Astron. Astrophys. 184, 1–6 (1987)

ASTRONOMY AND ASTROPHYSICS

Gravitational lensing effect on the fluctuations of the cosmic background radiation

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Received November 4, 1986; accepted March 13, 1987

Summary. The theory of optics in general relativity ensures that only redshift terms in gravitational lensing can induce fluctuations in a strictly uniform background. In this paper we investigate the possible consequences of randomly distributed deflectors on an initially non-uniform background. Starting with the conservation of the specific intensity, we find a general expression giving the power spectra of the perturbed background. This allows us to show that the variance of the fluctuating part of the background is conserved by any arbitrary physical deviation field. Using a model of the gravitational deviation field developed by Blandford and Jaroszynski, we apply our result to the cosmic background radiation, and we show that the angular fluctuations law could be modified at small angular scales. Finally we give a rough estimate of this effect. It appears that its strength depends strongly on the total mass present in a clumpy form and on the evolution of the correlation function at small scales. The amplitude of the distortion of the fluctuations law is expected to be small, but could be identified by its angular scaling. This provides a potential tool to get direct information on galaxy formation in the non-linear regime.

Key words: cosmology – cosmic background radiation – gravitational lenses

1. Introduction

Gravitational lensing on an uniform background and its application to the Cosmic Background Radiation (CBR) have been studied by several authors. It was early shown by Etherington (1933) in a different context that in this case, the only possible observational effect of lensing comes from the frequency shift of photons. Sachs and Wolf (1966) obtained the contribution to the fluctuations of the CBR from first order perturbations in the metric in a flat universe. The scalar term arises only from inhomogeneities located on the recombination shell (Doroshkevich and Zeldovich, 1983), i.e. the lenses located between us and the recombination surface do not give any first order contribution to the fluctuations of the CBR. The analysis of the effect of second order terms has been also performed (Rees and Sciama, 1966; Dyer, 1976; Nottale, 1984) and Nottale (1984) obtained the details of the different contributions to redshift effect in the vacuole model.

However, deflections can also lead to observable effects on the CBR Birkinshaw and Gull (1983) obtained that a moving lens can induce in principle an anisotropy and Mitrofanov (1981) pointed out that a single lens could lead to an effect in the case of a non-uniform background. The observational consequences of gravitational lensing are generally very interesting because they are sensitive to the whole mass present in the deflector. This makes hope to get information on dark matter.

In this paper our aim is to analyse the possible consequences of gravitational deviations on the angular law of the fluctuations of the CBR. This is possible because the CBR is not expected to be exactly uniform: since fluctuations on the mass distribution are needed at the time of recombination in order to allow for galaxy formation, this must leave some imprints on the CBR. Recent calculations of the fluctuations impose severe constraints on galaxy formation (see for instance N. Vittorio and J. Silk, 1985). In these calculations it is assumed that nothing happened after recombination, while clearly gravitational lensing effects are necessarily present. Here we want to investigate the mean effect of random deviations, and try to evaluate the order of magnitude of the induced correction.

In the forthcoming we will treat the temperature of the CBR as a random function $T(x)$, where x is a generic point on the sky. We do not expect the gravitational lensing to have any influence on large scales fluctuations, so it is justified to assume x to be a point on a plane. We will also assume a power spectrum for the fluctuations of the CBR of the form:

$$P(k) = \left|\frac{\delta \hat{T}}{T}(k)\right|^2 \propto \exp(-\theta_0^2 k^2/2) \quad (1)$$

The corresponding angular fluctuation law is given by:

$$\frac{\delta T}{T}(\theta) = [C(0) - C(\theta)]^{1/2} \propto (1 - \exp(\theta^2/2\theta_0^2)) \quad (2)$$

We have arbitrarily normalized the amplitude of the correlation function because we are not interested in its value here. In the same way angles are expressed in units of $\theta_0/\sqrt{2}$. Then the power spectrum is merely:

$$P(k) = \exp(-\tfrac{1}{2}k^2) \quad (3)$$

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Actually in our analysis, the main assumption is that small scale fluctuations obey the following law:

$$\frac{\delta T}{T}(\theta) \propto \theta \quad \text{when } \theta \lesssim \theta_0$$

and

$$\frac{\delta T}{T}(\theta) \sim \text{const} \quad \text{when } \theta \gg \theta_0 \tag{4}$$

We do not loose any generality as this corresponds to the usual behavior for the primordial fluctuations law. The angle θ_0 is expected to be of the oder of $\sqrt{\Omega_0}10'$ in the standard scenario. The value of θ_0 depends on the cosmical scenario. For instance a reionization (Davis, 1984) or a cosmological constant (Blanchard, 1984a) can significantly change its value. Such a possibility is not investigated in this paper.

In the second section we give the general formula which allows to calculate the power spectrum of the perturbed image from the initial one (the details of the demonstration of this formula are left in the Appendix). The apparent displacement of the position of point initially located at x due to gravitational bending is noted $\lambda(x)$. As we will not take into account redshift terms in our analysis, the conservation of the specific intensity tells us that the apparent temperature in the sky of the CBR, $\tilde{T}(x)$, i.e. the perturbed image in the direction x is the temperature that one would have seen at the point $x + \lambda(x)$ in the absence of any lensing. This reads:

$$\tilde{T}(x) = T(x + \lambda(x)) \tag{5}$$

The general formula giving the power $\tilde{P}(k)$ of $\tilde{T}$ is (7). This allows us to show that the variance of the perturbed process is unchanged whatever $\lambda(x)$ is. However, in practice, one needs all the statistics of the deviation field to calculate $\tilde{P}(k)$, so it is useful to assume the deviation field to be gaussian. This leads to a much simpler formula given by (9) which is used in the two last sections.

In the third sections, in order to analyse the lensing effect on the CBR we use a simple model of the deviation field, which is due to Blandford and Jaroszynski (1981). They found that the mean relative deviation σ_θ between two rays separated by an angle θ is proportional to:

$$\sigma_\theta \propto (\theta/1'')^{(3-\gamma)/2} \Omega_c S_\gamma^{1/2}(Z, \Omega_0, N) \tag{6}$$

where γ is the index of correlation function of the whole clustered mass, s_γ is a function which depends on γ and on the redshift z of the source from which the rays are coming and on the total mass density Ω_0 and on an index N related to the evolution of the correlation function. Using this result with $\gamma = 2.$, we find that small angular fluctuations $\delta\tilde{T}/T(\theta)$ of the CBR scale as $\theta^{1/2}$ rather than as θ (see Eq. 4). This shows that gravitational lensing can affect the statistics of the fluctuations at small scales in a characteristic way. In the fourth section we try to give an estimation of the numerical value of the amplitude of this effect. This amplitude readily depends on the level of the fluctuations present in a clumpy form, but it depends also on the evolution of the correlation function in the nonlinear regime. A contribution of a few percent of the large scale amplitude is possible on subarcmin scales, and then angular dependance makes it quite easy to identify. Even if such an observation is far from present possibilities, this indicates that such an effect will not be impossible to find and to identify in the future.

2. The power spectrum of the perturbed image

Standard calculations of the primordial fluctuations assume that intervening material between us and the recombination does not affect the CBR. The temperature on the sky then appears as a random function with a correlation function given in a first approximation by (2). However we have seen that, when gravitational lensing is taken into account, the temperature is:

$$\tilde{T}(x) = T(x + \lambda(x)) \qquad \text{See (5)}$$

From there we wish to compute the correlation function of $\tilde{T}(x)$ starting from the statistical properties of $T(x)$ and $\lambda(x)$. This is equivalent to know the power spectrum $\tilde{P}(u)$ of $\tilde{T}(x)$ from $P(k)$. In the Appendix we show that the power spectrum $\tilde{P}(u)$ of $T(x + \lambda(x))$ is given by:

$$\tilde{P}(u) = \frac{1}{4\pi^2} \int dk\, P(k) [FTC_\theta(k)]_{(u-k)} \tag{7}$$

where the Fourier Transform FT is performed relatively to the variable θ. In this formula $C_\theta(k)$ is the characteristic function of the relative deviation field $\Lambda_\theta(x)$:

$$\Lambda_\theta(x) = \lambda(x + \theta) - \lambda(x) \tag{8}$$

with:

$$\langle \Lambda_\theta^2 \rangle = \sigma_\theta^2$$

Equation (7) shows that the distortion in the transformation (5) does not depends directly on the statistics of $\lambda(x)$, but only on the statistics of the relative deviation field $\Lambda_\theta(x)$. This is satisfying because strictly speaking $\lambda(x)$ is not a well defined quantity as it is not invariant under a transformation of coordinates, while Λ_θ can be obtained by integrating the geodesic deviation equation along the rays and is therefore a well-defined quantity.

From (7) it is now easy to show that the variance of $\tilde{T}(x)$ does not depend on the statistics of $\lambda(x)$:

$$\tilde{C}(0) = \frac{1}{4\pi^2} \int \tilde{P}(u)\, du = \frac{1}{4\pi^2} \int \frac{dk}{4\pi^2} P(k) \int du\, [FTC_\theta(k)]_{(u-k)}$$
$$= \frac{1}{4\pi^2} \int P(k) \frac{dk}{4\pi^2} 4\pi^2 C_0(k) = \frac{1}{4\pi^2} \int P(k)\, dk = C(0)$$

This result contains the specific case of a uniform background: in such a case C is the delta function, and the variance trivially remains zero by (5).

Even if (7) is a rather short formula, it is not very easy to handle with: the characteristic function C_θ depends on the whole series of the moments of the field Λ_θ, something which is generally out of our knowledge. To get a more tractable formula it is convenient to assume the random field Λ_θ to be isotropic and gaussian. The characteristic function is then:

$$C_\theta(k) = \exp(-\tfrac{1}{2} k^2 \sigma_\theta^2)$$

The power spectrum $\tilde{P}(u)$ now reads:

$$\tilde{P}(u) = \frac{1}{4\pi^2} \int dk\, P(k) [FT(\exp(-k^2 \sigma_\theta^2))]_{(u-k)} \tag{9}$$

(One must keep in mind that in this formula the Fourier transform is performed on the variable θ). This last formula will now allow us to compute the effect of random deviations on the fluctuations of the CBR.

3. Gravitational lensing model

The distortion of the image of distant objects by gravitational bending of the light rays has been considered by Gunn (1966), by Hameury et al. (1981) and by Blandford and Jaroszynski (1981). They found that images of distant sources ($z \approx 1.$) could have a relative distortion of 0.5″. When one looks at the microwave background, one can hope the strength of gravitational bending to be larger, because of the high redshift of the source. For instance, Mitrofanov(1981) proposed to use a cluster as a lens to look at small scales fluctuations in the CBR. The consequences of random deviations are less easy to guess: We have already shown that the variance is unchanged and one can even wonder if such an effect could lead to any observable consequences. Equation (9) will be used here to compute this last effect. To this aim we need a model for the random gravitational deviations of rays reaching us from the recombination shell. Here we will use the model developed by Blandford and Jaroszynski (1981). The main assumption is the evolution and the shape assumed for the correlation function of the clustered component of the mass:

$$\xi(r) = \frac{1}{(1+z)^{N-6}}\left(\frac{r}{r_0}\right)^{-\gamma} \tag{10}$$

the index N being an evolution index.

Such a simple model for the evolution is questionable. Here we need a reliable model for the evolution of the correlation over a large range of time and the form (10) is not trivially satisfied both in the linear and in the non-linear regime. However we will use (10) as a first guess. It will allow us to understand how evolution can affect the results, more complicated evolution laws $f(z)$ being not difficult to include in the calculation.

According to Blandford and Jaroszynski, the mean square relative deviation between two rays coming from sources located at a redshift Z^* is given by

$$\sigma_\theta(Z^*) \sim 3''(\theta/1'')^{0.6}\Omega_c S^{1/2}(Z^*, \Omega_0, N) \tag{11}$$

(γ being taken equal to 1.8) Ω_c is the clustered component of the mass in the universe in units of the critical density. The quantity S is given by the integral:

$$\frac{1}{\mathscr{D}(Z^*,0)}\int_0^{Z^*} \frac{dZ'\,\mathscr{D}(Z^*,Z')\mathscr{D}^{3-\gamma}(Z',0)(1+Z')^{N-2}}{(1+\Omega_0 Z')^{1/2}} \tag{12a}$$

$\mathscr{D}(Z_2, Z_1)$ being the angular distance of an object located at Z_2 seen by an observer at the time corresponding to Z_1.

In order to apply their model to the CBR, we cannot assume Eq. (1) to apply from now up to the recombination epoch. This comes from the fact that (10) could not hold over such a large redshift interval. For instance an index N larger than 3 corresponds to a contracting period, and therefore this can only be relevant in the non-linear regime. We expect the contributions of deflectors located between $Z_{\rm rec}$ and some redshift Z^* corresponding more or less to the galaxy formation epoch not to be important. The value of S then is:

$$S(Z_{\rm rec}, Z^*, N) = \frac{1}{\mathscr{D}(Z_{\rm rec},0)}\int_0^{Z^*} \frac{\mathscr{D}(Z_{\rm rec},Z')\mathscr{D}^{3-\gamma}(Z',0)(1+Z')^{N-2}\,dZ'}{(1+\Omega_0 Z)^{1/2}} \tag{12b}$$

For practical computations, we have chosen $Z_{\rm rec} = 1067$ (Jones and Wyse, 1985), four values of N ($N = 2,3,4,5$) and two values of the present total density mass Ω_0 ($\Omega_0 = 0.1, 1.$). It appears that S is not very sensitive to Ω_0 (Figs. 1), but strongly depends on N and therefore on the scenario of galaxy formation. The fact that N could not be taken larger than 3 all the time is rather clear from the picture. However such a value of N may hold from the beginning of the galaxy formation. This suggests that a reasonable value of S could be larger than the value corresponding to $N = 3$. In any case we expect S to be in the range 0.1–10. This leaves us with large uncertainties on the amplitude of σ_θ.

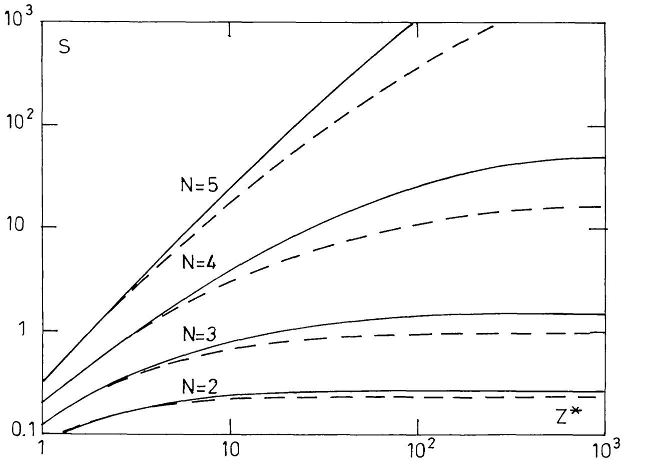

Fig. 1. Values of S as a function of Z^* from Blandford and Jaroszynski (see also (12a) in the text). The values of the index N (see Eq. (10)) are 2,3,4,5. The full lines correspond to $\Omega_0 = 0.1$ and the dashed ones correspond to $\Omega_0 = 1$

To carry out further our calculation, let us take a slightly different law for σ_θ:

$$\sigma_\theta^2 \sim a\theta \quad \text{for } \theta \le \theta_1$$

and

$$\sigma_\theta^2 \sim l^2 \quad \text{for } \theta \ge \theta_1 \tag{13}$$

The divergence of σ_θ in (11) at larges angles is clearly meaningless: the relative deviation between two rays is expected to be nearly constant at large separations. As we mentioned the index γ comes from the evolution law (11), and it was assumed that the slope of ξ did not change with time. A different choice for the evolution of ξ would probably lead to a different value of γ. The uncertainties on γ remain quite important.

Assuming (13) we can approximate directly the value of the quantity $\exp(k^2\sigma_\theta/2)$ we need to compute the power spectrum $\tilde{P}$ by using (9):

$$\exp(-k^2\sigma_\theta^2/2) = A\exp(-ak^2\theta/2) + \exp(-k^2 l^2/2) \tag{14}$$

where:

$$A = 1 - \exp(-k^2 l^2/2)$$

With this fit we can now directly get the Fourier transform of this quantity:

$$FT[\exp(-\tfrac{1}{2}k^2\sigma_\theta^2/2)](\omega) = e^{-k^2 l^2/2}4\pi^2\,\delta(\omega) + \frac{ak^2}{[\omega^2 + (ak^2/2A)^2]^{3/2}} \tag{15}$$

The perturbed power spectrum can now be expressed as:

$$\tilde{P}(u) = P(u)\exp(-u^2l^2/2) + \frac{1}{4\pi}\int \frac{dk\, ak^2 P(k)}{[(u-k)^2 + (ak^2/2A)^2]^{3/2}} \tag{16}$$

Using (3) as the power spectrum of the fluctuations of the CBR, we obtain:

$$\tilde{P}(u) = \exp(-u^2(1+l^2)/2) + \frac{1}{4\pi}\int_0^{2\pi} d\theta \int_0^{+\infty} \frac{dk\, ak^3 \exp(-\frac{1}{2}k^2)}{\left[u^2 + k^2 - 2uk\cos\theta + \left(\frac{ak^2}{2A}\right)^2\right]^{3/2}} \tag{17}$$

The numerical results allow us to discuss the qualitative features of $\tilde{P}(u)$: at small values of u, $\tilde{P}$ is mostly unchanged because $\exp(-u^2l^2/2) \sim 1$ while the integral in (17) is small. On the other hand, for large values of u the spectrum scales as u^{-3}. A typical spectrum is presented in Fig. 2. The behavior at large u is not very surprising, indeed. Even if the integral in (17) is not an exact convolution, the main contribution to $\tilde{P}(u)$ at large u comes from $P(k)$ at small values of k.

We have computed the power spectrum $\tilde{P}$ for a large set of the parameters l and a (Blanchard, 1984*b*). It appears that the amplitude of the tail at large u depends very weakly on l. This reflect the fact that l is a cutoff in the law (11) at large angular separation where the relative effect of the deflections is small. To understand more deeply the results we have plotted the resulting amplitude of the fluctuations of the CBR $\delta\tilde{T}/T(\theta)$. This is to be compared to the usual law given by (2). This represents the main output of our work. In Fig. 3, we have plotted the curves corresponding to different values of l ($l = 2.$, 0.5, 0.125, 0.031, 0.008) while a is kept constant ($a = 0.25$). From this picture it is clear that the effect of l is tiny unless l is much smaller than one.

We have also studied the effect of a which is a very sensitive parameter. In Fig. 4, l has been kept constant ($l = 0.5$) while a takes different values ($a = 0.05$, 0.1, 0.2, 0.4, 0.8). It can be seen that at small scales, $\delta\tilde{T}/T$ scales as $\theta^{1/2}$ rather like θ, as for the usual law. In addition the small scale behavior depends clearly on $a^{1/2}$. This allows us to fit the new fluctuations law as follows:

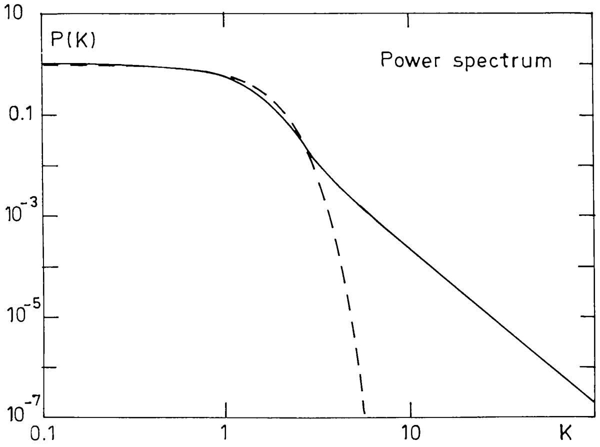

Fig. 2. An example of the perturbed spectrum (17) for ($a = 0.2$, $l = 0.5$). The dashed line corresponds to the initial spectrum

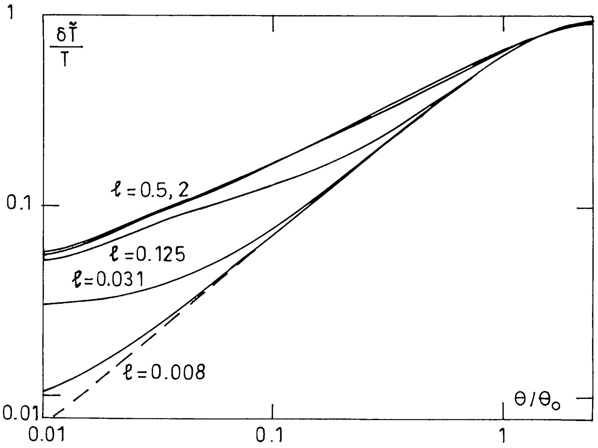

Fig. 3. The temperature $\frac{\delta\tilde{T}}{T}$ as a function of θ/θ_0 is plotted for different values of l($l = 2$, 0.5, 0.125, 0.031, 0.008) ($a = 0.25$). The maximum amplitude of temperature has been normalised to unity

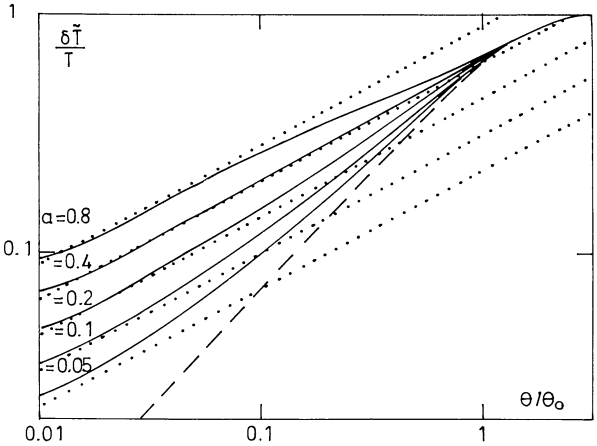

Fig. 4. l has been kept constant ($l = 0.5$) while a varies ($a = 0.05$, 0.1, 0.2, 0.4, 0.8). The point lines are the corresponding curves $a^{1/2}\theta^{1/2}$ for each value of a

– for small values of a (i.e. $2a \leq 1.$):

$$\begin{aligned} &\delta T/T(\theta) \sim a^{1/2}\theta^{1/2} \quad \text{for } \theta \leq \theta_* \\ &\delta T/T(\theta) \sim \delta T/T(\theta) \quad \text{for } \theta \leq \theta_* \end{aligned} \tag{18a}$$

with $\theta_* \sim 2a$

– for large values of a:

$$\begin{aligned} &\delta T/T(\theta) \sim a^{1/2}\theta^{1/2} \quad \text{for } \theta \leq \theta_* \\ &\delta T/T(\theta) \sim 1. \quad \text{for } \theta \leq \theta_* \end{aligned} \tag{18b}$$

with $\theta_* \sim 1/a$.

In the forthcoming discussion we will use slightly different formulae to take into account the fact that γ must be equal to 1.8 rather than 2.

4. Discussion

The spectrum (1) essentially means that there is no power in wavenumbers greater than $1/\theta_0$. Even if (17) is not a convolution the amplitude of $\tilde{P}(u)$ at large u comes from the tail of the function which multiplies the power spectrum in the integral. This leads $\tilde{P}(u)$ to scale as $FT(\sigma^2)$ at large u. Coming back to $\delta\tilde{T}/T(\theta)$, this implies that $\delta\tilde{T}/T$ will scale at small θ as σ_θ does. Therefore it seems quite reasonable to think that $\delta\tilde{T}/T$ will scale like $\theta^{0.6}$ if (11) is used instead of (13). In the forthcoming we will assume that (11) does lead to a fluctuaion law:

$$\delta\tilde{T}/T(\theta) \sim a^{1/2}\theta^{0.6} \quad \text{for } \theta \leq \theta_* \tag{19a}$$

and

$$\delta\tilde{T}/T(\theta) \sim \delta T/T(\theta) \quad \text{for } \theta > \theta_* \tag{19a}$$

with $\theta_* \sim (2a)^{5/2} \sim 2.4a^{1.25}$

In the same way, if $2a \geq 1$, we will assume:

$$\delta T/T(\theta) \sim a^{1/2}\theta^{0.6} \quad \text{for } \theta \leq \theta_* \tag{19b}$$

$$\delta T/T(\theta) \sim 1. \quad \text{for } \theta \leq \theta_* \tag{19b}$$

with $\theta_* = 1/a$.

The difference between (18) and (19) is small in regard of the uncertainties on the value of a and on the assumed law (11). Therefore this difference must not be taken too seriously. However in the present discussion we will use (19) to evaluate the numerical value of the effect.

As it will be seen, a is not expected to be larger than 0.1, so that only (19a) will be useful in numerical estimations. It must be noticed that we assume in (19a) that $\delta\tilde{T}/T$ is not perturbed for $\theta > \theta_*$, while it is in fact. This means that we underestimate the scale at which an effect is present, but on scales of the order of θ_* or larger the perturbation is no more a power law. This leaves little hope to the possibility of detecting such an effect on large scales.

The angle θ_0 in (1) is taken of the order of $\sqrt{\Omega_0}10'$ since we assume that no reionisation occurred and that the cosmological constant vanishes. In our units the value of a is obtained from (12):

$$a^{1/2} \sim 0.24\frac{\Omega_c}{\Omega_0^{0.2}}S^{1/2}(Z_{\rm rec}, Z^*, \Omega_0, N) \tag{20}$$

Let us now apply our results to specific evolution scenarios. As we mentioned in Sect. 3, large uncertainties lie in S and in Ω_0. This implies that we are not able to give a firm prediction on the amplitude of the effect. However we can get the typical numerical value of $a^{1/2}$ in two different scenarios to show how sensitive to the history of galaxy formation the effect is. Let us have a look at a standard hierarchical scenario with $\Omega_0 \sim 0.1$–0.3, starting from an isothermal or cold dark matter spectrum. In this scenario the structures are formed quite early at a redshift larger than 10, so that the evolution law (10) is expected to hold over a large range of redshifts. Therefore S is expected to be of the order of one or larger, and the corresponding value of a can be of the order of 0.1 or larger as $\Omega_c \sim \Omega_0$ in this kind of scenario.

Now, let us have a look at the recent "biased galaxy formation" scheme (Bardeen, 1985; Kaiser, 1985). In this scenario the preferred value of Ω_0 is 1, but this needs the mass distribution not to follow the light distribution:

$$\xi_{\rm mass} = \Omega_{\rm app}\xi_{\rm light} \quad \text{with } \Omega_{\rm app} \sim 0.2$$

The high value of Ω_0 implies that galaxy formation must have occurred very recently, and we expect S to be smaller than one. For instance if we choose $S \sim 0.1$, we obtain:

$$a \sim 0.001\text{–}0.01$$

However the values of a are not to be taken as firm values because of the large uncertainties. In the following we will keep these values as typical of the range of possibilities for the parameter a in a realistic scenario. We can now evaluate the value of θ_* for which the effect is expected to be identifiable. In the first case:

$$\theta_* \sim 2.4a^{1.25} \sim 30''$$

while in the second case θ_* is of the order of $2''$.

Clearly in the second case the gravitational lensing has virtually no effect on the CBR fluctuations, while in the former a noticeable difference is expected on subarcmin scales. Let us now assume that the fluctuations do have an amplitude of $5\,10^{-5}$ on scales larger than $10'$ (such a value is marginally consistent with the observations of Uson Wilkinson, 1985). In such a case the excess of fluctuations due to the lensing could be estimated as:

$$\text{for } \sim 0.01 \quad (\delta\tilde{T}/T)_{\theta \leq 2''} \leq 10^{-7}$$

while

$$\text{for } \sim 0.1 \quad (\delta\tilde{T}/T - \delta T/T)_{\theta < 30''} \leq 10^{-6}$$

(taking $\gamma = 2.$ leads to scales that are two times larger).

As we investigate lensing effects that only make a correction to the usual behavior, the resulting numerical values are very small indeed even in a favourable case. However the difference is of the order of the level of the fluctuations on these scales, and the specific scaling law ($\delta\tilde{T}/T \sim \theta^{0.5\text{-}0.6}$) may help to identify such an effect from different other effects (as radio-sources contamination). The main problem in our analysis remains the uncertainties lying in the estimation of a and γ.

5. Conclusion

In this paper we have investigated the statistical effect of gravitational random deviation on the fluctuations of the CBR. The first result we obtained is the general formula that allows one to calculate the correlation of the function perturbed by the means of (5). This permits also to show that the variance of the function is unchanged. When applied to the CBR our results show that the amplitude of the fluctuations at large scale could not be changed by gravitational lensing whatever its strength is. We also obtained that sub-arc minute fluctuations can be significantly altered, on this scale the amplitude of the primordial fluctuations is expected not to be larger than 10^{-6}, but an excess of fluctuations of the same order can be present. Even if it is far of present observational possibilities, a sensitivity of 10^{-6} on such scales is not excluded in the future. The specific scaling law makes hope that such an effect can be identified among other contaiminations. The possible detection of the lensing effect on the CMB will then give us a potential tool to get direct information on galaxy formation, as the amplitude is sensitive to the whole clustered mass and to its evolution in the non-linear regime.

Appendix

In this Appendix we give the details of the derivation of Eq. (7). The notation is slightly different from the text, and the derivation is done in the case of two dimensional space (the generalization is trivial). We recall that given a random function $T(x)$, we want to derive the power spectrum of the perturbed function $\tilde{T}(x) = T(x + \lambda(x))$. The Fourier transform of T is:

$$\hat{T}(k) = \int T(y)e^{-iky}\,dy$$

In this text we will also note indifferently $FT(A)$ the Fourier transform of the function A. The inverse Fourier transform is:

$$T(y) = \frac{1}{4\pi^2}\int \hat{T}(k)e^{iky}\,dy$$

Setting $y = x + \lambda(x)$ in this formula, we get the expression of $\tilde{T}(x)$:

$$\tilde{T}(x) = T(x + \lambda(x)) = \frac{1}{4\pi^2}\int \hat{T}(k)e^{ik(x+\lambda(x))}\,dk$$

We can now obtain the Fourier transform of $\tilde{T}$:

$$\hat{\tilde{T}}(u) = \int \tilde{T}(x)e^{-iux}\,dx = \frac{1}{4\pi^2}\int dk\,\hat{T}(k)\int e^{ik\lambda(x)}e^{-i(u-k)x}\,dx$$

Let us now define the auxiliary functions $f_k(x)$ by:

$$f_k(x) = e^{ik\lambda(x)}$$

(here k must be thought as a parameter rather than a point in the Fourier space). The Fourier transform of the f_k is:

$$\hat{f}_k(\omega) = \int e^{ik\lambda(x)}e^{-i\omega x}\,dx$$

From this, we can now write $\hat{\tilde{T}}(u)$ in the following way:

$$\hat{\tilde{T}}(u) = \frac{1}{4\pi^2}\int \hat{T}(k)\hat{f}_k(u-k)\,dk$$

This expression looks like a convolution but is not an exact convolution because of the dependence on k. However this way of writing the formula helps in understanding the results.
We can now write down the power spectrum of $\tilde{T}$ as:

$$\tilde{P}(u) = |\hat{\tilde{T}}(u)|^2 = \frac{1}{16\pi^4}\int \hat{T}(k)\bar{\hat{T}}(k')\hat{f}_k(u-k)\bar{\hat{f}}_{k'}(u-k')\,dk\,dk'$$

($\bar{A}$ denotes the hermitian conjugate of A). Assuming ergodicity, we can use the random phase formula:

$$\langle \hat{v}(k)\hat{v}(k')\rangle = 4\pi^2\delta(k-k')\hat{C}(k)$$

C being the autocorrelation of the process v. Then we get $P(u)$:

$$\tilde{P}(u) = \frac{1}{4\pi^2}\int P(k)\hat{C}(u-k)\,dk$$

where P is the power spectrum of T, and C_k is the correlation function of $f_k(x)$:

$$C_k(y) = \langle e^{-ik\lambda(x)}e^{ik\lambda(x+y)}\rangle = \langle e^{ik(\lambda(x+y)-\lambda(x))}\rangle$$

This last expression show that $C_k(y)$ is the characteristic function of the random function $\Lambda_y(x)$ defined by:

$$\Lambda_y(y) = \lambda(x+y) - \lambda(x)$$

This leads to a simpler formula for the power spectrum of $\tilde{T}$:

$$\tilde{P}(u) = \frac{1}{4\pi^2}\int P(k)[FT\theta_y(k)]_{(u-k)}\,dk$$

In this integral the Fourier transform FT must be taken relatively to the variable y, $\theta_y(k)$ is the characteristic function of the field $\Lambda_y(x)$.

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