THE CRITICAL INCLINATION IN ARTIFICIAL SATELLITE THEORY

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(Received 27 February; revised 1 November, 1986)

Abstract. Certain it is that the critical inclination in the main problem of artificial satellite theory is an intrinsic singularity. Its significance stems from two geometric events in the reduced phase space on the manifolds of constant polar angular momentum and constant Delaunay action. In the neighborhood of the critical inclination, along the family of circular orbits, there appear two Hopf bifurcations, to each of which there converge two families of orbits with stationary perigees. On the stretch between the bifurcations, the circular orbits in the planes at critical inclinmation are unstable.A global analysis of the double forking is made possible by the realization that the reduced phase space consists of bundles of two-dimensional spheres. Extensive numerical integrations illustrate the transitions in the phase flow on the spheres as the system passes through the bifurcations.

A delicacy so very susceptible of offence ...
— Hester Lynch PIOZZI, *Observations and Reflections made in the Course of a Journey through France, Italy and Germany* (1789)

1. Introduction

Orbits whose semi-major axis remains fixed on the average with respect to the line of nodes have drawn considerable attention in the theory of artificial satellites. The main problem, we recall, is the dynamical system with three degrees of freedom described in Cartesian coordinates (x, y, z) and momenta (X, Y, Z) by the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + J_2\mathcal{H}_1$. It is made of the Keplerian

$$\mathcal{H}_0 = \tfrac{1}{2}(X^2 + Y^2 + Z^2) - \mu/\rho$$

as its principal part, and of the term

$$\mathcal{H}_1 = (\mu/r)\,(\alpha/r)^2\,P_2(z/r)$$

as a first order perturbation in J_2. Here $P_2(z/r)$ denotes the Legendre polynomial of degree two in the variable z/r. The three parameters of the main problem are the Keplerian constant μ, the zonal oblateness coefficient J_2, and the length scale α which stands for the earth's equatorial radius.

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Celestial Mechanics **39** (1986) 365–406.

To our knowledge, Orlov was the first to signal that an unusual situation arises at the inclinations $I \equiv \pm \tan^{-1} 2 \pmod{\pi}$. He had set himself to the task of extending the periodic orbits of $\mathcal{H}_0$ into periodic orbits of $\mathcal{H}$, dealing first with the circular ones (Orlov, 1953), then proceeding to the general case of the elliptic orbits (Orlov, 1954). According to his setting of the periodicity criteria, the conventional Lindstedt-Poincaré algorithm —which he refers to as the Lyapunov-Poincaré algorithm —fails not only at zero inclination but also at inclinations such that $4 - 5 \sin^2 I = 0$. Orlov put aside these exceptional situations, and went on with the formal developments, thus leaving the reader in doubt. Is the critical inclination truly an essential feature in artificial satellite theory, or is it a spurious effect caused by the method of solution if not by the type of variables adopted in the analysis?

At about the same time, Krause (1952) was assessing the long term effects of J_2 on a Keplerian ellipse. His formula for the angular velocity of the mean perigee contains $(1 - 5 \cos^2 I)$ as a factor, a feature to which Krause attached no significance. Roberson (1957) who verified Krause's findings did not elaborate on that peculiarity either. Herget and Musen (1958) overlooked the difficulty in their treatment of the main problem by Hansen's method, at least not until it was pointed out to them by McVittie and Brouwer (1958, pp. 437--438). So casually did the enigma of the critical inclination slip in the theory of artificial satellites. Yet, right at the outset, appeared the main pieces to be fitted in the global portrait of the phase space: the possible existence of families of orbits with stationary perigees on the one hand, and the termination, to all seeming, of the families of circular orbits at the critical inclination on the other hand.

For a while Brouwer entertained the hope that Delaunay's method of eliminating periodic terms by canonical transformations would yield a solution free of singularities at small eccentricities and at critical inclinations (Brouwer 1958, p. 438, col. 1). Unlike Orlov who started with the Lagrangian equations in the map (a, e, I, h, g, ℓ) , Brouwer proceeded in the Delaunay variables (G, H, L, g, h, ℓ) from the Hamiltonian equations. Using Poincaré's *méthode nouvelle* rather than the conventional continuation adopted by Orlov, he removed the mean anomaly ℓ from the Hamiltonian $\mathcal{H}$ by means of a canonical transformation $(G, H, L, g, h, \ell) \rightarrow (G', H', L', g', h', \ell')$ developed in the powers of the small parameter J_2 (Brouwer, 1959). Distinct methods and coordinate systems led still to the same conclusion: because of the divisor $(1 - 5 \cos^2 I)$, neither Orlov's periodic series nor Brouwer's conditionally periodic developments make sense in the neighbourhood of the critical inclinations.

Incidentally, because we deal exclusively with the main problem after it has been normalized, from here on we shall drop the primes on the mean Delaunay phase variables. In addition, as we did in our previous publications, we shall use the notations

$$\eta = G/L, \qquad a = L^2/\mu, \qquad c = \cos I = H/G,$$
$$e = (1 - \eta^2)^{\frac{1}{2}}, \qquad p = G^2/\mu, \qquad s = \sin I = (1 - c^2)^{\frac{1}{2}},$$
$$n = (\mu/a^3)^{\frac{1}{2}}, \qquad q = H^2/\mu$$

throughout the paper without referring explicitly to their definitions.

Brouwer's elimination of the short period terms produced a Hamiltonian

$$\mathcal{H}' \equiv \mathcal{H}'_0 + J_2 \mathcal{H}'_1 + J_2{}^2\, \mathcal{H}'_2 + O(J_2{}^3) \tag{1}$$

in which the mean anomaly ℓ is ignorable. Since the main problem retains only the second zonal harmonic in the earth's gravity field, the perturbation function $\mathcal{H}_1$ is symmetric with respect to the earth's polar axis, which implies that the right ascension h of the ascending node is also ignorable in $\mathcal{H}'_1$. Accordingly the averaged Hamiltonian (1) defines an integrable system with only one degree of freedom. Brouwer approached it as he had done with great success when he used Brown-Hill's variational techniques to build a theory of Jupiter's theory of Io (Brouwer, 1946). Past the Delaunay normalization which he had compressed nicely into a unique operation, he believed that the solution could be completed with only one more elimination of a periodic argument. With the independent variable replaced by the long time scale τ such that

$$J_2\, dt = d\tau, \tag{2}$$

the flow on the phase cylinder (g, G) at the intersection of the integral manifolds L = constant and H = constant is determined by the canonical equations derived from the Hamiltonian

$$\mathcal{K} \equiv \mathcal{K}\,(L, G, H, g) = \mathcal{K}_0 + J_2\, \mathcal{K}_1 + \mathcal{O}(J_2{}^2) \tag{3}$$

whose terms are

$$\mathcal{K}_0 \equiv \mathcal{K}_0\,(L, G, H, g) = 4\, n\, G \left(\frac{\alpha}{p}\right)^2 (1 - 3\, c^2), \tag{4}$$

$$\mathcal{K}_1 = \frac{3}{128}\, n\, G \left(\frac{\alpha}{p}\right)^4 (m_{0,0} + m_{0,1}\, \eta + m_{0,2}\, \eta^2 + m_{2,0}\, s^2 e^2 \cos 2g), \tag{5}$$

the coefficients in (5) being the *inclination* polynomials

$$m_{0,0} = -5\,(8 - 16\,s^2 + 7\,s^4), \qquad m_{0,1} = -4\,(4 - 12\,s^2 + 9\,s^4),$$

$$m_{0,2} = 8 - 8\,s^2 - 5\,s^4, \qquad m_{2,0} = 2\,(14 - 15\,s^2).$$

Note that the right hand members of (4) and (5) are copied from (Deprit, 1981). They are arranged to display the algebraic structure of Hamiltonian $\mathcal{K}$. As a rule, state functions are decomposed into products of a factor carrying the physical dimensions in the most obvious way, and of a dimensionless expression involving only dimensionless quantities. For instance, at any order, a term in (3) is made to come out as a product of three factors: the energy nG, an even power of the parallax ratio α/p, and a finite trigonometric sum in multiples of 2g whose coefficients are polynomials in the dimensionless variables η, e^2, and s^2.

On the grounds that the angle g is ignorable in the principal term $\mathcal{K}_0$, it has been argued by analogy with the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\,\Theta^2 - \omega^2 \cos 2\theta$$

that the average main problem behaves essentially like a generalized pendulum. Admittedly incorrect from a global standpoint, the analogy is nonetheless acceptable in the region of the phase space where the motions are of a circulatory type, the argument of perigee g varying on the average monotonically with the long time τ. In that open domain, the particle with the kinetic energy $\mathcal{K}_0$ may be regarded as rotating at a fast rate (relatively speaking in the long time scale τ) on a circle while subject to an infinitesimal gradient arising from the perturbation $\mathcal{K}_1$. This, Tisserand (1868) had explained, is the foundation on which rests the most common Delaunay operation. In that context, elimination of the argument of perigee is legitimate everywhere in the plane (L, H) save at the points where the Lie derivative

$$\mathcal{L}_1 : F \to (F; \mathcal{K}_0) = -\frac{3}{4}\,n\,\left(\frac{\alpha}{p}\right)^2 (1 - 5\,c^2)\,\frac{\partial F}{\partial g}$$

is singular. Three kinds of exceptions arise thereby, not only the critical inclinations I such that $\tan I = \pm 2$, but also the equatorial orbits with $I \equiv 0 \pmod{\pi}$ filling the integral manifold H = L, as well as the circular orbits e = 0 since the Delaunay transformation $(R, \Theta, N, r, \theta, \nu) \to (L, G, H, \ell, g, h)$ leaves indeterminate the right ascension h for an equatorial orbit and the argument of perigee g for a circular orbit.

Astronomers are well acquainted with difficulties brought upon perturbed Keplerian problems by small inclinations and eccentricities; usually they can be disposed of by a change of coordinates. Singularities arising from non zero inclinations are of a different nature, though; they took everyone by surprise when they were discovered in the main problem of artificial satellite theory. The more so that engineers reported they did not detect adverse effects from the divisor $(1 - 5\,c^2)$ in their "guidance sensitivities" (Geyling, 1965) or in their numerical integrations (see e.g. Lubowe, 1969a and 1969b). Besides, answers the theorists gave the skeptics were not incontrovertibly convincing (see e.g. Message *et al.*, 1962; Kikuchi, 1967; Garfinkel, 1969; Allan, 1970; Hughes, 1981). All the same, while the debate lingered on, astronomers grew accustomed to the idea that singularities of that kind are common occurrences in perturbed Keplerian systems. Garfinkel (1959) discovered critical inclinations in his own separable intermediary

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - J_2\left(\frac{\Theta^2}{r^2}\right)\left(\frac{\alpha}{p}\right)^2\left[\,3\left(\frac{1}{2} - \frac{3}{4}s^2\right)\left(1 - \frac{p}{r}\eta\right) + \frac{3}{4}s^2\cos 2\theta\right],$$

or in the separable intermediary

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - J_2\left(\frac{\Theta^2}{r^2}\right)\left(\frac{\alpha}{p}\right)^2\left[\left(\frac{1}{2} - \frac{3}{4}s^2\right)\left(\frac{p}{r}\right) + \frac{3}{4}s^2\cos 2\theta\right], \tag{6}$$

proposed earlier by Sterne (1957). They were also found in the zonal intermediary (Aksnes 1965, 1966)

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - J_2\left(\frac{\Theta^2}{r^2}\right)\left(\frac{\alpha}{p}\right)^2\left[\left(\frac{1}{2} - \frac{3}{4}s^2\right) + \frac{3}{4}s^2\cos 2\theta\right], \tag{7}$$

and in the radial intermediary (Cid and Lahulla, 1969 and 1971)

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - J_2\left(\frac{\Theta^2}{r^2}\right)\left(\frac{\alpha}{p}\right)^2\left(\frac{p}{r}\right)\left(\frac{1}{2} - \frac{3}{4}s^2\right) \tag{8}$$

and its extensions (Cid *et al.* 1985). More significantly, critical inclinations appear in the quadratic Zeeman effect caused by weak magnetic fields (Delos *et al.* 1983a, 1983b). Undoubtedly genuine singularities at non zero inclinations and eccentricities are to be expected even in simple perturbed Keplerian systems. This being the case, past the Delaunay normalization and before reducing further the averaged problem, one should stop to analyze in detail the global phase space, locate its singularities, determine their stabilities, and then select the second reduction methods according to the overall chararcteristics of the flow in the mean phase space.

With the geometric structure of the phase space after a Delaunay normalization Brouwer had no concern; in particular he did not seem to have perceived a possible connection between the singularities at the critical inclinations and those at zero eccentricities. The first to tackle tl issue, Izsak (1963) questioned very harshly the unfounded, yet commonly held, allegation th the main problem, after it has been averaged over the mean anomaly, is assimilable to perturbed simple pendulum. As for himself, he contended that the phase space of $\mathcal{K}$ may t modelled after the perturbed harmonic oscillator

$$\mathcal{J} = 2\,a\,(X^2 + x^2) + 2\,b\,(X^2 - x^2) + 4\,c\,(X^2 + x^2)^2,$$

in an open region containing the critical inclinations at the exclusion, though, of the poin where $e = 0$ or $I = 0$. Garfinkel explored the possibility offered by the parameters a, b, and in Izsak's Hamiltonian to cover *indifferently* stable and unstable equilibria. Not until lon after Garfinkel closed his Ideal Resonance Theory did Cushman (1983) come with a glob: representation of the phase space along the lines of global geometric solutions proposed for tl Stark effect (Deprit, 1983) and the Blamont problem (Deprit, 1984). Relying on the invarianc of $\mathcal{K}$ with respect to the group of rotations around the polar axis, Cushman justifies the cycli character of the longitude of the node by way of a Meyer reduction (Meyer, 1973; see als Marsden and Weinstein, 1974—Quantum physicists name the technique after Kirillo\ Souriau and Kostant). Thus it turns out that, above each admissible point (L, H), the phas space is neither a cylinder as Brouwer trusted it to be, nor a plane as in Izsak's local mode but a two-dimensional *compact* manifold.

Cushman's extension of the momentum mapping based on the integral H is somewha awkward. No attention is paid to invariance with respect to the group of similitudes determine by the physical dimensions when defining the set of reducing coordinates. Indeed, one c Cushman's reducing coordinates is an action while the other two are squares of actions Insignificant as it may seem, the disparity causes unnecessary complications in th transformation formulas, the more so that it gives the phase space an unusual shape. In thi paper, we propose a set of coordinates homogeneous in physical dimensions, and an extensio of the momentum mapping that assimilates the phase space above each point (L, H) to a two dimensional sphere $\mathcal{S}$(L, H). Such representation makes it easy to explore how the singu larities change their relative positions in phase space as the base point moves in the plan (L,H), an aspect of the problem that has not yet been considered although it is of considerabl importance.

When only the principal term $\mathcal{K}_0$ is retained in Hamiltonian $\mathcal{K}$, there emanates from th critical inclination a continuum of singularities. In the local chart (G, g), it is represented by th line $G = H\sqrt{5}$, while, in our global model, it consists of a small circle on each sphere $\mathcal{S}$(L,H above each point of the plane (L,H) in the wedge defined by the inequality $0 \leq H \leq L$. This says that a *degeneracy* occurs at first order in the sense that the equilibria are not isolated points in the phase space [In some varieties of mathematical lingoes, one characterizes the situation by asserting that the Hamiltonian $\mathcal{K}_0$ is not a Morse function over $\mathcal{S}(L, H)$, only a Bott-Morse function!]. Now the perturbation $\mathcal{K}_1$ removes the degeneracy to leave only a finite number of isolated singularities. As long as the eccentricity is sufficiently far away from zero, there survive exactly four equilibria a shade above or below the criticial inclination; two of them, at $g = 0$ and π, are centers, and the other two, at $g = \pi/2$ and $3\pi/2$, are saddle points (Strubble, 1960; Petty and Breakwell, 1960; Izsak, 1963). Looking for a picture reminiscent of the phase space for a circular pendulum, Hori (1960a, 1960b) took only half of that figure for g between $\pi/2$ and $3\pi/2$. Carrving out the seccond center on the opposite side of the spehre $\mathcal{S}(L, H)$ may lead one to assimilate the phase space to that of a simple pendulum. Yet the analogy should not be taken at face value. In the first place: the phase space of the pendulum presents only one center and only one saddle point, and nothing in the main problem of artificial satellite theory justifies identifying opposite equilibria. Besides, like Strubble, Petty-Breakwell and Izsak, Hori excludes the small eccentricities, that is to say in our representation, an open neighbourhood of the pole $G = L$ on each sphere $\mathcal{S}(L, H)$. But that is precisely, our paper will reveal, the boundary layer where, in a two-step exchange of stability and instability, the saddle points at critical inclinations vanish first, and then the centers.

To be sure, the case of small eccentricities has been studied in the past. Authors like Aoki (1963a, 1963c) and Jupp (1980) developed their analysis along conventional lines on the basis of appropriate estimates for the values of the eccentricity and the divisor $(1 - 5c^2)$ compared to J_2. They took on more than the main problem. Aoki introduced the perturbations in J_4; Jupp, responding to indications given by Chapront (1965), added terms in J_6, and even the odd zonal harmonics since they affect significantly stationary perigees (see e.g. King-Hele et al., 1967, p. 761, also 1969, p. 642; Brookes, 1976; Lyddane and Cohen, 1978; Zeis and Cefola, 1980). It could be interesting to strip Jupp's formulas of these additional terms and then check whether his analysis in terms of the local variables

$$h = e \cos g = \frac{\xi_1 / L}{\xi_3 + (L^2 + H^2)/2}, \qquad k = e \sin g = \frac{\xi_2 / L}{\xi_3 + (L^2 + H^2)/2}$$

yields the same classification as this paper in the region of small eccentricities. What the global model gives—that cannot be obtained by treating the problem in separate local charts, one for

$e >> 0$ and one for $e \approx \mathcal{O}(\sqrt{J_2})$ — is the ability to follow how the families of stable and unstable orbits with stationary perigees evolve as H runs over the interval $0 \leq H \leq L$. The results may be phrased in Jupp's terminology as follows: the phase space in the neighborhood of $e = 0$ is of *type 3* when

$$0 \leq H < L/\sqrt{5}\,[1 - (J_2/10)\,(\alpha/a)^2 + \mathcal{O}(J_2^2)],$$

migrates through *type 4* in the interval

$$L/\sqrt{5}\,[1 - (J_2/10)\,(\alpha/a)^2 + \mathcal{O}(J_2^2)] \leq H \leq L/\sqrt{5}\,[1 + (J_2/10)\,(\alpha/a)^2 + \mathcal{O}(J_2^2)],$$

and ends up being of *type 1* when

$$L/\sqrt{5}\,[1 + (J_2/10)\,(\alpha/a)^2 + \mathcal{O}(J_2^2)] < H \leq L.$$

One could probably follow part of this evolution, at least when the eccentricity is sufficiently small, in Jupp's analysis were the coefficients of his cubic equation (34) given explicitly in terms of the integrals H and L.

Transitions from one type to the next are intrinsic, independent that is of the coordinate system in which one operates. They occur exactly where the families of orbits with stationary perigees branch off the family of circular orbits with non zero inclinations. It is remarkable that the origin of the critical inclinations can be traced back to a double bifurcation at the extremities of a short interval of instability along the sequence of mean circular orbits. From a practical standpoint, the most significant finding is the tiny gap of *unstable* circular orbits in the neighborhood of the critical inclination.

2. The Delaunay reduction

Because all orbits in bounded states of the Keplerian system are periodic, any smooth function F over its phase space may be decomposed in a unique way into a sum $F = F^\circ + F^\#$ with the following properties:

a) $(F^\circ; \mathcal{H}_0) = 0$, i.e. F° belongs to the kernel of the Lie derivative $\mathfrak{L}_0$: $F \rightarrow (F; \mathcal{H}_0)$;

b) There exists a smooth function F^* such that $(F^*; \mathcal{H}_0) = F^\#$, i.e. $F^\#$ belongs to the image of the operator $\mathfrak{L}_0$.

On this theorem proved by Cushman (1984) rests the concept of a Delaunay normalization (Deprit, 1982), that is a Lie transformation—in the sense of Deprit (1969)—to convert the perturbed Keplerian Hamiltonian $\mathcal{H}$ into a Hamiltonian $\mathcal{H}'$ belonging to the kernel of $\mathfrak{L}_0$. Analytically a Delaunay normalization is usually carried out, at least implicitly, in the Delaunay variables because they reduce the Lie derivative $\mathfrak{L}_0$ to the single partial derivative $n\partial/\partial\ell$. In the Delaunay chart, the perturbation stands as a periodic function of the mean anomaly ℓ, and the Delaunay normalization amounts to averaging the vector field derived from $\mathcal{H}$ over the mean anomaly. However, since the time when Fock (1936) discovered that Keplerian Hamiltonians are invariant with respect to the group SO(4) of rotations in a four-dimensional space, quantum physicists tend to look at Delaunay normalizations from a geometric standpoint with a view of characterizing the phase space of the averaged perturbed Keplerian system in a global way, rather than dealing with it through local coordinate charts as it is done in conventional celestial mechanics. From that viewpoint, a Delaunay normalization is no less than a *reduction* in the sense of Meyer (1973; see also Abraham and Marsden, 1978, 296 - 309). Cushman (1983) explains why it is so. Here is how we understand Cushman's geometric explanation in a somewhat intuitive way, at the risk though of sinning grievously against mathematical propriety.

In principle at least, after the mean anomaly has been removed, the motion of the satellite is obtained by first solving the differential system

$$\frac{dg}{d\tau} = \frac{\partial \mathcal{K}}{\partial G}, \qquad \frac{dh}{d\tau} = \frac{\partial \mathcal{K}}{\partial H}, \qquad \frac{dG}{d\tau} = -\frac{\partial \mathcal{K}}{\partial g}, \qquad \frac{dH}{d\tau} = -\frac{\partial \mathcal{K}}{\partial h}, \tag{9}$$

and then performing the quadrature

$$\ell = n\,t + \int \frac{\partial}{\partial L} \mathcal{K}(L, G(\tau), H, g(\tau))\, d\tau. \tag{10}$$

Thus, on the one hand, to each value of the (formal) integral L is attached the four-dimensional phase space $\mathcal{P}(L)$ of a reduced dynamical system described by the Hamiltonian $\mathcal{K}$ at that particular value of L; on the other hand, quadrature (10) reconstitutes any orbit in $\mathcal{P}(L)$ as an orbit on the integral surface L in the 6-dimensional phase space (L, G, H, ℓ, g, h). An essential charasteristic of the phase space $\mathcal{P}(L)$ might go unnoticed in this conventional treatment of an ignorable variable. A point of $\mathcal{P}(L)$ stands not for an individiual state in the original system but for the class of states in the manifold L whose Delaunay coordinates differ only in the mean anomaly. Thus, given the momentum mapping

$$\pi : (x, y, z, X, Y, Z) \to L : \mathbf{R}^6 \to \mathbf{R},$$

and the one-parameter group $\mathfrak{D}$ of transformations

$$\varphi: (L, G, H, \ell, g, h) \rightarrow (L, G, H, \ell + \epsilon, g, h) : \mathbf{R}^6 \rightarrow \mathbf{R}^6.$$

$\mathcal{P}(L)$ is the reduced space $\pi^{-1}(L)/\mathfrak{S}$. Along an orbit on $\mathcal{P}(L)$ that is a solution of the canonical system (9), the angular momentum and the Runge-Lenz vector

$$\mathbf{G} = \mathbf{x} \times \mathbf{X}, \qquad \mathbf{A} = (L/\mu)\,[\mathbf{X} \times \mathbf{G} - (\mu/r)\,\mathbf{x}] \tag{11}$$

are invariant with respect to the group $\mathfrak{S}$ since their Cartesian components

$$G_1 = G \sin I \sin h, \quad (12_1) \qquad A_1 = L\,e\,(\cos g \cos h - \sin g \cos I \sin h), \quad (13_1)$$

$$G_2 = -G \sin I \cos h, \quad (12_2) \qquad A_2 = L\,e\,(\cos g \sin h + \sin g \cos I \cos h), \quad (13_2)$$

$$G_3 = H, \quad (12_3) \qquad A_3 = L\,e \sin g \sin I \quad (13_3)$$

are independent of the mean anomaly. Furthermore these vectors are sufficient to determine unambiguously the equivalence classes that are the points of $\mathcal{P}(L)$. They may therefore serve as a coordinate system on the reduced manifold. In those coordinates, $\mathcal{P}(L)$ emerges as the algebraic surface satisfying the equations

$$\|\mathbf{G}\|^2 + \|\mathbf{A}\|^2 = L^2 \quad \text{and} \quad \mathbf{G} \cdot \mathbf{A} = 0. \tag{14}$$

Clearly the Delaunay elements (G, H, g, h) can be expressed in terms of **G** and **A**, so the reduced Hamiltonian $\mathcal{K}$ can be made into a function of G_1, G_2, G_3, A_1, A_2, and A_3 alone, hence the Hamiltonian system (9) is equivalent to the system

$$\frac{d\mathbf{G}}{d\tau} = \nabla_{\mathbf{G}}\mathcal{K} \times \mathbf{G} + \nabla_{\mathbf{A}}\mathcal{K} \times \mathbf{A}, \qquad \frac{d\mathbf{A}}{d\tau} = \nabla_{\mathbf{G}}\mathcal{K} \times \mathbf{A} + \nabla_{\mathbf{A}}\mathcal{K} \times \mathbf{G}. \tag{15}$$

In forming the right hand members of these equations one takes into account that all Poisson brackets $(G_i; G_j)$, $(G_i; A_j)$ and $(A_i; A_j)$ for $1 \le i, j \le 3$ are zero save

$$(G_1; G_2) = G_3, \qquad (G_2; G_3) = G_1, \qquad (G_3; G_1) = G_2,$$

$$(G_1; A_2) = A_3, \qquad (G_2; A_3) = A_1, \qquad (G_3; A_1) = A_2,$$

$$(A_1; A_2) = G_3, \qquad (A_2; A_3) = G_1, \qquad (A_3; A_1) = G_2.$$

Quantum phycisists arrive at a more intuitive description of $\mathcal{P}(L)$ by adopting the variables (due probably to Jauch and Hill):

$$\sigma_i = \tfrac{1}{2}(G_i + A_i), \qquad \delta_i = \tfrac{1}{2}(G_i - A_i), \qquad (1 \leq i \leq 3). \quad (16)$$

Owing to the relations

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \tfrac{1}{4}L^2, \qquad \delta_1^2 + \delta_2^2 + \delta_3^2 = \tfrac{1}{4}L^2, \quad (17)$$

equivalent to equalities (14), the reduced phase space $\mathcal{P}(L)$ above each point along the L-axis exhibits itself as a kind of dumb-bell [not in the sense of Algebraic Geometry!] consisting as it were of a pair of 2-dimensional spheres, the radius of each sphere being equal to L/2 (see Figure 1). Let it be mentioned that there is also a practical advantage in using the Cartesian components of the vectors $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\delta = (\delta_1, \delta_2, \delta_3)$ as the coordinates on the dumb-bell $\mathcal{P}(L)$ because all Poisson brackets $(\sigma_i; \sigma_j)$, $(\sigma_i; \delta_j)$, and $(\delta_i; \delta_j)$ are zero save

$$(\sigma_1; \sigma_2) = \sigma_3, \qquad (\sigma_2; \sigma_3) = \sigma_1, \qquad (\sigma_3; \sigma_1) = \sigma_2,$$
$$(\delta_1; \delta_2) = \delta_3, \qquad (\delta_2; \delta_3) = \delta_1, \qquad (\delta_3; \delta_1) = \delta_2,$$

hence fewer terms enter the right hand members of the reduced equations. In those variables,

$$\frac{d\sigma}{d\tau} = \nabla_\sigma \mathcal{K} \times \sigma, \qquad \frac{d\delta}{d\tau} = \nabla_\delta \mathcal{K} \times \delta. \quad (18)$$

In particular, the form given the right hand members of (18) makes it clear that

$$\sigma \cdot \frac{d\sigma}{d\tau} = \delta \cdot \frac{d\delta}{d\tau} = 0,$$

and so confirms that the orbits of the normalized system belong to the manifold $\mathcal{P}(L)$ at each level of the integral L, in other words, that each dumb-bell $\mathcal{P}(L)$ is an invariant manifold of the differential system (6).The reduction that ensued on the Delaunay normalization applies to any perturbed Keplerian system that is Hamiltonian in character, a wide class of dynamical sytems indeed, of which the main problem in artificial satellite is but an example. Further reduction of the phase space is usually not possible beyond this point unless the principal term $\mathcal{K}_0$ in the reduced Hamiltonian itself presents in its turn a dynamical symmetry.

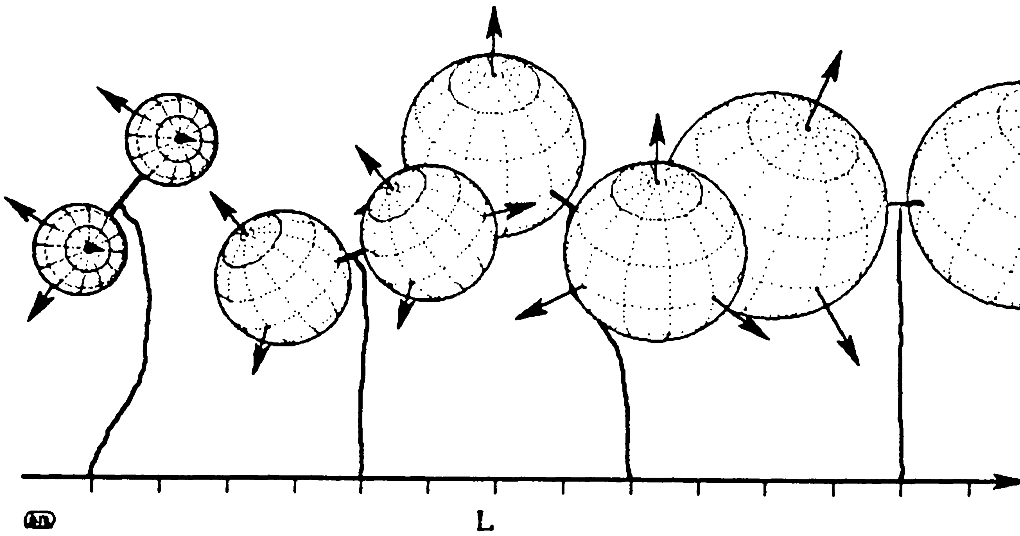

Figure 1. The phase space of a perturbed Keplerian system after a Delaunay normalization.

3. Elimination of the ascending node

The elimination of the mean anomaly consisted of two steps, first a Delaunay normalization to create a formal integral L by rendering the new mean anomaly ignorable in $\mathcal{K}$, then a Meyer reduction to partition the original six-dimensional phase space into a row of four-dimensional shells, one for each integral manifold. By contrast, the next reduction is simpler in that, owing to the longitude h being ignorable in $\mathcal{H}$, hence also in its average $\mathcal{K}$ over the mean anomaly, the Hamiltonian is invariant with respect to the group of rotations around the earth's polar axis. By virtue of Noether's theorem, the component H of the angular momentum is an integral. Although this is a well known fact in artificial satellite theory, there remains however, from a geometric standpoint, to explain how the conventional way of *ignoring* the longitude of the ascending node corresponds in fact to a Meyer reduction.

Let **G**' and **A**' be the images of the vectors **G** and **A** by a rotation of arbitrary amplitude ϵ around the earth's polar axis. Evidently $\|\mathbf{G}'\| = \|\mathbf{G}\|$ and $\|\mathbf{A}'\| = \|\mathbf{A}\|$; furthermore, $\|\mathbf{G}'\|^2 + \|\mathbf{A}'\|^2 = L^2$ and $\mathbf{G}' \cdot \mathbf{A}' = 0$, which proves that the transformation induced by the rotation leaves invariant the reduced phase space $\mathcal{P}(L)$. The Cartesian components of **G**' and

A' are linked to those of **G** and **A** by the relations

$$A_1 = A'_1 \cos\epsilon - A'_2 \sin\epsilon, \qquad G_1 = G'_1 \cos\epsilon - G'_2 \sin\epsilon,$$

$$A_2 = A'_1 \sin\epsilon + A'_2 \cos\epsilon, \qquad G_2 = G'_1 \sin\epsilon + G'_2 \cos\epsilon,$$

$$A_3 = A'_3, \qquad G_3 = G'_3.$$

There follows that the components of the image vectors σ' and δ' are related to those of the Jauch-Hill vectors σ and δ by the equalities

$$\sigma_1 = \sigma'_1 \cos\epsilon - \sigma'_2 \sin\epsilon, \qquad \delta_1 = \delta'_1 \cos\epsilon - \delta'_2 \sin\epsilon,$$

$$\sigma_2 = \sigma'_1 \sin\epsilon + \sigma'_2 \cos\epsilon, \qquad \delta_2 = \delta'_1 \sin\epsilon + \delta'_2 \cos\epsilon,$$

$$\sigma_3 = \sigma'_3, \qquad \delta_3 = \delta'_3.$$

Under the action of these rotations around the earth's polar axis, the tips of the vectors σ and δ describe small circles on $\mathcal{P}(L)$, one at the height σ_3 on the σ- sphere, and the other at the height δ_3 on the δ- sphere. Now the Meyer-reduction corresponding to the longitude of the node being ignorable in Hamiltonian $\mathcal{K}$ features a mapping of the dumb-bell $\mathcal{P}(L)$ into a two-dimensional manifold $\mathcal{P}(L, H)$ in such a way that the "orbits" on $\mathcal{P}(L)$ of the group SO(2) of rotations around the earth's axis coalesce into points of $\mathcal{P}(L, H)$. We shall now prove that the transformation $\pi : (L, G, H, g) \to \xi = (\xi_1, \xi_2, \xi_3)$ defined by the equations

$$\xi_1 \equiv \xi_1(L, G, H, g) = G\,L\,e\,\sin I \cos g, \tag{19$_1$}$$

$$\xi_2 \equiv \xi_2(L, G, H, g) = G\,L\,e\,\sin I \sin g, \tag{19$_2$}$$

$$\xi_3 \equiv \xi_3(L, G, H, g) = G^2 - \tfrac{1}{2}(L^2 + H^2) \tag{19$_3$}$$

has precisely that property. The coordinates (ξ_1, ξ_2, ξ_3) are homogeneous in dimension, being all three squares of actions. The inverse formulas

$$G = [\xi_3 + \tfrac{1}{2}(L^2 + H^2)]^{\frac{1}{2}}, \qquad L\,e = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}} / [\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}},$$

$$G \cos I = H, \qquad L\,e \cos g = \xi_1 / [\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}},$$

$$G \sin I = [\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}}, \qquad L\,e \sin g = \xi_2 / [\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}}.$$

serve to express the coordinates $(G_1, G_2, G_3, A_1, A_2, A_3)$ of any point on the dumb-bell

$\mathcal{S}(L)$ in terms of the variables ξ, L, H and h as follows:

$$\begin{aligned}
G_1 &\equiv G_1(\xi; L, H, h) &&= [\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}} \sin h, \\
G_2 &\equiv G_2(\xi; L, H, h) &&= -[\xi_3 + \tfrac{1}{2}(L^2 - H^2)]^{\frac{1}{2}} \cos h, \\
G_3 &\equiv G_3(\xi; L, H) &&= H, \\
A_1 &\equiv A_1(\xi; L, H, h) &&= (G\,\xi_1 \cos h - H\,\xi_2 \sin h) / [(\xi_3 + \tfrac{1}{2}L^2)^2 - \tfrac{1}{4}H^4]^{\frac{1}{2}}, \\
A_2 &\equiv A_2(\xi; L, H, h) &&= (G\,\xi_1 \sin h + H\,\xi_2 \cos h) / [(\xi_3 + \tfrac{1}{2}L^2)^2 - \tfrac{1}{4}H^4]^{\frac{1}{2}}, \\
A_3 &\equiv A_3(\xi; L, H) &&= \xi_2 / [\xi_3 + \tfrac{1}{2}(L^2 + H^2)]^{\frac{1}{2}}.
\end{aligned}$$

On account of (14), at every point of any dumb-bell $\mathcal{S}(L)$,

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \tfrac{1}{4}(L^2 - H^2)^2. \tag{20}$$

Thus, for any L and any H such that $0 \le H < L$, the transformation π projects the dumb-bell $\mathcal{S}(L)$ onto a two-dimensional sphere $\mathcal{S}(L, H)$ of radius $\frac{1}{2}(L^2 - H^2)$ centered at the origin in the three-dimensional space (ξ_1, ξ_2, ξ_3). Furthermore, by putting the projection π in vector form as the transformation

$$\xi_1 = (\mathbf{G} \times \mathbf{A}) \cdot \mathbf{k}, \tag{21$_1$}$$

$$\xi_2 = \|\mathbf{G}\| \, (\mathbf{A} \cdot \mathbf{k}), \tag{21$_2$}$$

$$\xi_3 = \tfrac{1}{2}\left(\|\mathbf{G} \times \mathbf{k}\|^2 - \|\mathbf{A}\|^2 \right) \tag{21$_3$}$$

with **k** as the unit vector in the direction of the rotation axis, we establish *ipso facto* that π is invariant, so to say, with respect to the group of rotations around the axis **k**. More precisely, all the points on a dumb-bell $\mathcal{S}(L)$ that are images of a given point (**G**, **A**) by the rotations around **k** are projected by π on the same point $\xi = \pi(\mathbf{G}, \mathbf{A})$ of the sphere $\mathcal{S}(L, H)$. Figure 2 purports to evoke what happens to the phase space in the course of the second Meyer above each point of an L-axis; the projection π rearranges it as a flight of balloons taking off the floor of a convention: the delegate in seat H of row L holds on to the sphere $\mathcal{S}(L, H)$ of his private interest no matter how the keynote speaker spins the rhetorical yarn around the themes of the convention.

Cushman (1983), we already said in the Introduction, deserves credit for having discovered that the elimination of the longitude of the ascending node amounts to a Meyer reduction. We only contribute a new version of the reducing projection. To recall, Cushman

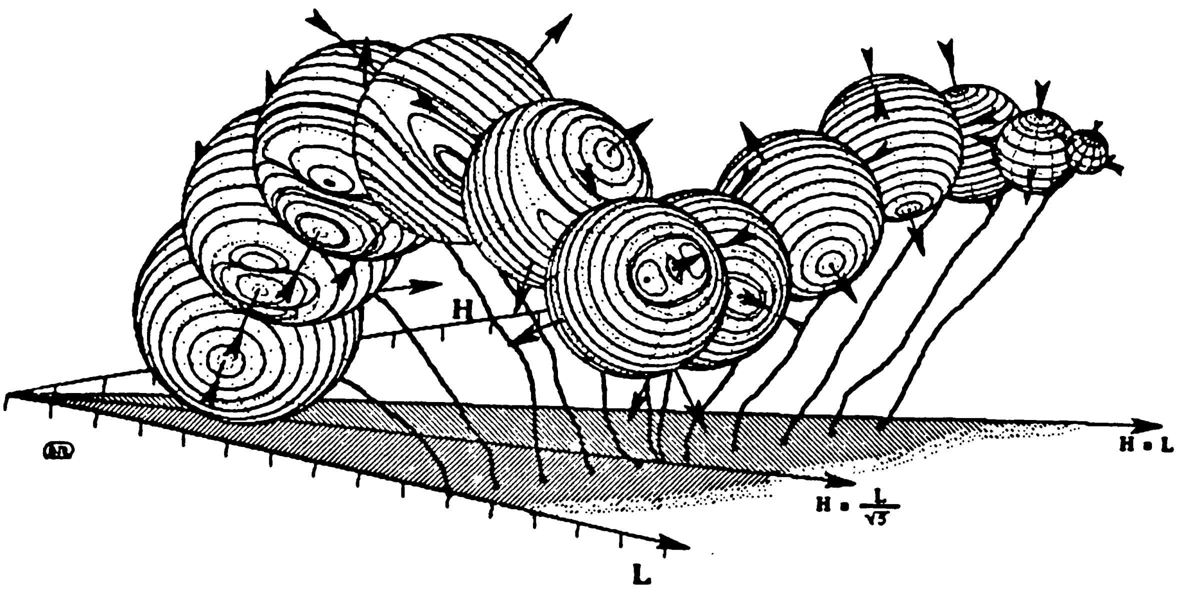

Figure 2. The phase space of a perturbed Keplerian system that is axially symmetric, after a double Meyer reduction.

proposed the transformation $(\sigma_1, \sigma_2, \sigma_3, \delta_1, \delta_2, \delta_3) \rightarrow (\eta_1, \eta_2, \eta_3)$ determined by the equations

$$\eta_1 = 2\,(\sigma_2\,\delta_1 - \sigma_1\,\delta_2) = G\,L\,E\cos g\sin I,$$

$$\eta_2 = 2\,(\sigma_1\,\delta_1 + \sigma_2\,\delta_2) = \tfrac{1}{2}\,(2\,G^2 - L^2 - H^2 + L^2\,e^2\sin^2 g\sin^2 I),$$

$$\eta_3 = \sigma_3 + \delta_3 = G.$$

It projects each dumb-bell $\mathcal{P}(L)$ onto a surface of revolution whose equation in the space (η_1, η_2, η_3) is

$$\eta_1^2 + \eta_2^2 = [L^2 - (H + \eta_3)^2]\,[L^2 - (H - \eta_3)^2].$$

From our coordinates to those of Cushman the conversion formulas are

$$\eta_1 = \xi_1, \quad \eta_2 = \xi_3 + \tfrac{1}{2}\,\frac{\xi_2^2}{\xi_3 + \frac{1}{2}\,(L^2 + H^2)}, \quad \eta_3 = [\xi_3 - \tfrac{1}{2}(L^2 + H^2)]^{\frac{1}{2}}.$$

By virtue of Liouville's theorem,

$$\frac{d\xi_1}{d\tau} = (\xi_1; \mathcal{K}), \qquad \frac{d\xi_2}{d\tau} = (\xi_2; \mathcal{K}), \qquad \frac{d\xi_3}{d\tau} = (\xi_3; \mathcal{K})$$

are the equations of motion on each individual sphere $\mathcal{S}$(L, H). In order to evaluate the Poisson brackets in the right hand members, we begin by observing that

$$\frac{\partial}{\partial G} G \sin I = \frac{1}{\sin I}, \quad \text{and} \quad \frac{\partial}{\partial G} L\, e = -\frac{G}{L\, e},$$

and deduce therefrom after a few straightforward manipulations that

$$\frac{\partial}{\partial G} G\, L\, e \sin I = -2 \frac{\xi_3}{L\, e \sin I},$$

hence the Poisson brackets

$$(\xi_1 ; \xi_2) = 2\, G\, \xi_3, \qquad (\xi_2 ; \xi_3) = 2\, G\, \xi_1, \qquad (\xi_3 ; \xi_1) = 2\, G\, \xi_2,$$

and, after the second Meyer-reduction, the equations of motion in a vectorial form

$$\frac{d\xi}{d\tau} = 2\, G\, \nabla_{\xi} \mathcal{K} \times \xi.$$

They prove in particular that, at each point (L, H), the average motions of the reduced system never leave the sphere $\mathcal{S}$(L, H).

Rather than face a redundant coordinate system subject to constraint (20), astronomers usually adopt (g, G) as a system of coordinates on the sphere. According to equations (19_1) - (19_3), the angle g is in effect the longitude along the equator in the plane $\xi_3 = 0$, and G being a function of ξ_3 alone measures the height above the equator . As one can see in Figure 3, the chart (g, G) is analogous to a Mercator map for the sphere $\mathcal{S}$(L,H); as such, it is singular at the poles $\xi_1 = \xi_2 = 0$ of the reduced phase space. It is now clear that the global flow induced by the perturbation is not determined adequately by the equations

$$\frac{dg}{d\tau} = \frac{\partial \mathcal{K}}{\partial G}, \qquad \frac{dG}{d\tau} = -\frac{\partial \mathcal{K}}{\partial g} \tag{22}$$

in the chart (g, G). While treating a symmetric perturbed Keplerian problem in the coordinates (g, G), one excludes *ipso facto* the solutions in the orbit classes represented by the north pole ($\xi_3 > 0$, $\xi_1 = \xi_2 = 0$) and by the south pole ($\xi_3 < 0$, $\xi_1 = \xi_2 = 0$) of each sphere $\mathcal{S}$(L, H).

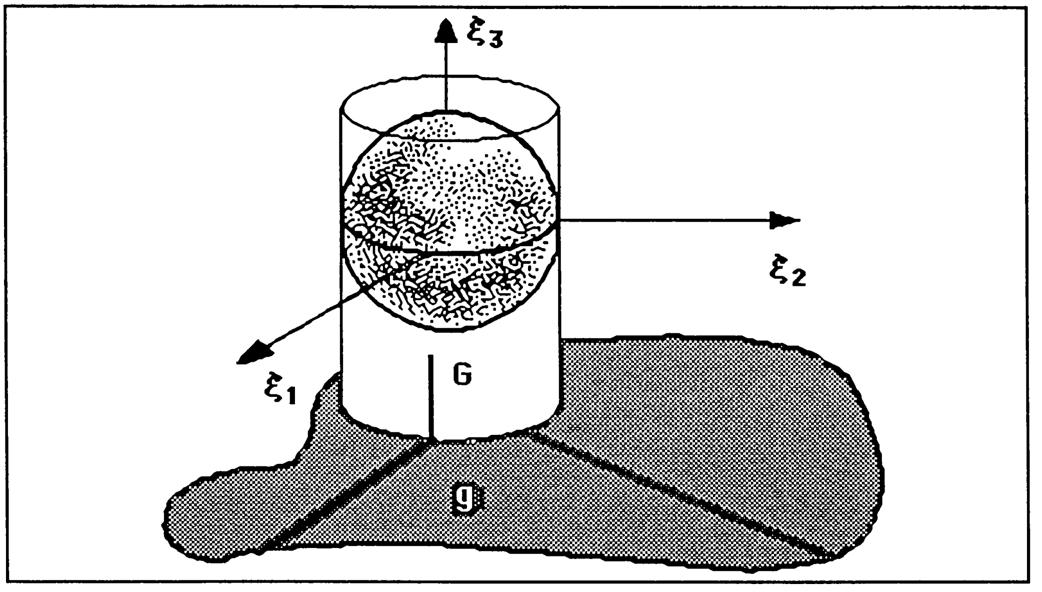

Figure 3. The angle and action (g,G) define a cylindrical projection of the sphere $\mathcal{S}$ (L,H)

The condition $\xi_1{}^2 + \xi_2{}^2 = 0$, equivalent to the condition $G^2 L^2 e^2 \sin^2 I = 0$, is satisfied either:

a) for G = 0. This means in particular that H = 0 or that the excepted solutions belong to the sphere $\mathcal{S}$(L, 0) representing the totality of average polar orbits in the integral manifold L. Owing to (19_1) - (19_3), there results also that $\xi_1 = \xi_2 = 0$, and $\xi_3 = -\frac{1}{2} L^2$; the exception is thus located at the south pole of $\mathcal{S}$(L,0), the point representing the class of linear orbits in the integral manifold L;

b) for sin I = 0. In those cases, G = H, and $\xi_3 = -\frac{1}{2}(L^2 - H^2) < 0$. As in the previous case, the exception is located at the south pole of $\mathcal{S}$(L, H), a point representing the average equatorial orbits of eccentricity $e = (1 - H^2/L^2)^{\frac{1}{2}}$ in the integral manifold L;

c) for e = 0. It means that G = L, hence that $\xi_3 = \frac{1}{2}(L^2 - H^2) > 0$. Therefore the exception is located at the north pole of $\mathcal{S}$(L, H), and represents the class of average circular orbits with an inclination $I = \cos^{-1}(H/L)$ over the earth's equator.

That the cylindrical chart (g, G) excludes both equatorial and circular orbits explains the difficulties encountered in dealing with small eccentricities and small inclinations while solving equations (22). Witness the *normality* conditions (Garfinkel 1972, 1973a-b) to be imposed when applying the Theory of the Ideal Resonance to the main problem in artificial satellite theory in the neighbourhood of the critical inclinations; they are called for cordoning off the north pole on each sphere $\mathcal{S}$(L, H).

4. The families of orbits with stationary perigees

As we enter a detailed analysis of the equations of motion, we face a fair amount of algebraic manipulations. Needless to say, they were performed by machine. Eliminations of the parallax and Delaunay normalizations were executed by MAO. This is a large package of functions written in an applicative language called LISP (McCarthy *et al* 1960, see also Winston and Horn, 1984). MAO's programming philosophy is documented elsewhere (Miller and Deprit, 1986). It is enough to say here that MAO specializes in handling generic operations in the abstract category of commutative algebras with a unit element over rings of coefficients that are domains of integrity. The concept of "object oriented programming" introduced by Dahl and Nygaard (1966) is the key structure on which MAO (Rom, 1970 and 1971) has been rebuilt.

Ever since Sconzo advocated using FORMAC (a macro processor to generate PL/I source programs), to reproduce Kozai's extension of Brouwer's solution, there have been various attempts at applying general purpose algebraic processors to solve analytically the main problem in artificial satellite theory. To our knowledge, the most advanced ones have been those of Zeis (1978) (see also Zeis and Cefola, 1978) with MACSYMA (Mathlab Group, 1983; Moses, 1974; see also Rand, 1984, and Sloane, 1986). For the extensive symbolic evaluations needed to weave chains of Poisson brackets in the development of Lie transformations, we prefer MAO to MACSYMA. Past that stage, the calculations become less extensive, but more varied and less structured algebraically; at that point we found MACSYMA convenient and resourceful, and we made extensive use of it.

We propose now to locate the classes of orbits with stationary perigees on the spheres $\mathcal{S}$(L, H). To this end, one could operate at once in the global coordinates (ξ_1, ξ_2, ξ_3). Instead, we shall proceed for a while in the cylindrical chart (g, G). In places where the analysis breaks down because of small eccentricities, we transfer to the global coordinate system. This compromise, we hope, makes it easy for the reader to relocate the classical results in our spherical model.

For the main problem in artificial satellite theory, the right hand members of the differential equations (22) are:

$$D_1 \equiv D_1(L, G, H, g) = -\frac{\partial K}{\partial g} = \frac{3}{32} J_2\, n\, G \left(\frac{\alpha}{p}\right)^4 s^2 (14 - 15 s^2)\, e^2 \sin 2g + \mathcal{O}(J_2^2), \qquad (23_1)$$

$$D_2 \equiv D_2(L, G, H, g) = \frac{\partial K}{\partial G} = \frac{3}{128}\, n \left(\frac{\alpha}{p}\right)^2 \{ 32 (4 - 5 s^2) + J_2 \left(\frac{\alpha}{p}\right)^2 \times$$

$$[\gamma_{0,0} + \gamma_{0,1}\, \eta + \gamma_{0,2}\, \eta^2 + (\gamma_{2,0} + \gamma_{2,2}\, \eta^2) \cos 2g] \} + \mathcal{O}(J_2^2), \qquad (23_2)$$

the coefficients in the right hand member of (23_2) being the inclination functions

$$\gamma_{0,0} = 440 - 860 s^2 + 385 s^4, \qquad \gamma_{2,0} = 56 - 372 s^2 + 330 s^4,$$

$$\gamma_{0,1} = 192 - 528 s^2 + 360 s^4, \qquad \gamma_{2,2} = -56 + 316 s^2 - 270 s^4,$$

$$\gamma_{0,2} = -56 + 36 s^2 + 45 s^4.$$

Coordinates of the critical points are solutions of the system of transcendental equations $D_1 = D_2 = 0$. At order 0, the system *degenerates* into the unique equation

$$D_{2,0} = \frac{3}{4}\, n \left(\frac{\alpha}{p}\right)^2 (4 - 5 s^2) = 0.$$

It would thus appear that, at order 0, all orbits at the critical inclination have stationary perigees. On the spherical model $\mathcal{S}(L, H)$, orbits of stationary perigees lie on the small circle of latitude at the height $\xi_3 = \xi_0$, where

$$\xi_0 \equiv \xi_0(L, H) = \tfrac{1}{2} (9 H^2 - L^2);$$

conversely, all points on the small circle at the height ξ_0 represent orbits with stationary perigees. In view of the inequalities $-\tfrac{1}{2}(L^2 - H^2) \le \xi_0 \le \tfrac{1}{2}(L^2 - H^2)$ expressing that the small circle belongs to the sphere $\mathcal{S}(L, H)$, there follows that orbits with stationary perigees exist if and only if

$$H \le H_0 \equiv H_0(L) = L/\sqrt{5},$$

i. e. only on spheres $\mathcal{S}(L, H)$ above points in the base plane (L, H) in the wedge between the lines $H = 0$ and $H = H_0$. One arrives at the same conclusion by observing that orbits of stationary perigees are ellipses with the eccentricity

$$e_0 \equiv e_0(L, H) = (1 - 5\,H^2/L^2)^{\frac{1}{2}}.$$

To our knowledge, Cushman (1983) was the first to state this restriction explicitly. It seems strange that it did not surface earlier, not even in the controversy between Lubowe (1969) and Garfinkel (1969), although, for orbits in Lubowe's Series II, $H > L/\sqrt{5}$, and thus Lubowe is vindicated in his claimi that the inclination $I = \tan^{-1}2$ is not critical in that series.

The history of the family of orbits with stationary perigees in a given integral manifold L is depicted schematically in Figure 4: the small circles of equilibria stem from the south pole representing a linear orbit on the sphere $\mathcal{S}(L,0)$, then bubble up on the spheres $\mathcal{S}(L,H)$ with decreasing eccentricities to terminate at the north pole of the sphere $\mathcal{S}(L, H_0)$ with a circular orbit.

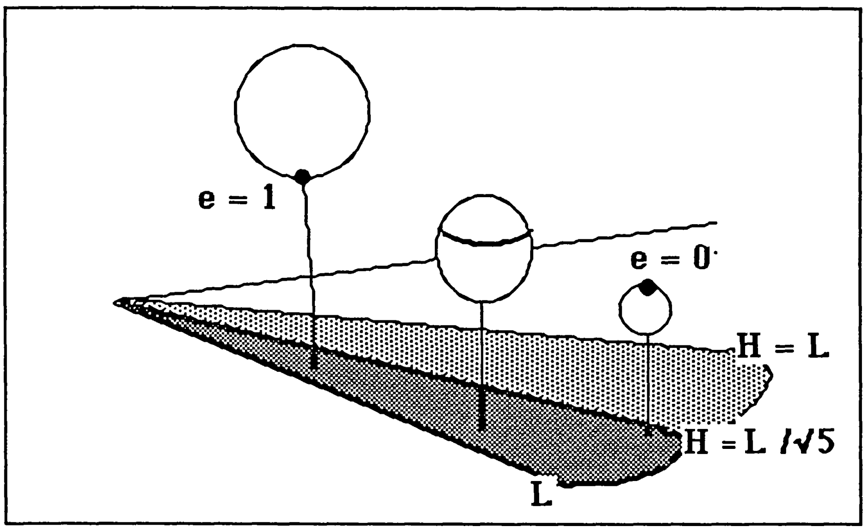

Figure 4. Evolution of the orbits with stationary perigees
(Degeneracy at order 0)

Broadly speaking, the perturbation terms do not alter this evolutionary pattern; they only remove the degeneracy by wiping out all but four at the most of the zero-order equilibria in the cylindrical chart above each point of the base plane (L, H). On the whole, they simplify the problem, yet, as we shall see in the next Section, at the cost of delicate complications in a thin layer on both sides of the separatrix $H = H_0$. The factor sin 2g in the right hand member of (23_1) is responsible for removing the degeneracy, while the first order term in (23_2) determines a correction to the value of the critical inclination. Two cases must be distinguished:

a) The equation $D_1 = 0$ is satisfied for $g \equiv 0 \bmod \pi$. In which case the value of G at both equilibria is a root of the equation

$$D_2(L, G, H, 0 \bmod \pi) = D_{2,0}(L, G, H) + J_2\, D_{2,1}(L, G, H) = 0,$$

the first order coefficient being

$$D_{2,1} = \frac{3}{128}\, n\, (\frac{\alpha}{p})^4\, [\gamma_{0,0} + \gamma_{2,0} + \gamma_{0,1}\, \eta + (\gamma_{0,2} + \gamma_{2,2})\, \eta^2].$$

The equation is readily solved to the first order in J_2 by a Newton-Raphson approximation. Thus, in the root Γ_1 as a series

$$\Gamma_1 \equiv G_1(L, H) = H\sqrt{5}\, [1 + J_2\, \gamma_1 + \mathcal{O}(J_2{}^2)], \tag{24_1}$$

the first order correction γ_1 is given by the formula

$$H\sqrt{5}\, \gamma_1 = -\, D_{2,1}(L, H\sqrt{5}, H) \,/ \frac{\partial}{\partial G} D_{2,0}(L, H\sqrt{5}, H).$$

With the help of MACSYMA, one calculates that

$$D_{2,1}(L, H\sqrt{5}, H) = -\frac{3}{2500}\, n\, (\frac{\alpha}{q})^4 (1 - 4\frac{H^2}{L^2}),$$

$$\frac{\partial}{\partial G} D_{2,0}(L, H\sqrt{5}, H) = -\frac{3\sqrt{5}}{250}\, \frac{n}{H}\, (\frac{\alpha}{q})^2$$

to arrive at the result

$$\gamma_1 \equiv \gamma_1(L, H) = -\frac{1}{50}\, (\frac{\alpha}{q})^2\, (1 - 4\frac{H^2}{L^2})\,. \tag{25_1}$$

Let S_1 and S_3 designate the equilibria at $g = 0$ and $g = \pi$ respectively. In the interval $0 \le H \le H_0$, the factor $(1 - 4\, H^2/L^2)$ in (25_1) is > 0, hence γ_1 is < 0 and Γ_1 is $\le H\sqrt{5}$, which means that, for the orbits whose perigees lie permanently on the line of nodes, the inclination is slightly less than the so-called critical inclination $I_c = \tan^{-1} 2$. Geometrically, on the sphere $\mathcal{S}(L, H)$, the equilibria S_1 and S_3 fall slightly below the critical parallel at the height ξ_0.

b) Another way of satisfying the equation $D_1 = 0$ is by taking $g \equiv \pi/2 \bmod \pi$. For a point in the meridian perpendicular to the line of nodes to be an equilibrium, the norm of the angular momentum must be a root of the equation

$$D_2(L, G, H, \pi/2 \bmod \pi) = D_{2,0}(L, G, H) + J_2\, D'_{2,1}(L, G, H) = 0,$$

the first order coefficient being this time

$$D'_{2,1} = \frac{3}{128}\, n \left(\frac{\alpha}{p}\right)^4 [\, \gamma_{0,0} - \gamma_{2,0} + \gamma_{0,1}\,\eta + (\gamma_{0,2} - \gamma_{2,2})\,\eta^2].$$

For the Newton-Raphson approximation, one calculates by MACSYMA that

$$D'_{2,1}(L, H\sqrt{5}, H) = -\frac{3}{25000}\, n \left(\frac{\alpha}{q}\right)^4 (9 - 35\,\frac{H^2}{L^2}),$$

and thus find that, in the power series

$$\Gamma_2 \equiv \Gamma_2(L, H) = H\sqrt{5}\,[1 + J_2\, \gamma_2 + \mathcal{O}(J_2{}^2)], \qquad (24_2)$$

the first order coefficient is

$$\gamma_2 \equiv \gamma_2(L, H) = \frac{1}{500} \left(\frac{\alpha}{q}\right)^2 (9 - 35\,\frac{H^2}{L^2}). \qquad (25_2)$$

By S_2 and S_4 we designate the equilibria respectively at $g = \pi/2$ and $g = 3\pi/2$. In the interval $0 \le H \le H_0$, the factor $(9 - 35\, H^2/L^2)$ in (25_2) is > 0, hence γ_2 is > 0, $\Gamma_2 \ge H\sqrt{5}$, and, for the orbits whose perigees are permanently perpendicular to the line of nodes, the inclination is slightly greater than the critical inclination.

Not for all values of H do these points belong to the phase shells $\mathcal{P}(L, H)$ even though they belong to the chart (g, G). The value H_1 at which the equilibria S_1 and S_3 reach the north pole of the sphere $\mathcal{P}(L, H_1)$ is the root of the equation $\Gamma_1 = L$ We solve it by a Newton-Raphson approximation to find that

$$H_1 \equiv H_1(L) = H_0\,[\,1 + \frac{1}{10}\, J_2 \left(\frac{\alpha}{a}\right)^2 + \mathcal{O}(J_2{}^2)]. \qquad (26)$$

Likewise we find that the equilibria S_2 and S_4 coalesce at the north pole of the sphere

$\mathcal{G}(L,H_2)$, where H_2 is the root of the equation $\Gamma_2 = L$. A Newton-Raphson approximation yields that

$$H_2 \equiv H_2(L) = H_0 \left[1 - \frac{1}{10} J_2 \left(\frac{\alpha}{a}\right)^2 + \mathcal{O}(J_2^2)\right]. \tag{27}$$

Cushman (1983) makes two statements concerning the families of orbits with stationary perigees:

(i) they do not exist in the interval $H_0 \le H < L$;

(ii) there are exactly four of them at every point in the interval $0 < H < H_0$.

Cushman's statements, our analysis shows, ought to be amended in the following way:

(i) there are no families of orbits with stationary perigees in the interval $H_1 \le H \le L$;

(ii) there are two such families in the interval $H_2 \le H < H_1$;

(iii) there are four of them in the interval $0 < H < H_2$.

We believe Cushman has overstated what he proved in his theorem. Concerning the equilibria S_1 - S_4, one should observe that Cushman's proof is merely local. In order to by-pass the involved differentiations leading to our equations (23_1) and (23_2), Cushman proposed to construct a tubular neighbourhood of the small circle of latitude $\xi_3 = \xi_0$, and a function N depending on the small parameter J_2 which, although simpler than the Hamiltonian $\mathcal{K}_0 + J_2 \mathcal{K}_1$, would have the same critical points. We surmise that a careful review of Cushman's estimate for the width δ of the tubular neighbourhood would restrict his construction to be valid only in the interval $0 < H < H_2$. As for the equilibria S_0 and S_5, one should note that Cushman's analysis is limited to order zero. Had he thrown in $\mathcal{K}_1$ as well, Cushman would have discovered the segment of instability along the families S_0.

5. Stability and bifurcations

Stability at the equilibria is decided by solving the variational equations

$$\frac{d}{d\tau}\,\delta G = -\,nG\left(P\,\delta g + Q\,\frac{\delta G}{G}\right), \qquad \frac{d}{d\tau}\,\delta g = n\left(Q\,\delta g + R\,\frac{\delta G}{G}\right), \tag{28}$$

Straightforward differentiations executed by means of MACSYMA deliver the dimensionless coefficients

$$P \equiv P(L, H, G, g) = \frac{1}{nG}\frac{\partial^2 K}{\partial g^2} = \frac{3}{16} J_2 \left(\frac{\alpha}{p}\right)^4 s^2 (-14 + 15 s^2)\, e^2 \cos 2g + \mathcal{O}(J_2^2), \qquad (29_1)$$

$$Q \equiv Q(L, H, G, g) = \frac{1}{n}\frac{\partial^2 K}{\partial G\, \partial g} = \frac{3}{32} J_2 \left(\frac{\alpha}{p}\right)^4 [(-28 + 186 s^2 - 165 s^4) + (28 - 158 s^2 + 135 s^4)\, \eta^2] \sin 2g + \mathcal{O}(J_2^2)], \qquad (29_2)$$

$$R \equiv R(L, H, G, g) = \frac{G}{n}\frac{\partial^2 K}{\partial G^2} = \left(\frac{\alpha}{p}\right)^2 \{\frac{3}{2}(-13 + 15 s^2) + \frac{3}{64} J_2 \left(\frac{\alpha}{p}\right)^2 \times [\zeta_{0,0} + \zeta_{0,1}\eta + \zeta_{0,2}\eta^2 + (\zeta_{2,0} + \zeta_{2,2}\eta^2)\cos 2g] + \mathcal{O}(J_2^2)\}, \qquad (29_3)$$

the coefficients in the right hand member of (29_3) being the inclination functions

$$\zeta_{0,0} = -2620 + 5070 s^2 - 2310 s^4, \qquad \zeta_{2,0} = -596 + 2520 s^2 - 1980 s^4,$$
$$\zeta_{0,1} = -1200 + 3096 s^2 - 1980 s^4, \qquad \zeta_{2,2} = 484 - 1804 s^2 + 1350 s^4,$$
$$\zeta_{0,2} = 204 - 54 s^2 - 225 s^4.$$

With g and G being given their values at equilibrium in the partial derivatives, the eigenvalues at the equilibria are the roots of the characteristic equation

$$\lambda^2 + n^2 (P R - Q^2) = 0$$

On the one hand, the argument of perigee being a multiple of $\pi/2$ for the equilibria at critical inclinations, there follows that $\sin 2g = 0$, hence that $Q(L,G,H,0 \bmod \pi/2) = \mathcal{O}(J_2^2)$, and the characteristic equation reduces to

$$\lambda^2 + n^2 P R = 0.$$

when terms of order 2 and higher are omitted. On the other hand,

$$R(L, H, G, 0 \bmod \pi) = \left(\frac{\alpha}{p}\right)^2 \{\frac{3}{2}(-13 + 15 s^2) + \frac{3}{64} J_2 \left(\frac{\alpha}{p}\right)^2 \times [\zeta_{0,0} + \zeta_{2,0} + \zeta_{0,1}\eta + (\zeta_{0,2} + \zeta_{2,2})\eta^2] + \mathcal{O}(J_2^2)\},$$

$$R(L, H, G, \pi/2 \bmod \pi) = (\frac{\alpha}{p})^2 \{ \frac{3}{2}(-13 + 15 s^2) + \frac{3}{64} J_2 (\frac{\alpha}{p})^2 \times$$

$$[\zeta_{0,0} - \zeta_{2,0} + \zeta_{0,1}\eta + (\zeta_{0,2} - \zeta_{2,2}) \eta^2] + \mathcal{O}(J_2^2) \};$$

hence, in the vicinity of each equilibrium,

$$R(L, H, H\sqrt{5}, 0 \bmod \pi/2) = -\frac{3}{50}(\frac{\alpha}{q})^2 + \mathcal{O}(J_2),$$

is negative. Consequently the question of deciding whether the equilibrium is a center or a saddle point on the sphere $\mathcal{S}(L, H)$ rests with the sign of the partial derivative P. In this regard, one sees immediately that

$$P(L, H, G, 0 \bmod \pi) = -\frac{3}{16} J_2 (\frac{\alpha}{p})^4 e^2 s^2 (14 - 15 s^2) + \mathcal{O}(J_2^2),$$

$$P(L, H, G, \pi/2 \bmod \pi) = \frac{3}{16} J_2 (\frac{\alpha}{p})^4 e^2 s^2 (14 - 15 s^2) + \mathcal{O}(J_2^2),$$

from which there follows that, at the equilibria,

$$P(L, H, H\sqrt{5}, 0 \bmod \pi) = -\frac{3}{6250} J_2 (\frac{\alpha}{q})^4 e_0^2 + \mathcal{O}(J_2^2),$$

$$P(L, H, H\sqrt{5}, \pi/2 \bmod \pi) = \frac{3}{6250} J_2 (\frac{\alpha}{q})^4 e_0^2 + \mathcal{O}(J_2^2).$$

This means that, the terms of order 2 and higher being omitted, P is ≤ 0 at S_1 and S_3, and ≥ 0 at S_2 and S_4, or that S_1 and S_3 are stable equilibria whereas S_2 and S_4 are unstable (Strubble, 1960). Needless to say, the stability may change drastically when higher zonal harmonics are incorporated into the perturbation, not only J_4 (Hori, 1960b; Aoki 1962, 1963b) but also the harmonics of odd degree (Petty-Breakwell, 1960; Aoki, 1963 a, 1963c; Jupp, 1975; Lyddane and Cohen, 1978) or even the attractions by sun and moon (Hough, 1981).

In the global model set up for the phase space in the main problem of artificial satellite theory, we can sharpen these classical results by showing that the families S_1 and S_3 of stable equilibria as well as the families S_2 and S_4 of unstable equilibria bifurcate from the family S_0

of equilibria corresponding to the circular orbits. Bifurcations might indeed occur at the points in the base plane (L, H) where the families terminate if the characteristic exponents of the boundary orbits are zero (see e.g. Deprit and Henrard, 1968).

In the coordinates (ξ_1, ξ_2, ξ_3),

$$P \equiv P(L, H, \xi_1, \xi_2, \xi_3) = -\frac{3}{16} J_2 \left(\frac{\alpha}{p}\right)^4 (15\, c^2 - 1) \frac{\xi_1^2 - \xi_2^2}{G^2 L^2} + \mathcal{O}(J_2^2).$$

The equilibria S_1 and S_3 belong to the meridian plane $\xi_2 = 0$, the equilibria S_2 and S_4 to the meridian $\xi_1 = 0$; in those meridian planes,

$$P \equiv P(L,H, \xi_1, 0, \xi_3) = +\frac{3}{16} J_2 \left(\frac{\alpha}{p}\right)^4 (15\, c^2 - 1) \quad \frac{\frac{1}{4}(L^2 - H^2)^2 - \xi_3^2}{G^2 L^2} + \mathcal{O}(J_2^2),$$

$$P \equiv P(L, H, 0, \xi_2, \xi_3) = -\frac{3}{16} J_2 \left(\frac{\alpha}{p}\right)^4 (15\, c^2 - 1) \quad \frac{\frac{1}{4}(L^2 - H^2)^2 - \xi_3^2}{G^2 L^2} + \mathcal{O}(J_2^2).$$

With each family terminating at the north pole of the spheres $\mathcal{S}(L, H_0)$, there follows that P = 0, hence that the characteristic exponents vanish at the boundary orbits. There remains now to check that, along the family S_0 itself, the characteristic exponents are flipping from stability to instability or vice versa precisely at the bifurcation orbits.

6. Bifurcations on the family of circular orbits

At the time it was discovered, we said in the introduction, the critical inclination manifested itself in two ways, through the existence of manifolds of orbits with stationary perigees, on the one hand, and as the spot in phase space where conventional continuation methods fail in extending families of circular orbits, on the other hand. Having completed a detailed analysis of the cross-sections of the manifolds of orbits with stationary perigees on the orbit spheres $\mathcal{S}$(L, H), we now turn to the families of circular orbits. We shall find that there is a geometric reason, independent of the variables used and of the method employed, why the families of circular orbits cannot be continued analytically beyond the critical inclination.

The cylindrical chart (g, G) being inappropriate for orbits of zero eccentricity, we shall operate instead in the Cartesian coordinates (ξ_1, ξ_2, ξ_3) since they are defined globally on the spheres $\mathcal{S}$(L, H). In those coordinates the reduced Hamiltonian (5) becomes the function

$$\mathcal{K}_1 = \frac{1}{128} nG \left(\frac{\alpha}{p}\right)^4 \left(m_{0,0} + m_{0,1}\eta + m_{0,2}\eta^2 + m_{2,0}\frac{\xi_1^2 - \xi_2^2}{L^2 G^2}\right). \tag{30}$$

In application of Liouville theorem's, the global equations of motion are

$$\frac{d}{d\tau}\xi_1 = (\xi_1; \mathcal{K}), \qquad \frac{d}{d\tau}\xi_2 = (\xi_2; \mathcal{K}), \qquad \frac{d}{d\tau}\xi_3 = (\xi_3; \mathcal{K}).$$

Building the right hand members of these canonical equations is a rather surprisingly simple task in global coordinates. The Poisson brackets are evaluated as sums of the form

$$(\xi_k; \mathcal{K}) = (\xi_k; G)\frac{\partial \mathcal{K}}{\partial G} + (\xi_k; p)\frac{\partial \mathcal{K}}{\partial p} + (\xi_k; \eta)\frac{\partial \mathcal{K}}{\partial \eta}$$

$$+ (\xi_k; s^2)\frac{\partial \mathcal{K}}{\partial s^2} + (\xi_k; \xi_1)\frac{\partial \mathcal{K}}{\partial \xi_1} + (\xi_k; \xi_2)\frac{\partial \mathcal{K}}{\partial \xi_2}$$

in which all partial derivatives are taken for $\mathcal{K}$ as an explicit function of the variables ($G, p, \eta, s^2, \xi_1, \xi_2$). The Poisson brackets ($\xi_j; \xi_k$) have been evaluated in Section 3. The others are easily deduced from the basic identities

$$(\xi_1; G) = -\xi_2, \qquad (\xi_2; G) = \xi_1, \qquad (\xi_3; G) = 0.$$

On account of the formulas $\partial p/\partial G = 2\,p/G$, $\partial\eta/\partial G = 1/L$, and $\partial s^2/\partial G = 2\,c^2/G$, there follows at once that

$$(\xi_1; p) = -2\frac{p}{G}\xi_2, \qquad (\xi_2; p) = 2\frac{p}{G}\xi_1, \qquad (\xi_3; p) = 0,$$

$$(\xi_1; \eta) = -\frac{\xi_2}{L}, \qquad (\xi_1; \eta) = \frac{\xi_1}{L}, \qquad (\xi_3; \eta) = 0,$$

$$(\xi_1; s^2) = -2\,c^2\frac{\xi_2}{G}, \qquad (\xi_2; s^2) = 2\,c^2\frac{\xi_1}{G}, \qquad (\xi_3; s^2) = 0.$$

After introducing the dimensionless functions

$$M_1 = -\frac{3}{4}\left(\frac{\alpha}{p}\right)^2(1 - 5c^2) + \frac{3}{128}J_2\left(\frac{\alpha}{p}\right)^4\left(x_{0,0} + x_{0,1}\eta + x_{0,2}\eta^2 + x_{2,0}\frac{\xi_1^2}{G^2 L^2}\right) + \mathcal{O}(J_2^2),$$

$$M_2 = -\frac{3}{4}\left(\frac{\alpha}{p}\right)^2(1 - 5c^2) + \frac{3}{128} J_2 \left(\frac{\alpha}{p}\right)^4 \left(y_{0,0} + y_{0,1}\eta + y_{0,2}\eta^2 + y_{2,0}\frac{\xi_2^2}{G^2 L^2}\right) + \mathcal{O}(J_2^2),$$

$$M_3 = \frac{3}{8} J_2 \left(\frac{\alpha}{p}\right)^4 \frac{1}{L^2} z_{2,0} + \mathcal{O}(J_2^2)$$

involving the inclination polynomials

$$x_{0,0} = -49 + 378\, c^2 + 55\, c^4, \qquad y_{0,0} = -21 - 198\, c^2 + 715\, c^4,$$

$$x_{0,1} = 24 - 192\, c^2 + 360\, c^4, \qquad y_{0,1} = 24 - 192\, c^2 + 360\, c^4,$$

$$x_{0,2} = 35 - 350\, c^2 + 315\, c^4, \qquad y_{0,2} = 15 - 98\, c^2 - 225\, c^4,$$

$$x_{2,0} = 36 - 660\, c^2, \qquad y_{2,0} = -36 + 660\, c^2, \qquad z_{2,0} = 1 - 15\, c^2,$$

one finds that the differential equations in the global variables are of the form

$$\frac{d\xi_1}{d\tau} = -n\, \xi_2 M_1, \qquad \frac{d\xi_2}{d\tau} = n\, \xi_1 M_2, \qquad \frac{d\xi_3}{d\tau} = -n \frac{\xi_1 \xi_2}{L^2} M_3, \tag{31}$$

The right hand members of equations (31) vanish together at $\xi_1 = \xi_2 = 0$, that is to say, at the poles of the spheres $\mathcal{S}(L, H)$. If the reader will remember, the north pole of $\mathcal{S}(L, H)$ in an averaged perturbed Keplerian problem corresponds to the class of circular orbits in orbital planes whose inclination is equal to $\cos^{-1} H/L$, whereas the south pole stands for the ellipses of eccentricity $(1 - H^2/L^2)^{\frac{1}{2}}$ in the plane whose inclination is I mod $\pi = 0$. These are precisely the additional classes of equilibria needed to account for the phase flow on each sphere $\mathcal{S}(L, H)$ in a manner consistent with Morse's Theory in Global Analysis. In fact little of Global Analysis is needed beyond the intuitive idea that, if a function is sufficiently smooth and regular, its level contours on a sphere must evoke a relief much like a landscape on earth. The Meyer-reduction having converted the averaged main problem into a system with only one degree of freedom, hence integrable, the curves described by the solutions of the global equations are the level contours of the Hamiltonian $\mathcal{K}$ on the sphere $\mathcal{S}(L, H)$, hence $\mathcal{K}$ defines the height in a kind of topography on the sphere. North and south poles are readily shown to be minima of $\mathcal{K}$. When the first order term $\mathcal{K}_1$ is omitted, the small circle of degenerate equilibria at the critical inclination delineates the ridge of a zonal bulge. The second order term in cos 2g destroys the axial symmetry in the first order profile. By raising the bulge at one place while lowering it in compensation elsewhere, it creates a dome at S_1 and another one of equal height at S_3. The equilibria S_2 and S_4, when they exist, are located at the top of the mountain passes connecting the domes in one direction and the polar depressions in the other direction. When the families S_2 and S_4 terminate, the passes have come to coincidence at the north pole of the sphere $\mathcal{S}(L, H_2)$. As H increases from H_2 to H_1, the top of the gap at the north pole rises with respect to the domes, hence its floor gets flatter and

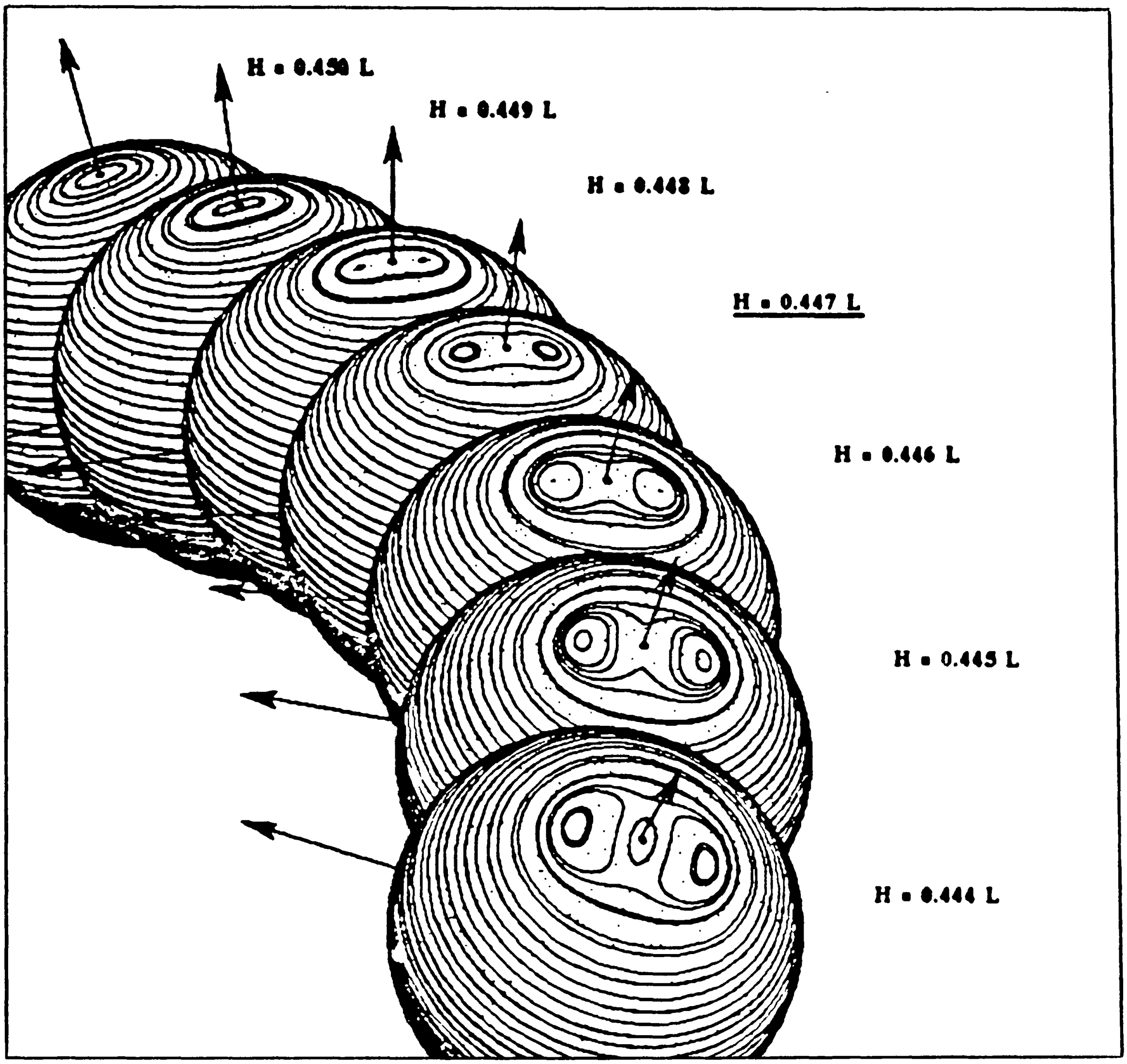

Figure 5. Phase flow in the orbit space.

flatter. Finally, at H_1, the domes are smoothed out, and the north pole has become the point where Hamiltonian $\mathcal{K}$ now reaches its maximum. (For a formal survey of the terrain at the north pole, see Cushman, 1986).The curves plotted on the spheres in Figure 5 were not obtained by connecting points of equal values of $\mathcal{K}$ over the sphere — a technique proposed first by Kozai(1963) in relation with lunar orbiters, apparently re-discovered by J.C. Smith (1986) for designing orbits for TOPEX satellites. Rather they result from integrating numerically equations (31), since the level contours of $\mathcal{K}$ are precisely the orbits followed by the solutions of the equations of motion. As long as the perturbation $\mathcal{K}_1$ is ignored in the averaged Hamiltonian, the flow on each sphere $\mathcal{S}(L, H)$ acts in the manner of a differential rotation around the ξ_3–axis with the angular velocity

$$\omega = - \frac{3}{4} n \left(\frac{\alpha}{p}\right)^2 (1 - 5 c^2) \mathbf{b},$$

where $\mathbf{b}$ is the direction of the ξ_3–axis. The poles as well as the points on the small circle of latitude at the height $\xi_3 = \xi_0$ are fixed under that rotation. The rotation is direct in the equatorial zone $-\xi_0 < \xi_3 < \xi_0$, and retrograde on the polar caps. The perturbation $\mathcal{K}_1$ affects the global circulation pattern by fostering counterclockwise eddies at the equilibria S_1 and S_3. The evolution in the circulation patterns on the spheres $\mathcal{S}(L, H)$ for a given L as H goes from 0 to L is depicted in Figure 5. For each value of H, the bit–map created to draw the integrated curves, sequentially as they were actually computed, in an orthographic projection of the sphere on the screen was stored in the memory of the workstation. Fetching them back from memory, and using drawing tools designed by B. R. Miller, we arranged them immediately on the screen of the workstation to compose Figure 5.

Miller's package of LISP graphics functions proved an invaluable instrument in the course of the research leading to this paper. The *Mathematical Doodler,* as we came to call this program, is more than an artist's toolbox for illustrating concepts and results; it is an instrument for numerical explorations, with the operator at the console of the terminal passing graphic or mathematical commands as he reacts on the spot to the results unfolding in front of his eyes.

In principle, the variations from an orbit in the global coordinate system are of the form

$$\frac{d}{d\tau}\delta\xi_1 = -n M_1 \delta\xi_2 - n\xi_2\left(\frac{\partial M_1}{\partial\xi_1}\delta\xi_1 + \frac{\partial M_1}{\partial\xi_2}\delta\xi_2 + \frac{\partial M_1}{\partial\xi_3}\delta\xi_3\right), \qquad (32_1)$$

$$\frac{d}{d\tau}\delta\xi_2 = n M_2 \delta\xi_1 + n\xi_1\left(\frac{\partial M_2}{\partial\xi_1}\delta\xi_1 + \frac{\partial M_2}{\partial\xi_2}\delta\xi_2 + \frac{\partial M_2}{\partial\xi_3}\delta\xi_3\right), \qquad (32_2)$$

$$\frac{d}{d\tau}\delta\xi_3 = -n\frac{M_3}{L^2}(\xi_1\delta\xi_2 + \xi_2\delta\xi_1) - n\frac{\xi_1\xi_2}{L^2}\left(\frac{\partial M_3}{\partial\xi_1}\delta\xi_1 + \frac{\partial M_3}{\partial\xi_2}\delta\xi_2 + \frac{\partial M_3}{\partial\xi_3}\delta\xi_3\right). \qquad (32_3)$$

Of the solutions of this differential system, only the variational curves lying on the spheres $\mathscr{S}(L, H)$ must be considered, that is to say, only those satisfying identically the constraint

$$\xi_1\delta\xi_1 + \xi_2\delta\xi_2 + \xi_3\delta\xi_3 = 0.$$

Along a circular orbit, $\xi_1 = \xi_2 = 0$, hence $\delta\xi_3 = 0$, and the variational equations reduce to the elementary system

$$\frac{d}{d\tau}\delta\xi_1 = -n M_1 \delta\xi_2, \qquad \frac{d}{d\tau}\delta\xi_2 = n M_2 \delta\xi_1. \qquad (33)$$

There follows that the characteristic exponents for a circular orbit are the solutions of the quadratic equation:

$$\lambda^2 + n^2 M_1 M_2 = 0. \qquad (34)$$

Thus the stability of a circular orbit is decided by the sign of the product $M_1 M_2$. At the north pole,

$$M_1 = -\frac{3}{4}\left(\frac{\alpha}{a}\right)^2\left(1 - 5\frac{H^2}{L^2}\right) + \frac{3}{64}J_2\left(\frac{\alpha}{a}\right)^4\left(5 - 82\frac{H^2}{L^2} + 365\frac{H^4}{L^4}\right) + \mathcal{O}(J_2^2),$$

$$M_2 = -\frac{3}{4}\left(\frac{\alpha}{a}\right)^2\left(1 - 5\frac{H^2}{L^2}\right) + \frac{3}{64}J_2\left(\frac{\alpha}{a}\right)^4\left(9 - 146\frac{H^2}{L^2} + 425\frac{H^4}{L^4}\right) + \mathcal{O}(J_2^2).$$

The terms of order 1 and higher being omitted, the graphs of the coefficients M_1 and M_2 as functions of H are parabolas reaching their minimum at $H = 0$ and crossing the abscissa axis at $H = H_0$. It would then seem that the product $M_1 M_2$ is always ≥ 0 in the interval $0 \leq H \leq L$, or that the characteristic exponents are purely imaginary so that the circular orbits are always stable. But the first order corrections in M_1 and M_2 force us to revise our conclusion substantially in the neighbourhood of H_0. Indeed it is readily checked that M_1 vanishes at $H = H_1$, and M_2 at $H = H_2$. So, whereas the product $M_1 M_2$ is > 0 in the intervals $0 < H < H_1$ and $H_2 < H \leq L$, it is ≤ 0 in the interval $H_1 \leq H \leq H_2$, and there appears a segment on the family S_0 where the circular orbits are unstable. This portion of instability ties in with the fact that the families S_1 and S_3 terminate at H_1 on an orbit belonging to S_0 , and so do the families S_2 and S_4 at H_2. In summary, for each value of $L > 0$, pitchfork bifurcations occur along the family $S_0(L, H)$, one at $H = H_1$, and the other one at $H = H_2$.

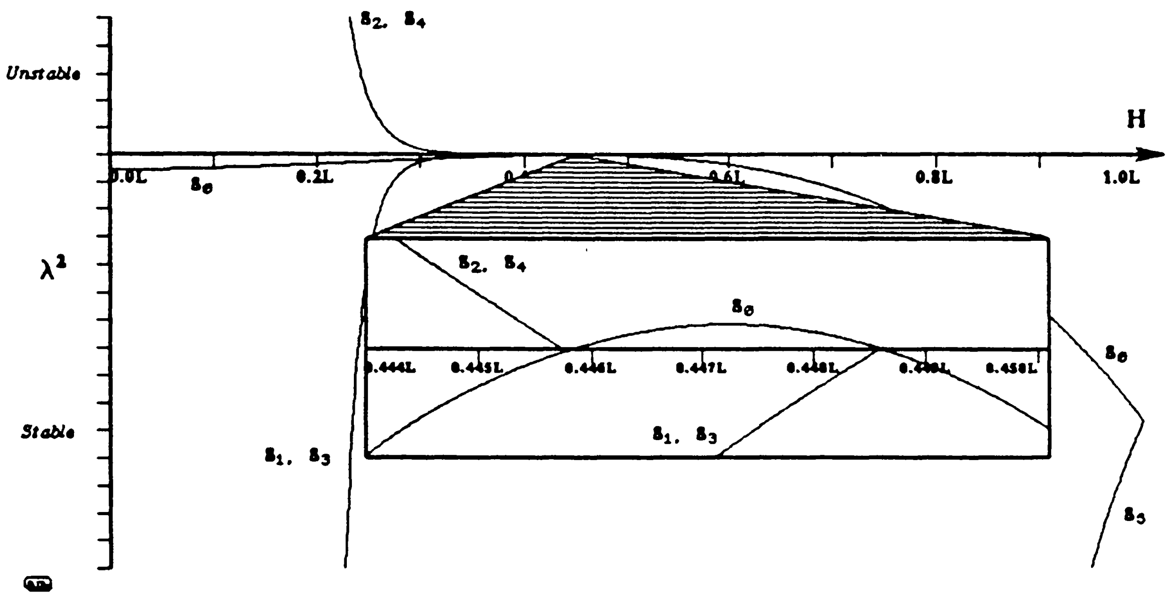

Figure 6. Stability of families of periodic orbits: λ^2 vs H

At the south pole,

$$M_1 = 3 \left(\frac{\alpha}{q}\right)^2 \left[1 + \frac{3}{2} J_2 \left(\frac{\alpha}{q}\right)^2 \left(2 + \frac{H}{L}\right) + \mathcal{O}(J_2^2) \right],$$

$$M_2 = 3\left(\frac{\alpha}{q}\right)^2 \left[1 + \frac{1}{8} J_2 \left(\frac{\alpha}{q}\right)^2 \left(31 - 12\frac{H}{L} - 7\frac{H^2}{L^2}\right) + \mathcal{O}(J_2^2) \right],$$

hence the product $M_1 M_2$ is always positive, the characteristic exponents stay purely imaginary in the interval $0 \leq H \leq L$, and all the orbits in the family S_5 are stable. Nothing like the double bifurcation we encountered at the north pole occurs at the south pole. There, as H tends to zero, all four families S_1 - S_4 converge jointly toward the same collinear orbit, and they disappear simultaneously at $H = 0$.The curves λ^2 *versus* H plotted in Figure 6, especially the enlargement obtained by zooming onto a neighbourhood of H_0, encapsulate the interplay of stability and instability between the six families of critical orbits in the main problem of artificial satellite theory.

7. Adequacy of some intermediaries

Far from being an artifact due to the choice of coordinates or the selection of the perturbation algorithm, the circulation patterns depicted in Figure 5 belong intrinsically to the averaged flow of the main problem in the theory of artificial satellites. Therefore it stands to reason that, when choosing an intermediary Hamiltonian as an integrable approximation of the main problem, one should examine how closely the phase flow of the averaged intermediary resembles that of the averaged main problem. This issue has never been raised, perhaps because it is of a global nature whereas most of the attention has been directed toward assessing the merits of intermediaries in terms of formal asymptotic expansions and of analytic simplifications. At the very least an intermediary for the main problem of artificial satellite theory should induce a differential rotation on the phase spheres about the north-south axis, preferably one that leaves invariant a small circle of latitude in the vicinity of the critical inclination. Ideally one should also find in the second order part of the intermediary Hamiltonian a term in cos 2g in order to dispel the first order degeneracy and leave only four isolated equilibria in addition to the fixed points at the poles of the rotation axis. The latter condition may be exorbitant. Among the classical intermediaries we have studied, the one coming the closest to the ideal is the very first separable Hamiltonian (see Equation 6) proposed to coincide at the first order, after the Delaunay normalization, with the averaged main problem in artificial satellite theory.

Sterne (1957) was interested in a closed form approximation to the solutions of the main problem, and he found it in a complete solution by separation of the coordinates of the Hamilton-Jacobi equation based on (6). For our part we are interested in showing how closely the flow induced by Sterne's averaged Hamiltonian resembles what we have seen in the full main problem of artificial satellite theory. To this end, overlooking the fact that $\mathcal{J}$ in (6) is separable; we shall treat the system by conventional perturbation techniques.

First, in order to make the Delaunay normalization simpler to execute by machine, we remove the argument of latitude θ by an operation called the elimination of the parallax (Deprit, 1981). The Lie transformation is generated by a series

$$\mathcal{W}^\dagger = \mathcal{W}^\dagger_1 + J_2\, \mathcal{W}^\dagger_2 + \tfrac{1}{2} J_2^2\, \mathcal{W}^\dagger_3 + \mathcal{O}(J_2^3).$$

It has been calculated by machine to the third order. After seeing the first two terms,

$$\mathcal{W}^\dagger_1 = \frac{1}{8}\, \Theta \left(\frac{\alpha}{p}\right)^2 [(4 - 6s^2)(S\cos\theta - C\sin\theta) - 3 s^2 \sin 2\theta],$$

$$\begin{aligned}\mathcal{W}^\dagger_2 = \frac{1}{128}\, \Theta \left(\frac{\alpha}{p}\right)^4 \{ & S(256 - 504 s^2 + 288 s^4)\cos\theta \\ & + C(-256 + 1032 s^2 - 864 s^4)\sin\theta \\ & + S\,C(+48 - 144 s^2 + 108 s^4)\cos 2\theta \\ & + [144 s^2 c^2 + (S^2 - C^2)(24 - 72 s^2 + 54 s^4)]\sin 2\theta \\ & + s^2(72 - 96 s^2)(S\cos 3\theta - C\sin 3\theta) - 9 s^4 \sin 4\theta\},\end{aligned}$$

the reader will readily admit that the third order generator is boring to anyone but a handful of MAO partisans operating in LISP country. After all, the elimination of the parallax should have been stopped after the third order term had been extracted from the transformed Hamiltonian before performing the quadrature yielding $\mathcal{W}_3$. The quantities C and S in the generator are the elements labelled C* and S* in (Deprit, 1981, p. 115), that is to say

$$C = e\cos g, \qquad\qquad S = e\sin g.$$

From that paper we also borrow the notations

$$\beta = (1 + \eta)^{-1}, \qquad\qquad \varphi = f - \ell$$

with f standing for the true anomaly so that φ designates the so-called equation of the center. The elimination of the parallax converts Sterne's intermediary Hamiltonian into a series

$$\mathcal{J}^\dagger \equiv \mathcal{J}^\dagger(L, G, H, \ell, g) = \mathcal{J}^\dagger_0 + J_2\, \mathcal{J}^\dagger_1 + \tfrac{1}{2} J_2^2\, \mathcal{J}^\dagger_2 + (1/6) J_2^3\, \mathcal{J}^\dagger_3 + \mathcal{O}(J_2^4) \qquad (35)$$

whose terms are:

$$\mathcal{J}^\dagger_0 = \mathcal{H}_0,$$

$$\mathcal{H}^\dagger_1 = \frac{1}{4}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2(-2+3s^2), \tag{36}$$

$$\mathcal{H}^\dagger_2 = \frac{1}{32}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^4[-40+84s^2-9s^4+(-12+36s^2-27s^4)(C^2+S^2)],$$

$$\mathcal{H}^\dagger_3 = \frac{1}{128}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^6[-1248+5832s^2-9558s^4+4860s^6$$
$$+S^2(-696+3888s^2-6264s^4+3159s^6)$$
$$+C^2(-696+2376s^2-3132s^4+1539s^6)].$$

The Hamiltonian $\mathcal{H}^\dagger$ still depends on the mean anomaly, but only through the factors Θ^2/r^2. Thus it is still amenable to a Delaunay normalization, A quick glance at the first order solution developed in full by Sterne (1960, pp. 121-126) with the collaboration of L. E. Cunningham is enough, it is hoped, to convince the reader that averaging after the elimination of the parallax is considerably simpler than a straightforward elimination performed immediately on the original Hamiltonian (6). The result is once again a power series in the small parameter J_2. Dropping the principal term $\mathcal{H}^\dagger_0$, and adopting the long time scale τ, we bring it to the form

$$\mathcal{H}^* \equiv \mathcal{H}^*(L, G, H, g) = \mathcal{H}^*_0 + J_2\,\mathcal{H}^*_1 + \tfrac{1}{2}J_2^2\,\mathcal{H}^*_2 + \mathcal{O}(J_2^3) \tag{37}$$

with the components

$\mathcal{H}^*_0 = \mathcal{K}_0$, as in Equation (4)

$$\mathcal{H}^*_1 = \frac{3}{64}\,n\,G\left(\frac{\alpha}{p}\right)^4[-20+48s^2-18s^4$$
$$+(-8+24s^2-18s^4)\eta+(4-12s^2+9s^4)\eta^2],$$

$$\mathcal{H}^*_2 = \frac{3}{128}\,n\,G\left(\frac{\alpha}{p}\right)^6[-280+1260s^2-1936s^4+936s^6$$
$$+(-120+468s^2-540s^4+162s^6)\eta$$
$$+(72-324s^2+486s^4-243s^6)\eta^2$$
$$+(40-180s^2+270s^4-135s^6)\eta^3$$
$$+(-84+174s^2-90s^4)s^2e^2\cos 2g].$$

Basically, for Sterne's intermediary like it is for the main problem in artificial satellite theory, the flow on the spheres $\mathcal{S}$(L, H) is the differential rotation induced by the Hamiltonian $\mathcal{K}_0$; the first order terms introduce corrections to the height of the small circle of degenerate equilibria and to the angular velocity of rotation. Belatedly, at the second order appears the term in cos 2g responsible for dissipating the degeneracy. In that regard, the intermediaries of Aksnes and Cid–Lahulla(1969) are definitely more rudimentary. When treated by perturbation techniques, both yield asymptotic approximations that depend exclusively on the actions L, G, and H, at least as far as the second order.

In the case of Aksnes' intermediary, we find that the elimination of the parallax applied to Hamiltonian (7) yields a series of the type (35) with the same coefficients $\mathcal{J}^{\dagger}_0$ and $\mathcal{J}^{\dagger}_1$, but with

$$\mathcal{J}^{\dagger}_2 = -\frac{9}{32}\,\frac{\Theta^2}{r^2}\,(\frac{\alpha}{p})^4\, s^2\,(9\, s^2 - 4),$$

$$\mathcal{J}^{\dagger}_3 = \frac{9}{64}\,\frac{\Theta^2}{r^2}\,(\frac{\alpha}{p})^6 s^2\,(60 - 303\, s^2 + 270\, s^4).$$

As a side effect of the elimination of the parallax, the argument of perigee is made ignorable in Aksnes' intermediary at least up to the third order. Hence, without having to average over the mean anomaly, we can already conclude that, to the third order, the phase flow in Aksnes' system boils down to a mere differential rotation about the ξ_3–axis on the spheres $\mathcal{S}$(L, H). Nevertheless, for the benefit of the reader interested in comparing Brouwer's solution with the one proposed in Aksnes (1970; see also Deprit and Richardson, 1982), we shall outline our procedure in full. The Lie transformation that eliminates the argument of latitude is generated here by a series in the small parameter of the form

$$\mathcal{W}^{\dagger} = \mathcal{W}^{\dagger}_1 + J_2\, \mathcal{W}^{\dagger}_2 + \tfrac{1}{2}\, J_2^2\, \mathcal{W}^{\dagger}_3 + \mathcal{O}(J_2^3),$$

the coefficients being

$$\mathcal{W}^{\dagger}_1 = -\frac{3}{8}\,\Theta\,(\frac{\alpha}{p})^2\, s^2 \sin 2\,\theta,$$

$$\mathcal{W}^{\dagger}_2 = \frac{3}{128}\,\Theta\,(\frac{\alpha}{p})^4\,[\,(80 - 96\, s^2)\, s^2 \sin 2\,\theta - 3\, s^4 \sin 4\,\theta\,],$$

$$\mathcal{W}^{\dagger}_3 = \frac{9}{512}\,\Theta\,(\frac{\alpha}{p})^6\,[\,(-896 + 2712\,s^2 - 2175\,s^4)\,s^2 \sin 2\,\theta + (380 - 432\,s^2)\,s^4 \sin 4\,\theta - 3\,s^6 \sin 6\,\theta\,].$$

Then the Delaunay averaging leads to a reduced Hamiltonian of the form (37). As expected, Aksnes' averaged intermediary agrees with Sterne's in its principal term, and differs from it by the perturbation terms which are here:

$$\mathcal{J}^*_1 = \frac{1}{64}\,n\,G\,(\frac{\alpha}{p})^4\,[\,-8 - 12\,s^2 + 63\,s^4) + \eta\,(-24 + 72\,s^2 - 54\,s^4)\,],$$

$$\mathcal{J}^*_2 = \frac{1}{384}\,n\,G\,(\frac{\alpha}{p})^6\,[\,(-16 + 360\,s^2 - 1656\,s^4 + 1431\,s^6) + \eta\,(-48 + 486\,s^4 - 567 s^6) + \eta^2\,(-64 + 288\,s^2 - 432\,s^4 + 216\,s^6)\,].$$

The coefficients in the generator of the Lie transformation which effected the Delaunay normalization are:

$$\mathcal{W}^*_1 = \frac{1}{4}\,G\,(\frac{\alpha}{p})^2\,\varphi\,(-2 + 3\,s^2),$$

$$\mathcal{W}^*_2 = \frac{1}{32}\,G\,(\frac{\alpha}{p})^4\,[\,\varphi\,(-8 - 12\,s^2 + 63\,s^4) + \beta\,(-16 + 48\,s^2 - 36\,s^4)\,e \sin f + \beta\,(-4 + 12\,s^2 - 9\,s^4)\,e^2 \sin 2f\,],$$

$$\begin{aligned}\mathcal{W}^*_3 = \frac{1}{512}\,G\,(\frac{\alpha}{p})^6\,\{\,&\varphi\,(-192 + 4320\,s^2 - 19872\,s^4 + 17172\,s^6)\\ &+ [\,\beta\,(-640 + 1152\,s^2 + 2160\,s^4 - 3672\,s^6)\\ &\quad + \eta^{-1}\,(192 - 864\,s^2 + 1296\,s^4 - 648\,s^6)\,]\,e \sin f\\ &+ [\,\beta\,(-256 + 720\,s^2 - 108\,s^4 - 594\,s^6)\\ &\quad + \beta^2\,(-128 + 576\,s^2 - 864\,s^4 + 432\,s^6)\\ &\quad + \eta^{-1}\,(176 - 792\,s^2 + 1188\,s^4 - 594\,s^6)\,]\,e^2 \sin 2f\\ &+ (\,\beta + \beta^2 - \eta^{-1})\,(-64 + 288\,s^2 - 432\,s^4 + 216\,s^6)\,e^3 \sin 3f\\ &+ (\,\beta + \beta^2 - \eta^{-1})\,(-8 + 36\,s^2 - 54\,s^4 + 27\,s^6)\,e^4 \sin 4f\,\}.\end{aligned}$$

The same techniques will now be applied to the Hamiltonian $\mathcal{H}_0 + J_2\,\mathcal{H}_1$ with

$$\mathcal{H}_1 = \frac{\mu}{r}\left(\frac{\alpha}{r}\right)^2\left(\frac{1}{2} - \frac{3}{4}s^2\right),$$

which Cid (1969) proposed as an intermediary for the main problem in satellite theory. Thus a preliminary elimination of the parallax will yield a series like (35) with the coefficients

$$\mathcal{J}'_0 = \mathcal{H}_0,$$

$$\mathcal{J}'_1 = \frac{1}{4}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^2(-2 + 3s^2),$$

$$\mathcal{J}'_2 = \frac{1}{32}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^4[-40 + 120s^2 - 90s^4 + (-12 + 36s^2 - 27s^4)(C^2 + S^2)],$$

$$\mathcal{J}'_3 = \frac{1}{128}\frac{\Theta^2}{r^2}\left(\frac{\alpha}{p}\right)^6[-1248 + 5616s^2 - 8424s^4 + 4212s^6 + (-696 + 3132s^2 - 4698s^4 + 2349s^6)(S^2 + C^2)].$$

Exactly like it did for Aksnes' intermediary, the elimination of the parallax removes the argument of perigee from Cid's Hamiltonian. After the Delaunay normalization, one ends up therefore with a Hamiltonian that, at least to the third order in J_2, depends exclusively on the momenta L, G, and H. In the long time scale τ after omitting the Keplerian part, one finds that

$$\mathcal{J}^*_0 = \mathcal{K}_0,$$

$$\mathcal{J}^*_1 = \frac{3}{64}nG\left(\frac{\alpha}{p}\right)^4[-20 + 60s^2 - 45s^4 + (-8 + 24s^2 - 18s^4)\eta + (4 - 12s^2 + 9s^4)\eta^2],$$

$$\mathcal{J}^*_2 = \frac{3}{128}nG\left(\frac{\alpha}{p}\right)^6[-280 + 1260s^2 - 1890s^4 + 945s^6 + (-120 + 540s^2 - 810s^4 + 405s^6)\eta + (72 - 324s^2 + 486s^4 - 243s^6)\eta^2 + (40 - 180s^2 + 270s^4 - 135s^6)\eta^3].$$

It is not our task here to evaluate intermediaries for the main problem in artificial satellite theory. We focused rather on the global properties of the major representatives. Our trio of intermediaries reveals, despite their differences in analytical purposes, a common virtue and a common deficiency: they all include the first order term of the main problem, and thus agree with it in determining a differential rotation in the reduced phase space, and they all fail in capturing at the second order the crucial term in cos 2g responsible for driving out the first order degeneracy. [*Note* . Most recently, in collaboration with Ferrer and Sein-Echaluce, Cid (1986) has extended his intermediary to the second order. We found that the additional terms, when averaged over the mean anomaly, introduce the critical terms in cos 2g already at order two.]

Celestial mechanics, whatever mathematics happen to govern it, must be worked out in a specific computing milieu. The period following the theories of Brouwer and Vinti for artificial satellites witnessed a triumphant revival of analytical developments carried out manually. This is the context in which Sterne, Garfinkel, and their followers developed the concept of intermediaries with a view of alleviating the burden of developing asymptotic solutions by hand.The environment, since then, has changed dramatically. It took a little less than two hours, at the console of a LISP workstation, not only to perform all the reductions reported in this Section, but also to edit them in a readable form, Greek symbols, fractions - all reduced to their simple form -, subscripts as well as superscripts, all properly aligned the way one is used to finding them in a conventional mathematical text. If simplifications of long hand developments is no longer an issue, what then is the purpose of an intermediary when it fails to induce the global flow of the problem it claims to approximate?

8. Conclusions

A method has been designed to analyze a certain class of perturbed Keplerian systems after they have been reduced by a Delaunay normalization, namely those systems whose averaged form is invariant with respect to the group of rotations around a fixed axis. In which case, the fact that the right ascension of the ascending node is invariable permits a second Meyer reduction by which the averaged perturbed system is made equivalent to a Hamiltonian flow on a two-dimensional sphere.

The method worked very well for the main problem in artificial satellite theory. It led to to the discovery of the phase events responsible for the critical inclination. At the points in phase space where the families of circular orbits become unstable, there stems a pair of families of stable orbits with perigees lying permanently on the line of nodes. A little later, when the families of circular orbits return to stability, a second bifurcation occurs, giving rise to a pair of families of unstable orbits with their perigees permanently perpendicular to the line of nodes.

We shall report elsewhere how the method has been applied with equal success to the Zeeman problem (a charged particle in a Coulomb field immersed in a small uniform magnetic field).

In artificial satellite theory, we have considered only the perturbation due to the zonal harmonic of degree two. One of us (S.L.C.) is presently incorporating the zonal harmonics from degree 3 to 5; apparently the extension presents no difficulty other than a rapid complication of the analytical expansions.

Nowhere in this paper have we touched the thorny question of developing analytically the solutions of the main problem in the neighbourhood of its singularities. When H is $> H_1$, the flow on the sphere $\mathcal{S}(L, H)$ is globally a differential circulation admitting two fixed points, hence it looks as though Brouwer's third reduction—a Delaunay operation to eliminate the argument of perigee—might be adequate for any eccentricity. In the interval $0 \leq H \leq H_2$, such a Delaunay operation would be adequate only over the polar capst, it would have to exclude a certain zone inluding the equilibria S_1 - S_4. Is it true that it amounts to a Whittaker-Birkhoff normalization about the equilibrium S_5 at the south pole, and about the equilibrium S_0 at the north pole? How should one cover the zone of turbulence containing the equilibria S_1 - S_4? What should be done in the open interval $H_2 < H < H_1$ is not altogether clear. Yet to engineers designing clusters of artificial satellites, this is a problem of the utmost practical importance. At this stage we can only hope our analysis will assist in identifying the issues at hand.

Acknowledgements

Dr. Coffey expresses his thanks to the Center for Applied Mathematics for its warm hospitality in the year 1984-1985 while he was on a sabbatical leave from the U.S. Naval Research Laboratory. Dr. Miller is grateful to the National Research Council of the U.S. National Academy of Sciences for a two-year post-graduate research assistantship at the National Bureau of Standards. The authors benefited from consultations with Dr. Richard Cushman of the Rijksuniversiteit Utrecht, Professor Kenneth R. Meyer of the University of Cincinnati, Dr. Jan A. Sanders of the Vrije Universiteit Amsterdam. They followed the irenic suggestions made by Professor Carol Williams of the University of South Florida on the last version of the manuscript. There were lively discussions with Professor Sebastian Ferrer when the material of this research was presented at the University of Zaragoza.

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