^4\left[1 - \frac{3\Delta r}{r_0} + \left(\frac{3\Delta r}{r_0}\right)\left(\frac{\Delta r}{\lambda}\right) + \left(\frac{4\Delta T_m}{T_R}\right)\left(\frac{\Delta r}{\lambda}\right) + O(3)\right] , \tag{18}$$

where the first-order term is due to dilution alone, and all higher order terms due to emission. This simple analysis shows that

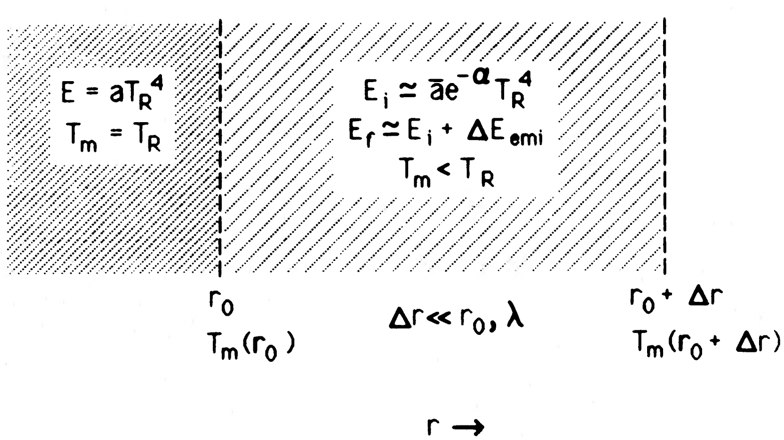

FIG. 2.—Starting with a diluted radiation density E_i in the region Δr, where $\Delta r \ll r_0$, we consider the net emission from matter in LTE over the time interval $\Delta t \approx \Delta r/c$, which contributes positively to the energy density an amount $\Delta E_{\rm emi}$.

emission can be neglected, to first order, in the dilution regime. However, the more significant result that can be inferred from the above is that the dilution scenario, which allows the general application of equation (14), is valid to this level of approximation. Of course, since we usually cannot determine such a transition region *a priori*, the source terms, as in equation (15), shall generally be retained.

VII. ON APPLICATIONS TO HIGH-ENERGY ASTROPHYSICS

The assumptions outlined above have immediate relevance to the study of high-energy flows in astrophysics. Consider, for instance, the study of radiative processes in accretion flows onto a compact object. The simplest possible scenario is a spherically symmetric, optically thin, Bondi-type flow illuminated by a point source of X-rays. Interior to the sonic point the material is in free fall, and the density ρ varies as $r^{-3/2}$. If electron scattering is the dominant transport opacity, then from mass conversation, $M = 4\pi r^2 \rho v$, and the free-fall velocity $v \propto \infty^{-1/2}$, we find that

$$\rho \kappa_{\rm T} = \frac{\alpha}{r^{3/2}} ,$$

where $\kappa_{\rm T}$ is the Thomson opacity and $\alpha^2 \equiv r_s$ is a scale radius outside of which the photon mean free path exceeds unity. For a solar mass neutron star, r_s is given by

$$r_s \approx 1.0 \times 10^5 \left(\frac{\dot{M}}{10^{-9} M_\odot \, {\rm yr}^{-1}}\right)^2 \ {\rm cm} .$$

Since this is actually less than the size of the neutron star, the general prediction is that the region of diffusive radiative transfer ($\lambda \ll r$) is exceedingly compact. By contrast, we note that the sonic radius is given by

$$r_{\rm sonic} \approx 7.5 \times 10^{13} \left(\frac{T_{\rm sonic}}{10^4 \ {\rm K}}\right)^{-1} \ {\rm cm} .$$

The value of r_s given above is an approximate lower limit, for r_s would increase somewhat if a mean photoionization cross section is included. Nevertheless, it would still be true that $r_s \ll r_{\rm sonic}$. Therefore, in a first attempt to calculate the radiative transfer (an "optically thick" calculation) for this situation, we would approximate the transport geometry as an infinite radius "Kosirev sphere" with $n = 3/2$, where the Kosirev transfer problem is defined by

$$\rho \kappa = \frac{\alpha}{r^n}$$

for $n > 1$. Here $r_s = \alpha^{1/(n-1)}$. For r_s, λ/r increases with r. It is known that a general feature of such systems is an extreme, rapidly developing forward peaking of the radiation field as r increases. However, a comparison of the free-fall time scale $t_{\rm ff}$ and the Compton heating time scale t_c shows that $t_c < t_{\rm ff}$ for $r < 10^{10}$ cm; hence, that Compton heating of the flow is significant out to large distances from the central X-ray source. This optically thin heating can significantly affect the nature of the flow, even "choking off" the flow at fairly large radii (see, for instance, Buff and McCray 1974). The global structure of the flow $[\rho(r), T(r)]$ is then determined by the nonlinear coupling between radiation and matter embodied in the equations of radiation hydrodynamics. Because radiative heating is significant at intermediate and low optical depths, this coupling is highly nonlocal. Moreover, we might identify the compact diffusive region as the region of complete Comptonization, in which the photon spectrum is thermalized to a Bose-Einstein distribution with nonzero chemical potential μ. This "thermal dilution" is to be contrasted with the geometric dilution which occurs at larger radii as the photons and matter decouple. Since both of these effects are parametrized by μ, the latter effect is admittedly modeled artificially. The strength of this approach, however, lies in its ability to track the photon temperature independently of the total photon number and energy. This, among other things, allows for an accurate calculation of the local Compton heating rate. Therefore, its use is plausible as long as we do not derive detailed spectra from it.

The above example demonstrates general characteristics of radiative transfer that we would expect to prevail during a gravitationally controlled flow. Since, however, the problem of X-ray illumination of spherical atmospheres with and without hydrodynamic accretion flow is thoroughly studied, this situation per se is certainly not the most viable case for current investigation. Thus a significant application is the transport of neutrinos during the gravitational collapse of an envolved massive stellar core. Calculations by Arnett (1987) show that neutrino heating produced by convective overturn shortly after core bounce provides an efficient mechanism for tapping the enormous reservoir of thermal energy in the hot core, and is the most likely mechanism for a Type II supernova explosion. A time-dependent calculation of the neutrino transfer through the overlying, infalling material is necessary to determine whether this heating can reverse the infall and blow off the mantle and envelope of the star. Arnett has shown that the convective heat engine churns at $\tau_\nu \approx 1$ for the dominant neutrino energies. Hence, as in the previous case, the important radiative transfer occurs above the photosphere. Also, the atmosphere is spherical since $\tau = 1$ occurs deep within the flow.

A particularly exciting application is the case of radiative transfer through the hot coronae thought to exist above the accretion disks surrounding compact X-ray sources. Following the theoretical work of Begelman, McKee, and Shields (1983) and Begelman and McKee (1983), and most recently Melia (1987) a numerical study of X-ray burst induced coronae above accretion disks spurned by the successful priminary model of Melia provides excellent grounds for the development and exploitation of the methods discussed in this paper. Of special interest are the time-dependent shape of the envelope of the corona (defined by the isothermal scale height; see Melia 1987), and the time-dependent scattered component of the radiation field, which is crucial for the formation of a triple-peaked burst of the type observed, for example, from 4U/MXB 1636−53. Although beyond the scope of this paper, a complete model would also entail a two-dimensional adaptation of the transfer techniques; such an adaptation for the type of variable Eddington factor discussed in the next section should be fairly simple to formulate (Minerbo 1978). As is relevant for the physical assumptions presented here, the transfer of radiation in this case occurs in a divergent, optically thin, electron scattering medium.

We see that the situations just described contain the features which warrant the assumptions discussed in this section. It is clear that we need to take into account simultaneously the effects of (1) geometric dilution and (2) forward peaking caused by divergent geometry with steep gradients; (3) thermal dilution caused by noncoherent electron scattering (Compton recoil); and (4) nonlocal coupling caused by radiative heating/cooling above the photosphere.

VIII. ON NEUTRINO TRANSPORT

Since we wish to apply the formalism presented above to the problem of neutrino transport during the gravitational collapse of stars, an important question arises: is the "Bose-Einstein angular distribution" for photons valid for neutrinos as well? In order to answer this question, we must define very carefully what we mean by an "angular state." Previously, this construct was not explicitly defined; nevertheless, in the theoretical development it was fairly clear that an angular state designated by θ_i represented merely a fraction of 4π at the location θ_i on the unit sphere. Therefore, this narrow cone of solid angle encompassed many quantum energy states in phase space oriented in the direction θ_i. It follows that a similar treatment for fermions would result in a *Bose-Einstein* angular distribution. If the Pauli exclusion pprinciple were to hold for the angular distribution of neutrinos, then no two particles could be located at the same θ_i; obviously, this cannot be the case, since any number of particles up to N, each with a different energy ϵ_j (corresponding to a distinct quantum energy state) may occupy the same angular state θ_i. However, the condition of *indistinguishability* of the particles is still valid in the statistical analysis since the interchange of any two particles does not lead to a new state of the gas. Hence, under the initial set of premises outlined in § II, both photons and neutrinos are "bosons" in angle space.

If we wish to investigate this matter with more care, we might attempt to find a set of *antisymmetric* wave functions for the gas, since it is the requirement of antisymmetry which leads to the Pauli exclusion principle. We might first consider the simplest possible case, that of a two-particle gas. For an isotropic Fermi gas, the total wave function Ψ is a linear combination of product terms of the form

$$\Psi^{\epsilon} = \psi_{\epsilon_i}(\boldsymbol{x}_A)\psi_{\epsilon_j}(\boldsymbol{x}_B) ,$$

where $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are the coordinates of the two particles, ϵ_i and ϵ_j denote distinct energy states, and ψ is a single particle wave function. Suppose there are three allowed energy states ϵ_1, ϵ_2, and ϵ_3. Then the only antisymmetric wave functions of the gas are

$$\begin{aligned}
\Psi^{\epsilon}_{12} &= \psi_{\epsilon_1}(\boldsymbol{x}_A)\psi_{\epsilon_2}(\boldsymbol{x}_B) - \psi_{\epsilon_1}(\boldsymbol{x}_B)\psi_{\epsilon_2}(\boldsymbol{x}_A) ,\\
\Psi^{\epsilon}_{13} &= \psi_{\epsilon_1}(\boldsymbol{x}_A)\psi_{\epsilon_3}(\boldsymbol{x}_B) - \psi_{\epsilon_1}(\boldsymbol{x}_B)\psi_{\epsilon_3}(\boldsymbol{x}_A) ,\\
\Psi^{\epsilon}_{23} &= \psi_{\epsilon_2}(\boldsymbol{x}_A)\psi_{\epsilon_3}(\boldsymbol{x}_B) - \psi_{\epsilon_2}(\boldsymbol{x}_B)\psi_{\epsilon_3}(\boldsymbol{x}_A) ,
\end{aligned} \tag{19}$$

For our problem, the statistical independence of the angular and energy distribution suggests that a single particle wave function is of the form

$$\psi_{\epsilon_i\theta_j}(\boldsymbol{x}) = \psi_{\epsilon_i}(\boldsymbol{x})\psi_{\theta_j}(\boldsymbol{x}) = \psi_{ij}(\boldsymbol{x}) ,$$

and we seek a set of wave functions $\Psi^{\epsilon\theta}$ which are linear combinations of the products $\psi_{ij}(\boldsymbol{x}_A)\psi_{kl}(\boldsymbol{x}_B)$, where $i = k$ to satisfy the Pauli principle in energy space, and which are antisymmetric under interchange of x_A and x_B. Let us assume that the neutrinos are "fermions" in angle space. Then, for three allowed angular states θ_1, θ_2, θ_3, there would be three microstates of the two-particle gas described by a set of wave functions Ψ^{θ} of exactly the same form as equation (19) above, constructed from the single-particle wave functions $\psi_{\theta_i}(\boldsymbol{x}_A)$ and ψ_{θ_j} and $(\boldsymbol{x}_B)$. However, the nine possible combinations of the form $\Psi^{\epsilon\theta} = \psi^{\epsilon}\,\psi^{\theta}$, giving all the complexions of the gas, are *symmetric* functions, since they are the products of two antisymmetric functions. Therefore, a *sufficient* condition for the total wave function to be antisymmetric, under the assumption of statistical independence of energy and angle, is that the particles are "bosons" in angle space. In other words, the angular microstates of the two-particle gas are described by the six *symmetric* wave functions of the form

$$\begin{aligned}
&\Psi^{\theta}_{11} = \psi_{\theta_1}(\boldsymbol{x}_A)\psi_{\theta_1}(\boldsymbol{x}_B) , \quad \Psi^{\theta}_{22} = \psi_{\theta_2}(\boldsymbol{x}_A)\psi_{\theta_2}(\boldsymbol{x}_B) , \quad \Psi^{\theta}_{33} = \psi_{\theta_3}(\boldsymbol{x}_A)\psi_{\theta_3}(\boldsymbol{x}_B) ,\\
&\Psi^{\theta}_{12} = \psi_{\theta_1}(\boldsymbol{x}_A)\psi_{\theta_2}(\boldsymbol{x}_B) + \psi_{\theta_1}(\boldsymbol{x}_B)\psi_{\theta_2}(\boldsymbol{x}_A) , \quad \Psi^{\theta}_{13} = \psi_{\theta_1}(\boldsymbol{x}_A)\psi_{\theta_3}(\boldsymbol{x}_B) + \psi_{\theta_1}(\boldsymbol{x}_B)\psi_{\theta_3}(\boldsymbol{x}_A) ,\\
&\Psi^{\theta}_{23} = \psi_{\theta_2}(\boldsymbol{x}_A)\psi_{\theta_3}(\boldsymbol{x}_B) + \psi_{\theta_2}(\boldsymbol{x}_B)\psi_{\theta_3}(\boldsymbol{x}_A) .
\end{aligned} \tag{20}$$

The 18 possible combinations of equation (19) with equation (20) would then give all the complexions of the gas.

We make the tentative conclusion that a gas of identical, indistinguishable particles with no restriction on the occupation of states is a valid model for the angular statistical problem. Thus in forthcoming work the results for the Bose-Einstein angular distribution shall be considered valid for neutrino transport. Accordingly, as an initial attempt to improve the treament of neutrino transport during gravitational stellar collapse, Arnett (1987) has employed the interpolation $f_{\rm E} = \frac{1}{3} + \frac{2}{3}(f_{\rm stream})^2$ in a detailed numerical simulation of an evolved, collapsing stellar core. The transport code is stable, as fast as the flux-limiting techniques used previously, and has yielded encouraging results. This, however, precludes an especially interesting possibility, that of a *degenerate angular distribution*, characterized by an "angular Fermi level" θ_c, above which the particles are distributed uniformly in angle, and below which no particles are found. The transition from optical thickness to thinness is envisioned in phase space as a radially symmetric array of momentum vectors from which we remove larger and larger cones of vectors, centered at $\cos\theta = -1$, each cone having an opening angle θ_c. The variable Eddington factor associated with this picture is interesting since f_E initially goes *below* $\frac{1}{3}$ as we move away from the isotropic limit, reaching a minimum value of $\frac{1}{4}$ at $\cos\theta_c = -\frac{1}{2}$. This relation $\theta_c = 0$ is a special instance of the two-stream limit, with $I_- = 0$, and where $K/J = \frac{1}{3}$. The normalized angular distribution is easy to derive; the result is

$$\Phi(\theta) = \frac{1}{2\pi}\frac{1}{1-\cos\theta_c}\,.$$

These considerations raise an interesting, although perhaps obvious point: a degenerate gas, by definition, must be isotropic. Yet we can imagine situations where nondegenerate particles are distributed only in a fairly well defined cone of solid angle, and resemble an angularly degenerate condition: an example is neutrino transport through a cold spherical medium with a central source and strongly forward scattering. In such a state, the neutrinos can be collimated no more than that defined by a "packing angle," θ_p, given by

$$\frac{2}{h^3}\frac{4\pi p^2 dp}{e^{\beta\epsilon+\alpha}+1} = \frac{2}{h^3}\,p^2 dp 2\pi\int_0^{\theta_p}(-d\cos\theta)\,,$$

or

$$\cos\theta_p = 1 - \frac{2}{e^{\epsilon\beta+\alpha}+1}\,,$$

which defines the minimum solid angle into which nondegenerate fermions can be packed or redistributed as a function of ϵ. Of course, this particular definition of θ_p also assumes that the particles are separately in a Fermi-Dirac energy distribution. This assumption is central to the whole theoretical framework presented thus far. The above arguments, however, seem to rule out the possibility of angular fermions. These arguments follow directly from the separability assumption. Therefore, the theory does not predict the existence of an angular Fermi distribution. Nevertheless, the above discussion motivates us to consider the apparent physical dilemma which arises when neutrinos, propagating from an optically thick source to small optical depths, are "redistributed" into the forward directed energy states in phase space. Using the example of a nondegenerate Fermi distribution, we can say that this limitation is expressed in terms of θ_p.

It is difficult to say whether this effect can be demonstrated by analysis of state-of-the-art neutrino transport calculations. No such analysis is attempted here. Nevertheless, we note that the work of Schinder and Shapiro (1982) is relevant for this purpose. The authors solve the angle-dependent equation of transfer for a static neutrino atmosphere, employing a complete linearization scheme presented by Mihalas (1978). Since the calculations do not employ variable Eddington factor techniques, the methods presented in this paper cannot be directly compared with those of the authors. However, since Fermionic blocking effects (stimulated absorption) have been included in these frequency-dependent calculations, a search for angle-dependent blocking might seem possible; the mild forward peaking in these planar models may be an obstacle to this search.

IX. SUMMARY AND CONCLUSION

We have presented a theory of photon transport which may be adapted for use in radiation hydrodynamics calculations. We have devised a method from basic principles, with an emphasis on assembling a coherent physical structure rather than on computational efficiency. The essence of this approximation is the inclusion of the angular moments in a more generalized set of thermodynamic variables for the radiation field. A set of *extensive thermodynamic quantities* which takes into account the anisotropic, nonequilibrium nature of the photon gas is thereby obtained. These quantities are then used as macroscopic *constraints* in the construction of a microcanonical statistical ensemble for the photon gas, from which the most probable distribution function, or specific intensity, follows, In accomplishing this, we have made the assumption that the distribution of the photons over their energy states is statistically *independent* of their distribution over angle. This is justified because frequency-dependent opacities seldom correlate photon energies and angles *locally* (e.g., isotropic cross sections), but only do so *globally*, over a distance of many mean free paths. Therefore, the assumption is valid for the entire energy spectrum if the opacity is gray, and is a good approximation for each energy group in a nongray, multigroup method.

Only the lowest order angular information has been incorporated into our approximation, viz. that contained in the first-order angular moment. Better approximations, both for the closure scheme and for the generalized thermodynamic relations, might be obtained by considering higher orders as well. Each of these schemes has the property that all of the moments, both in number and energy, can be determined only by the inclusion of the *number conservation equation* in the set of equations to be solved.

We have established a theoretical connection between variable optical depth radiative transfer and the science of nonequilibrium thermodynamics. An equation of entropy balance, which follows from a generalized Gibbs relation, has been derived for an anisotropic radiation field. Besides including the nonzero chemical potential for the photons in the entropy generation, this equation has introduced new constributions to the entropy generation and the entropy flux due to the anisotropy. Specifically, entropy can be generated by sources, sinks and gradients in the *directionality* of the radiation field, F_N/c^2, and entropy can flow into a system via the directionality *flux*, cP_N.

The practical utility of this theory is apparent from an investigation of the Eddington factors associated with limiting cases of the angular distribution and remains to be demonstrated in detail. Plots of the variable Eddington factor $f_E = K/J$ versus the variable streaming factor $f_{stream} = H/J$ show that the curves for the limits of small and large angular chemical potential η bound a region of space in which acceptable "Eddington trajectories" lie in this approximation. This is not to say that trajectories outside of the allowed space are to be automatically or wholly rejected; rather, the value of this plot is that it provides some physical underpinning for variable Eddington factors which do lie within the region, thus providing a criterion for choosing among various schemes. Until now, some of these curves have been viewed as totally ad hoc prescriptions. In subsequent work, the theory shall be tested by using it to solve a standard problem, the gray atmosphere in planar and spherical geometry.

The advantage of a moment method in radiation hydrodynamics problems is that it allows for a proper coupling of the radiative energy and momentum to the hydrodynamic flow, while forgoing the detailed angular information contained in the transfer equation which may not be of immediate interest. This information, however, may not be merely discarded or forgotten. In a moment equation for a certain order, it is plainly expressed through the appearance of higher order moments in the same equation; the same angular information is contained in the set of moment equations, carried out to infinite order. Hence, practically speaking, it is a chosen method of truncation or closure of the equations which affords the desired computational simplicity; in order to make a decent choice, it then becomes necessary to know *enough* angular information to close the equations at the desired order, to the desired accuracy.

Unfortunately, in astrophysics, the closure problem has often been viewed as a secondary one, and so there has been a tendency to emphasize efficiency of the closure scheme at the expense of its physical consistency, meaning, or content. Consequently, many schemes have been devised which utilize only the angular information at the extremes of diffusion and streaming, with the idea that the intermediate regime can be left to an ad hoc interpolation. However, we have taken the view that there is much physics to be learned in the systems with anisotropic, nonequilibrium radiative transport, from which more efficient ways of doing radiation hydrodynamics in complex astronomical environments may be discovered.

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