

DIFFUSION COEFFICIENTS FOR STELLAR PLASMAS

C. PAQUETTE, C. PELLETIER, G. FONTAINE,¹ AND G. MICHAUD

Département de Physique and Observatoire du mont Mégantic, Université de Montréal

Received 1985 July 16; accepted 1985 November 11

ABSTRACT

An approximate method of computing diffusion coefficients for the relatively dense plasmas characteristic of white dwarf envelopes is presented. The method is based on the numerical evaluation of collision integrals for a screened Coulomb potential, and becomes rigorously valid in the limit of a dilute plasma. In this limit, it is shown that the usual analytic formulae of Chapman and Cowling (1970), which have been widely used in astrophysical applications, give rise to systematic errors in the diffusion coefficients. These errors can become considerable even in low-density stellar plasmas such as those found in the atmospheres of nondegenerate stars. The plasmas encountered in white dwarf envelopes are neither weakly nor strongly coupled, and only provisional results can be claimed for the diffusion coefficients in this difficult regime. However, a comparison with the results of rigorous Monte Carlo calculations (applicable at very high densities) indicates that the region of intermediate coupling is probably reasonably bridged. The main results are presented in the form of high-accuracy analytic fits for the collision integrals. The fits can be used within the framework of Chapman-Enskog's theory or Burgers's method of solution of the Boltzmann equation.

Subject headings: diffusion — plasmas — stars: white dwarfs

1. INTRODUCTION

Diffusion coefficients for stellar plasmas have customarily been evaluated in the limit of the dilute-gas approximation. In this limit, the collisions are treated classically and transport properties are computed from approximate solutions of the Boltzmann equation of the kinetic theory of gases. Two different methods have been used. The Chapman-Enskog theory (Chapman and Cowling 1970) assumes that the total distribution function of a given species can be written as a convergent series, each term representing a successive approximation to the distribution function. Transport properties are usually computed from velocity moments of the first-order approximation to the distribution function. Another level of approximation is introduced by expanding the first-order distribution function on the basis of Sonine polynomials. For ionized dilute gases, this series has recently been shown (Roussel-Dupré 1981, 1982) to converge very rapidly; terms beyond the second one introduce only small corrections to the diffusion coefficients, of the order of 10%–20%. Good estimates of the diffusion coefficients are therefore given by the so-called first and second approximations to transport properties as introduced by Chapman and Cowling (1970).

The second method has been developed by Burgers (1960, 1969) and is based on the Grad 13 moment approximation and the use of a Fokker-Planck collision term in the Boltzmann equation. In this approach, the emphasis is on the computation of higher order moments of the Boltzmann equation, which allows a more direct evaluation of physical quanti-

ties of interest than the Chapman-Enskog theory. The main advantage of Burgers's method over that of Chapman and Cowling is that it provides a more convenient way for handling multicomponent gases. In the limit where collisions are very frequent and the temperatures of the various species are the same (collision-dominated plasma), the two methods are equivalent (Burgers 1969; Roussel-Dupré 1981). Thus, transport properties computed using the network of equations put forward by Burgers must, ultimately, be the same as those obtained by Chapman and Cowling.

In both methods, the diffusion coefficients are expressible in terms of the so-called *collision integrals*, which, in turn, depend on the exact nature of the interaction between colliding particles. For a stellar plasma, the physics of collision is specified by some form of the Coulomb interaction. In particular, for a dilute gas, the choice of a *pure* Coulomb potential can provide a good approximation, although, strictly speaking, such a potential is not an exact description of the interaction between two charged particles embedded in a plasma. The interaction cross section for the pure Coulomb potential is well known to diverge, but this divergence is removed by introducing the concept of a long-range cutoff distance to take approximately into account the effects of the surrounding plasma. The actual value of this cutoff distance is somewhat arbitrary, but the Debye length, λ_D , is usually chosen. To the extent that the derived transport properties are not too sensitive to the actual choice of the cutoff distance, the procedure is justified. However, the use of a *screened* Coulomb potential would permit, in principle, a better description of the collective effects of the surrounding charges and remove the need for an ad hoc cutoff distance because the cross section

¹E. W. R. Steacie Memorial Fellow.

integrals converge naturally in such a case.² The disadvantage of such an approach is that, in general, the collision integrals cannot be evaluated analytically.

Chapman and Cowling (1970) have provided formulae for the diffusion coefficients in the case of the pure Coulomb potential. It has not been generally realized that, beyond the usual arbitrary choice of the long-range cutoff distance, they have made further approximations in order to evaluate the collision integrals *analytically*. These additional approximations are not essential and introduce significant systematic errors in the diffusion coefficients, even in cases for which the dilute-gas approximation is excellent—as in the atmospheres of main-sequence stars. The analytic formulae of Chapman and Cowling have nevertheless been very widely used in the astrophysical literature. In particular, in addition to numerous applications to main-sequence conditions, these formulae have also been used in exploratory studies of diffusion processes in the atmospheres and envelopes of white dwarf stars (Fontaine and Michaud 1979a; Vauclair, Vauclair, and Greenstein 1979; D'Antona and Mazzitelli 1979; Alcock and Illarionov 1980). However, in such cases (as pointed out, for example, by Fontaine and Michaud 1979a), the relevant densities are so high that the validity of the dilute-gas approximation itself becomes questionable, the pure Coulomb potential becomes a poor interaction model, and the formulae of Chapman and Cowling may give rise to serious errors in the evaluation of diffusion coefficients. The preliminary calculations of Fontaine and Michaud (1979b) have indicated that this is indeed the case. These calculations suggest that, under white dwarf conditions, much more reliable diffusion coefficients could be obtained through the numerical evaluation of collision integrals for *screened* Coulomb potentials.

In the present paper, we follow such an approach and compute collision integrals appropriate for stellar plasmas in general. This has been motivated primarily by our ongoing desire and efforts to understand better the atmospheric compositions of white dwarfs. Diffusion is the most important mechanism in the spectral evolution of these stars and, as indicated by the preliminary results of Fontaine and Michaud (1979b), there is a clear need for improved estimates of diffusion coefficients in their envelopes. Our computations are also relevant to other types of stars. In particular, the collision integrals evaluated here become very accurate for low-density stellar plasmas such as those encountered in the atmospheres of nondegenerate stars. For applications requiring a high level of precision, these integrals are to be preferred to the analytic formulae given by Chapman and Cowling (1970).

² The situation is directly analogous to the problem of the number of electronic bound states in an atom. For an *isolated* system, described by the pure Coulomb potential, it is well known that solving Schrödinger's equation leads to an infinite number of bound states. Several ad hoc prescriptions have been put forward to keep the internal partition function finite. All of these prescriptions involve cutting off the higher energy levels. A much more elegant solution is to invoke the fact that an atom is never isolated in nature and that a *screened* Coulomb potential is a more realistic description of the interaction between an electron and the nucleus. For such a potential, there is only a finite number of bound states (cf. Rogers, Graboske, and Harwood 1970). This leads to a natural convergence of the internal partition function.

The essential results of the paper are given in the form of high-accuracy fits for the collision integrals. We have developed such fits in order to make our numerical results more flexible and generally available to other users. The fits are valid in a broad range of stellar parameters (see below for a discussion) and can be used within the framework of Chapman-Enskog's theory or Burgers's method of solution of the Boltzmann equation. In related publications, we have used them to discuss in some detail the phenomenon of thermal diffusion in stars (Pelletier *et al.* 1986) and to present a comparative study of diffusion time scale estimates in white dwarfs (Paquette *et al.* 1986). We have also used the same fits in a number of astrophysical applications ranging from the main sequence to the white dwarf regime. Other authors, such as Muchmore (1984) and Iben and MacDonald (1985), have similarly used fits to diffusion coefficients for screened potentials in their independent discussions of diffusion in white dwarfs.

In the next section, the assumptions and hypotheses that form the basis of our method are discussed. Collision integrals for screened potentials are next evaluated (§ III). Section IV is concerned with a discussion of the derived diffusion coefficients under a broad range of conditions. In particular, a comparison with Monte Carlo results (applicable at very high densities) is presented. We conclude (§ V) with a few brief remarks.

II. BASIC ASSUMPTIONS AND FORMULAE

a) Hypotheses

The fundamental premise of our approach is the assumption that reasonable estimates of transport properties can be obtained from Boltzmann's equation according to the theory of Chapman and Enskog or the method of Burgers. This is certainly the case for dilute plasmas but remains difficult to justify for the relatively dense plasmas characteristic of white dwarf envelopes. Thus, it is assumed that the collisions are dominated by the *classical* interaction between two point-charge particles. It is possible to estimate the conditions under which quantum effects can become important by comparing the thermal wavelength λ_t [$\equiv (2\hbar^2/mkT)^{1/2}$] of a particle with various characteristic lengths. For example, if λ_t is much smaller than the average interparticle distance, degeneracy effects can be neglected. Electrons can become degenerate in white dwarf envelopes, although they remain only weakly so ($\eta \leq 5$; Fontaine and Van Horn 1976) at the base of the superficial convection zones where diffusion time scales are evaluated. By contrast, because of their much larger masses, ions remain nondegenerate. Likewise, a comparison of λ_t with the characteristic length of the interaction potential (λ_D) indicates that wave mechanical effects can be neglected for ionic collisions ($\lambda_t \ll \lambda_D$). The condition $\lambda_t \sim \lambda_D$ corresponds to the equality of the Debye temperature with the local temperature for ions; such conditions are usually encountered only in the interiors of white dwarfs, not in their outermost layers. However, electron-ion collisions can be affected by both degeneracy and quantum diffraction effects in white dwarf envelopes. To first order, such collisions do not affect

the binary diffusion coefficients for ion-ion interactions. Because interionic diffusion is usually the interesting quantity, it seems sensible to ignore the quantum corrections arising from the electron-ion collisions. A quantity such as the total thermal diffusion coefficient—which involves an electronic contribution—is, however, affected. We present a discussion of these effects in Pelletier *et al.* (1986).

Another complication due to quantum mechanics is the possible presence of bound states, especially for the heavy elements which may not be completely ionized. These ions, contrary to classical particles, have finite sizes ($\gg \lambda_i$) which could be handled by introducing the concept of a short-range cutoff distance. For charged particles in a relatively dilute plasma, this does not have much importance because the long-range nature of the Coulomb interactions dominates the collision process. Bound states could, however, lead to inelastic collisions; this is also briefly discussed in Pelletier *et al.* (1986) and shown to be potentially important in dense plasmas. However, no precise estimate of the magnitude of these effects is available, and inelastic collisions have been ignored here.

Other assumptions inherent in our use of the methods developed by Chapman and Cowling (1970) and Burgers (1969) are that (1) the temperatures of all the species are the same, (2) the thermal velocity is much larger than the diffusion velocities, (3) the ideal gas equation of state applies, and (4) the collisions are dominated by binary encounters. The first two conditions are completely satisfied in white dwarf envelopes, but nonideal effects and multiple collisions can also be quite important. To the extent that the concept of partial pressures is still reasonably valid under such conditions, assumption 3 could be somewhat relaxed. Also, as discussed below, the use of *screened* potentials can be expected to mimic—again to a certain extent—multiple collisions. Admittedly, in view of these assumptions, our approach can only be regarded as yielding provisional results for the transport properties of dense plasmas. It nevertheless appears to be the best method currently available for describing the difficult regime of physical conditions encountered in white dwarf envelopes.

b) Treatment of the Intermediate Regime

In a dilute plasma, it can be shown (cf. DeWitt 1961, 1969) that an excellent representation of the interaction between two classical particles of charge Z_s and Z_t , respectively, is given by a static screened Coulomb potential (SSCP) of the Debye-Hückel type:

$$V_{st}(r) = Z_s Z_t \frac{e^2 e^{-r/\lambda_D}}{r}, \quad (1)$$

where r is the distance and λ_D is the Debye screening length, given by

$$\lambda_D = \left(\frac{kT}{4\pi e^2 \sum_i n_i Z_i^2} \right)^{1/2}, \quad (2)$$

where T is the temperature, e the unit charge, and n_i the

number density of charged particles of species i . The summation is taken over all types of particles, including electrons. Equation (1) remains appropriate as long as the plasma remains weakly coupled. This coupling is measured by the so-called plasma parameter, given by

$$\Lambda = \frac{e^2}{kT\lambda_D} \left(\frac{\sum_i n_i Z_i^2}{\sum_i n_i} \right). \quad (3)$$

Its physical interpretation is straightforward: Λ is the ratio of an average classical distance of closest approach in the plasma to the screening length. Equivalently, it is the ratio of the potential to the kinetic energy in the plasma. As long as $\Lambda \ll 1$, the plasma is weakly coupled; the kinetic energy dominates over the interaction energy, Boltzmann's equation is rigorously valid, and the ideal gas equation of state applies in this regime. Such conditions are encountered in the atmospheres of main-sequence stars, for example.

Transport coefficients have also been computed in quite a different regime. The theoretical model is the so-called one-component plasma in which classical ions interact strongly in a uniform, noninteracting, neutralizing background of degenerate electrons (cf. DeWitt 1976; Hansen 1978). The astrophysical conditions under which this model is applicable are encountered in the deep interiors of white dwarf stars. In this regime, the coupling of the dense plasma is measured by the dimensionless parameter

$$\Gamma = \frac{Z_i^2 e^2}{kT\lambda_i}, \quad (4)$$

where Z_i is the charge of one ion and $\lambda_i = (3/4\pi n_i)^{1/3}$ is the average interionic distance or ion-sphere radius. The parameter Γ represents the ratio of the classical distance of closest approach of two like charges in a thermal distribution to the average interparticle distance. It also gives the ratio of the average potential to kinetic energy in the plasma. The latter is strongly coupled if $\Gamma \gg 1$. Because the motions of the ions are strongly correlated in this regime, Boltzmann's equation—which intrinsically assumes the predominance of two-body encounters—is totally inadequate. Instead, molecular dynamic techniques and Monte Carlo calculations have been used to describe the thermodynamic and transport properties of the one-component plasma.

In the intermediate regime in which the plasma coupling is neither weak nor strong ($\Lambda \sim \Gamma \sim 1$), there exists, to our knowledge, no reliable description of the transport properties of matter. Boltzmann's equation is, at best, questionable, and the assumption of a uniform background of electrons fails. Astrophysical environments characteristic of such a difficult regime are found in the envelopes of white dwarfs and the interiors of low-mass main-sequence stars. Not surprisingly, comparable difficulties are encountered in equation-of-state studies under similar conditions. For example, Fontaine, Graboske, and Van Horn (1977) were forced to *interpolate* thermodynamic surfaces across the region of intermediate coupling between two asymptotically rigorous models: the

ideal Maxwell-Boltzmann gas at low densities and the Fermi-Dirac electron gas at high densities. Despite this approximate treatment, their equation-of-state surfaces are still the best available in this regime and have been routinely used in the construction of models of white dwarf stars.

In the same spirit, but following a different approach, we *extrapolate* the techniques that are rigorously valid for dilute gases only, to compute diffusion coefficients for the intermediate coupling region. In effect, we are pushing the Chapman-Enskog and Burgers solutions of Boltzmann's equation to their limits with the use of screened potentials. Some justification for this procedure comes by noting, after Burgers (1960), that screened potentials—in the context of the Boltzmann equation—represent a crude way of taking into account multiple collisions. We assume that the appropriate interaction potential is of the Debye-Hückel type. However, a physical interpretation of equations (3) and (4) clearly indicates that while λ_D is an appropriate screening distance at low densities ($\lambda_D \gg \lambda_i$), the Debye sphere loses its significance at very high densities ($\lambda_D \ll \lambda_i$), and a more relevant screening distance is then λ_i itself. Exactly the same considerations apply for theories of electron screening for nuclear reactions in stars (cf. DeWitt, Graboske, and Cooper 1973). In practice, we take the larger of λ_D or λ_i . As discussed below, a comparison with the Monte Carlo results of Hansen at very high densities indicates that our technique of coupling Boltzmann's equation with screened potentials gives surprisingly good agreement and good overlap of the two regimes. This gives us confidence that the intermediate coupling region is reasonably bridged.

c) Diffusion and Resistance Coefficients

For completeness we reproduce here the basic formulae used to compute the diffusion coefficients for collisions involving particles of species s and t . According to Chapman and Cowling (1970), the first approximations to the diffusion coefficient and the thermal diffusion coefficient are, respectively,

$$[D_{st}]_1 = \frac{3E}{2nm} \quad (5)$$

and

$$[\alpha_{st}]_1 = \frac{5C(x_s S_s - x_t S_t)}{x_s^2 Q_s + x_t^2 Q_t + x_s x_t Q_{st}}, \quad (6)$$

where $n = n_s + n_t$, the total particle number density; $m = m_s + m_t$, the sum of the masses of the particles; and $x_i = n_i/n$ ($i = s, t$), the number concentration for particles of species i . The second approximation to the diffusion coefficient is given by

$$[D_{st}]_2 = \frac{[D_{st}]_1}{1 - \Delta}, \quad (7)$$

where

$$\Delta = \frac{5C^2(M_s^2 P_s x_s^2 + M_t^2 P_t x_t^2 + P_{st} x_s x_t)}{x_s^2 Q_s + x_t^2 Q_t + x_s x_t Q_{st}}, \quad (8)$$

with $M_i = m_i/m$ ($i = s, t$). The other quantities are defined as follows:

$$P_i = \frac{8M_i E \Omega_{ii}^{(22)}}{5kT} \quad (i = s, t), \quad (9)$$

$$P_{st} = 3(M_s - M_t)^2 + 4M_s M_t A, \quad (10)$$

$$Q_s = P_s(6M_t^2 + 5M_s^2 - 4M_s^2 B + 8M_s M_t A), \quad (11)$$

$$Q_{st} = 3(M_s - M_t)^2(5 - 4B) + 4M_s M_t A(11 - 4B) + 2P_s P_t, \quad (12)$$

$$S_s = M_s P_s - M_t [3(M_t - M_s) + 4M_s A], \quad (13)$$

$$A = \Omega_{st}^{(22)} / (5\Omega_{st}^{(11)}), \quad (14)$$

$$B = (5\Omega_{st}^{(12)} - \Omega_{st}^{(13)}) / (5\Omega_{st}^{(11)}), \quad (15)$$

$$C = (2\Omega_{st}^{(12)} / 5\Omega_{st}^{(11)}) - 1, \quad (16)$$

$$E = kT / (8M_s M_t \Omega_{st}^{(11)}). \quad (17)$$

The quantities Q_t and S_t are, respectively, obtained from equations (11) and (13) by interchanging the indices s and t . The quantities $\Omega_{st}^{(ij)}$ ($= \Omega_{ts}^{(ij)}$) are the *collision integrals*, given by

$$\Omega_{st}^{(ij)} = \left(\frac{kT}{2\pi m M_s M_t} \right)^{1/2} \int_0^\infty e^{-g^2} g^{2j+3} \phi_{st}^{(i)} dg, \quad (18)$$

with

$$\phi_{st}^{(i)} = 2\pi \int_0^\infty (1 - \cos^i \chi_{st}) b db, \quad (19)$$

and

$$\chi_{st} = \pi - 2 \int_{r_{st}^{\min}}^\infty b dr \left\{ r^2 \left[1 - \frac{b^2}{r^2} - \frac{V_{st}(r)}{g^2 kT} \right]^{1/2} \right\}^{-1}, \quad (20)$$

where $V_{st}(r)$ is the interaction potential and $r_{st}^{\min}$ is the distance of closest approach given by the solution of the equation

$$1 - \frac{b^2}{(r_{st}^{\min})^2} - \frac{V_{st}(r_{st}^{\min})}{g^2 kT} = 0. \quad (21)$$

These triple integrals describe the *classical* collisions between particles of species s and t interacting via the potential $V_{st}(r)$. Their physical interpretation is straightforward: the $\Omega_{st}^{(ij)}$ are related to the total cross sections after integrating over a Maxwellian velocity distribution; g is a dimensionless velocity. The $\phi_{st}^{(i)}$ are the collision cross sections for a given energy; the integration is over the impact parameter b . Finally, χ_{st} is the scattering angle whose evaluation requires an integration over the distance between the colliding particles.

The same collision integrals are used in the formalism of Burgers (1960, 1969). His network of diffusion equations involves the so-called *resistance coefficients*. These are given by

$$K_{st} = x_s x_t n \frac{kT}{[D_{st}]_1}, \quad (22)$$

$$z_{st} = -C, \quad (23)$$

$$z'_{st} = -2B + 2.5, \quad (24)$$

$$z''_{st} = 5A, \quad (25)$$

where the terms on the right-hand side have been defined previously.

III. EVALUATION OF THE COLLISION INTEGRALS

a) Pure Coulomb Potential

It is instructive to consider first the case of the pure Coulomb potential in order to appreciate the approximations that are made in obtaining analytic expressions for the diffusion coefficients. The scattering problem in a potential $V_{st}(r) = Z_s Z_t e^2 / r$ is a standard one in classical mechanics, and it is easy to show that the $\phi_{st}^{(i)}$ integrals assume their usual logarithmic form:

$$\phi_{st}^{(1)} = 2\pi \left(\frac{Z_s Z_t e^2}{2kTg^2} \right)^2 \ln(1 + \beta_{st}^2), \quad (26)$$

$$\phi_{st}^{(2)} = 4\pi \left(\frac{Z_s Z_t e^2}{2kTg^2} \right)^2 \left[\ln(1 + \beta_{st}^2) - \frac{\beta_{st}^2}{1 + \beta_{st}^2} \right], \quad (27)$$

where

$$\beta_{st} = \frac{2kTg^2 \lambda_D}{Z_s Z_t e^2}. \quad (28)$$

The integrations over the impact parameter have been stopped at a maximum value of b equal to λ_D , the Debye length, in order to control the well-known divergences of the Coulomb cross sections. The actual choice of this long-range cutoff distance constitutes a *first approximation*. To the extent that $\beta_{st} \gg 1$, as must be the case for dilute gases ($\beta_{st} \propto 1/\Lambda$), this choice is not too critical because it enters in a logarithmic way into the equations.³ A point of interest here is the fact that the scattering angle (in absolute value) is the same irrespective of the signs of the charges for given energy and impact parameter. This is not generally true for screened Coulomb potentials, and the cross sections for the attractive and repulsive collisions differ in such cases.

To obtain collision integrals in closed forms, a *second approximation* is made. It consists of ignoring the velocity dependence of the β_{st} terms in the expressions for the mono-

energetic cross sections (eqs. [26] and [27]). Chapman and Cowling (1970) replace the g^2 term by an average value for the square of the dimensionless relative velocity ($\langle g^2 \rangle = 2$). With this approximation, the cross sections become

$$\phi_{st}^{(1)} = 2\pi \left(\frac{Z_s Z_t e^2}{2kT} \right)^2 \frac{A_{st}^{(1)}}{g^4}, \quad (29)$$

$$\phi_{st}^{(2)} = 2\pi \left(\frac{Z_s Z_t e^2}{2kT} \right)^2 \frac{A_{st}^{(2)}}{g^4}, \quad (30)$$

where

$$A_{st}^{(1)} = \ln(1 + \gamma_{st}^2), \quad (31)$$

$$A_{st}^{(2)} = 2 \left(A_{st}^{(1)} - \frac{\gamma_{st}^2}{1 + \gamma_{st}^2} \right), \quad (32)$$

and

$$\gamma_{st} = \frac{4kT\lambda_D}{Z_s Z_t e^2}. \quad (33)$$

The integrations over the velocity distribution can next be performed, and it can be shown that the relevant collision integrals assume the following forms:

$$\Omega_{st}^{(11)} = \pi \left(\frac{kT}{2\pi m M_s M_t} \right)^{1/2} \left(\frac{Z_s Z_t e^2}{2kT} \right)^2 A_{st}^{(1)}, \quad (34)$$

$$\Omega_{st}^{(12)} = \Omega_{st}^{(11)}, \quad (35)$$

$$\Omega_{st}^{(13)} = 2\Omega_{st}^{(11)}, \quad (36)$$

$$\Omega_{st}^{(22)} = 5\bar{A}_{st} \Omega_{st}^{(11)}, \quad (37)$$

where

$$\bar{A}_{st} = \frac{A_{st}^{(2)}}{5A_{st}^{(1)}} = 0.4 \left[1 - \frac{\gamma_{st}^2}{(1 + \gamma_{st}^2) \ln(1 + \gamma_{st}^2)} \right]. \quad (38)$$

A *third approximation* is to take $A_{ss}^{(1)} = A_{st}^{(1)}$, on the assumption that these logarithmic terms are large and are not sensitive to the product of the charges. This ignores the differences between self-diffusion and collisions between particles of different species. The approximation leads to

$$\Omega_{ss}^{(22)} = 5\bar{A}_{st} \left(\frac{Z_s}{Z_t} \right)^2 (2M_t)^{1/2} \Omega_{st}^{(11)}. \quad (39)$$

Equations (34)–(39), with the addition of a *fourth approximation* (the neglect of terms of order m_e/m_i , the electronic mass in front of the ionic mass), constitute the basis for the analytic expressions of the diffusion coefficients for a pure Coulomb potential.

³Very often, one writes $\frac{1}{2} \ln(1 + \beta_{st}^2) \approx \ln \beta_{st}$. The last expression is then known as the *Coulomb logarithm*.

For example, consider the case of *trace* ions of species 2 diffusing in a background of electrons (e) and ions of species 1, a situation often encountered in astrophysical problems. In such a case ($n_2 \ll n_1 = n_e/Z_1$), the binary diffusion coefficients of Chapman and Cowling can be written

$$[D_{12}]_1 = \frac{3(2kT)^{5/2}}{16(\pi m M_1 M_2)^{1/2} n_1 Z_1^2 Z_2^2 e^4 A_{12}^{(1)}}, \quad (40)$$

$$[\alpha_{12}]_1 = 3 \left\{ \left(\frac{Z_2}{Z_1} \right)^2 \frac{M_2^{3/2}}{(\sqrt{2}) \bar{A}_{12}} [3(2M_2 - 1) + 4\bar{A}_{12} M_1] - M_1 \right\} \\ \times (6M_2^2 + 2.6M_1^2 + 8M_1 M_2 \bar{A}_{12})^{-1}, \quad (41)$$

$$[\alpha_{1e}]_1 = -\frac{3(1 + Z_1)}{2.6 + 2.828\bar{A}_{1e}/Z_1}, \quad (42)$$

$$[\alpha_{e2}]_1 = \frac{1.061 Z_2^2}{\bar{A}_{e2}}. \quad (43)$$

The resistance coefficients of Burgers can be written

$$K_{12} = \frac{n_2 kT}{[D_{12}]_1}, \quad (44)$$

$$z_{st} = 0.6, \quad (45)$$

$$z'_{st} = 1.3, \quad (46)$$

$$z''_{12} = \bar{A}_{12}. \quad (47)$$

It is worthwhile to point out that further approximations have very often found their way into the astrophysical literature. For instance, the logarithmic terms $A_{st}^{(1)}$ have been considered very large in the third approximation, but a *fifth approximation* makes them infinite, so that $\bar{A}_{st} = 0.4$ according to equation (38). A *sixth approximation* is to assume that the mass of ions of species 2 is much larger than the mass of ions of species 1. Under such circumstances, some transport coefficients reduce to

$$[\alpha_{12}]_1 = 2.65 \left(\frac{Z_2}{Z_1} \right)^2, \quad (48)$$

$$[\alpha_{1e}]_1 = -\frac{3(1 + Z_1)}{2.6 + 1.131/Z_1}, \quad (49)$$

$$[\alpha_{e2}]_1 = 2.65 Z_2^2, \quad (50)$$

$$z''_{st} = 0.4, \quad (51)$$

while the others do not change. In the *vast majority* of astrophysical applications of diffusion, these last expressions have been used.

b) Static Screened Coulomb Potential

We have computed the collision integrals (eqs. [18]–[21]) for a static screened Coulomb potential of the form

$$V_{st}(r) = Z_s Z_t e^2 \frac{e^{-r/\lambda}}{r}, \quad (52)$$

where the screening length λ is taken as the larger of λ_D or λ_i .⁴ The numerical calculations were carried out with a computer code which was initially developed by Fontaine and Michaud (1979b) but which needed substantial improvements. The code can now handle collision integrals for any spherically symmetric potential that is attractive, repulsive, or both. In the cases of Coulomb potentials, we introduce a dimensionless distance $R = r/\lambda_c$ and a dimensionless impact parameter $B = b/\lambda_c$, where

$$\lambda_c = \frac{e^2}{kT}. \quad (53)$$

In these units the minimum distance of closest approach, R_0 , is given by the solution of

$$1 - \frac{B^2}{R_0^2} - \frac{U_{st}(R_0)}{g^2 kT} = 0, \quad (54)$$

where

$$U_{st}(R_0) = V_{st}(r = R_0 \lambda_c). \quad (55)$$

Because the integrand of χ_{st} has a singularity at $R = R_0$ (see eq. [20]), we introduce a change of variable:

$$z = \left(\frac{1}{R_0} - \frac{1}{R} \right)^{1/2}. \quad (56)$$

The scattering angle is then given by

$$\chi_{st} = \pi - 4 \int_0^{z_0} dz \left[\frac{U_{st}(R_0) - W_{st}(z)}{B^2 g^2 kT z^2} + \frac{2}{R_0} - z^2 \right]^{-1/2}, \quad (57)$$

with

$$z_0 = \left(\frac{1}{R_0} \right)^{1/2} \quad (58)$$

and

$$W_{st}(z) = V_{st} \left(r = \frac{\lambda_c}{1/R_0 - z^2} \right). \quad (59)$$

The integral of equation (57) is then evaluated with a 6-point Gauss-Legendre formula.

⁴Note that for a multicomponent plasma, λ_i is still given by $(3/4\pi n_i)^{1/3}$, where n_i is now the *total* number density of ions. It represents an *average* interionic distance.

TABLE 1
SPLINE COEFFICIENTS (REPULSIVE POTENTIAL): $j=1$

n	$c_{1n}^{(1)}$	$c_{2n}^{(1)}$	$c_{3n}^{(1)}$	$c_{4n}^{(1)}$
1	1.19599E-02	-2.39198E-02	-3.02547E+01	-2.94860E+01
2	-2.39198E-02	-1.48010E-02	-2.94860E+01	-2.87231E+01
3	-1.48010E-02	-1.77390E-02	-2.87231E+01	-2.79637E+01
4	-1.77390E-02	-1.74423E-02	-2.79637E+01	-2.72086E+01
5	-1.74423E-02	-1.80040E-02	-2.72086E+01	-2.64576E+01
6	-1.80040E-02	-1.83218E-02	-2.64576E+01	-2.57110E+01
7	-1.83218E-02	-1.86847E-02	-2.57110E+01	-2.49688E+01
8	-1.86847E-02	-1.90073E-02	-2.49688E+01	-2.42310E+01
9	-1.90073E-02	-1.93026E-02	-2.42310E+01	-2.34978E+01
10	-1.93026E-02	-1.95555E-02	-2.34978E+01	-2.27693E+01
11	-1.95555E-02	-1.97557E-02	-2.27693E+01	-2.20454E+01
12	-1.97557E-02	-1.98886E-02	-2.20454E+01	-2.13263E+01
13	-1.98886E-02	-1.99373E-02	-2.13263E+01	-2.06120E+01
14	-1.99373E-02	-1.98810E-02	-2.06120E+01	-1.99024E+01
15	-1.98810E-02	-1.96948E-02	-1.99024E+01	-1.91976E+01
16	-1.96948E-02	-1.93486E-02	-1.91976E+01	-1.84975E+01
17	-1.93486E-02	-1.88059E-02	-1.84975E+01	-1.78021E+01
18	-1.88059E-02	-1.80227E-02	-1.78021E+01	-1.71112E+01
19	-1.80227E-02	-1.69459E-02	-1.71112E+01	-1.64246E+01
20	-1.69459E-02	-1.55109E-02	-1.64246E+01	-1.57421E+01
21	-1.55109E-02	-1.36394E-02	-1.57421E+01	-1.50633E+01
22	-1.36394E-02	-1.12361E-02	-1.50633E+01	-1.43878E+01
23	-1.12361E-02	-8.18466E-03	-1.43878E+01	-1.37150E+01
24	-8.18466E-03	-4.34258E-03	-1.37150E+01	-1.30441E+01
25	-4.34258E-03	4.65253E-04	-1.30441E+01	-1.23743E+01
26	4.65253E-04	6.45493E-03	-1.23743E+01	-1.17044E+01
27	6.45493E-03	1.38941E-02	-1.17044E+01	-1.10329E+01
28	1.38941E-02	2.31151E-02	-1.10329E+01	-1.03581E+01
29	2.31151E-02	3.45317E-02	-1.03581E+01	-9.67777E+00
30	3.45317E-02	4.86585E-02	-9.67777E+00	-8.98913E+00
31	4.86585E-02	6.61321E-02	-8.98913E+00	-8.28881E+00
32	6.61321E-02	8.77309E-02	-8.28881E+00	-7.57261E+00
33	8.77309E-02	1.14383E-01	-7.57261E+00	-6.83537E+00
34	1.14383E-01	1.47142E-01	-6.83537E+00	-6.07066E+00
35	1.47142E-01	1.87092E-01	-6.07066E+00	-5.27065E+00
36	1.87092E-01	2.35096E-01	-5.27065E+00	-4.42573E+00
37	2.35096E-01	2.91268E-01	-4.42573E+00	-3.52439E+00
38	2.91268E-01	3.53977E-01	-3.52439E+00	-2.55315E+00
39	3.53977E-01	4.18217E-01	-2.55315E+00	-1.49695E+00
40	4.18217E-01	4.73499E-01	-1.49695E+00	-3.40379E-01
41	4.73499E-01	5.02343E-01	-3.40379E-01	9.29832E-01
42	5.02343E-01	4.82140E-01	9.29832E-01	2.32060E+00
43	4.82140E-01	3.92303E-01	2.32060E+00	3.82709E+00
44	3.92303E-01	2.20401E-01	3.82709E+00	5.42773E+00
45	2.20401E-01	-5.31156E-02	5.42773E+00	7.08127E+00
46	-5.31156E-02	-3.94063E-01	7.08127E+00	8.72205E+00
47	-3.94063E-01	-5.99574E-01	8.72205E+00	1.02683E+01
48	-5.99574E-01	-4.71033E-01	1.02683E+01	1.16706E+01
49	-4.71033E-01	-4.68969E-01	1.16706E+01	1.29598E+01
50	-4.68969E-01	2.34484E-01	1.29598E+01	1.41366E+01

The next step is to compute the $\phi_{st}^{(i)}$ integrals. The integration over the impact parameter B leads to certain difficulties because it becomes inefficient to search for the root, R_0 , of equation (54) with a Newton-Raphson technique for very small values of B . We avoid these problems by integrating over R_0 instead of B . Equation (54) yields

$$B = R_0 \left[1 - \frac{U_{st}(R_0)}{g^2 kT} \right]^{1/2}. \quad (60)$$

In terms of this new variable, equation (19) now reads:

$$\phi_{st}^{(i)} = 2\pi\lambda_c^2 \int_{R_c}^{\infty} (1 - \cos^i \chi_{st}) R_0 \times \left\{ \left[1 - \frac{U_{st}(R_0)}{g^2 kT} \right] - \frac{R_0}{2g^2 kT} \frac{dU_{st}}{dR_0} \right\} dR_0, \quad (61)$$

where R_c is equal to zero for an attractive potential and is given by $U_{st}(R_c) = g^2 kT$ for a repulsive potential. We handle the infinite upper limit of the integral by empirically determining the value, R_l , for which the integral has converged. This limiting value of R_0 is related to the limiting value of the impact parameter, B_l , by

$$1 - \frac{B_l^2}{R_l^2} - \frac{U_{st}(R_l)}{g^2 kT} = 0. \quad (62)$$

For the SSCP, one can expect that very little contribution to the integral will come from collisions with impact parameters much larger than several screening lengths. In practice, we have found that the integrals are the same (to the roundoff errors) by choosing $\lambda_c B_l = 10 \lambda$ or $\lambda_c B_l = 20 \lambda$. With these upper limits, a 30-point Gauss-Legendre formula is used to evaluate the integrals of equation (61).

TABLE 2
 SPLINE COEFFICIENTS (REPULSIVE POTENTIAL): $j = 2$

n	$c_{1n}^{(2)}$	$c_{2n}^{(2)}$	$c_{3n}^{(2)}$	$c_{4n}^{(2)}$
1	1.34102E-02	-2.68205E-02	-2.55941E+01	-2.48408E+01
2	-2.68205E-02	-1.66309E-02	-2.48408E+01	-2.40939E+01
3	-1.66309E-02	-1.99547E-02	-2.40939E+01	-2.33511E+01
4	-1.99547E-02	-1.96575E-02	-2.33511E+01	-2.26130E+01
5	-1.96575E-02	-2.03264E-02	-2.26130E+01	-2.18796E+01
6	-2.03264E-02	-2.07272E-02	-2.18796E+01	-2.11511E+01
7	-2.07272E-02	-2.11843E-02	-2.11511E+01	-2.04276E+01
8	-2.11843E-02	-2.16034E-02	-2.04276E+01	-1.97091E+01
9	-2.16034E-02	-2.20005E-02	-1.97091E+01	-1.89959E+01
10	-2.20005E-02	-2.23602E-02	-1.89959E+01	-1.82879E+01
11	-2.23602E-02	-2.26732E-02	-1.82879E+01	-1.75853E+01
12	-2.26732E-02	-2.29256E-02	-1.75853E+01	-1.68881E+01
13	-2.29256E-02	-2.31017E-02	-1.68881E+01	-1.61965E+01
14	-2.31017E-02	-2.31822E-02	-1.61965E+01	-1.55103E+01
15	-2.31822E-02	-2.31439E-02	-1.55103E+01	-1.48298E+01
16	-2.31439E-02	-2.29591E-02	-1.48298E+01	-1.41548E+01
17	-2.29591E-02	-2.25945E-02	-1.41548E+01	-1.34853E+01
18	-2.25945E-02	-2.20100E-02	-1.34853E+01	-1.28212E+01
19	-2.20100E-02	-2.11574E-02	-1.28212E+01	-1.21624E+01
20	-2.11574E-02	-1.99787E-02	-1.21624E+01	-1.15087E+01
21	-1.99787E-02	-1.84040E-02	-1.15087E+01	-1.08598E+01
22	-1.84040E-02	-1.63488E-02	-1.08598E+01	-1.02153E+01
23	-1.63488E-02	-1.37108E-02	-1.02153E+01	-9.57474E+00
24	-1.37108E-02	-1.03658E-02	-9.57474E+00	-8.93745E+00
25	-1.03658E-02	-6.16232E-03	-8.93745E+00	-8.30266E+00
26	-6.16232E-03	-9.15489E-04	-8.30266E+00	-7.66934E+00
27	-9.15489E-04	5.60137E-03	-7.66934E+00	-7.03625E+00
28	5.60137E-03	1.36664E-02	-7.03625E+00	-6.40181E+00
29	1.36664E-02	2.36209E-02	-6.40181E+00	-5.76409E+00
30	2.36209E-02	3.58820E-02	-5.76409E+00	-5.12070E+00
31	3.58820E-02	5.09546E-02	-5.12070E+00	-4.46870E+00
32	5.09546E-02	6.94376E-02	-4.46870E+00	-3.80447E+00
33	6.94376E-02	9.20141E-02	-3.80447E+00	-3.12357E+00
34	9.20141E-02	1.19403E-01	-3.12357E+00	-2.42059E+00
35	1.19403E-01	1.52233E-01	-2.42059E+00	-1.68896E+00
36	1.52233E-01	1.90763E-01	-1.68896E+00	-9.20788E-01
37	1.90763E-01	2.34333E-01	-9.20788E-01	-1.06834E-01
38	2.34333E-01	2.80392E-01	-1.06834E-01	7.63360E-01
39	2.80392E-01	3.23003E-01	7.63360E-01	1.70085E+00
40	3.23003E-01	3.51091E-01	1.70085E+00	2.71586E+00
41	3.51091E-01	3.47669E-01	2.71586E+00	3.81513E+00
42	3.47669E-01	2.93155E-01	3.81513E+00	4.99784E+00
43	2.93155E-01	1.77064E-01	4.99784E+00	6.25091E+00
44	1.77064E-01	1.68817E-02	6.25091E+00	7.54647E+00
45	1.68817E-02	-1.37836E-01	7.54647E+00	8.84609E+00
46	-1.37836E-01	-2.36519E-01	8.84609E+00	1.01126E+01
47	-2.36519E-01	-2.49035E-01	1.01126E+01	1.13224E+01
48	-2.49035E-01	-1.95933E-01	1.13224E+01	1.24724E+01
49	-1.95933E-01	-1.60453E-01	1.24724E+01	1.35754E+01
50	-1.60453E-01	8.02267E-02	1.35754E+01	1.46398E+01

The infinite upper limit could have been handled by introducing a trigonometric change of variable, as is done routinely in such instances. However, we found that our technique is actually better than the use of angular variables, mostly because of the presence of the exponential in the SSCP. The trigonometric technique would produce small oscillations in the cross sections as a function of the energy, whereas our method led to perfectly monotonic relationships, as it should. For other types of potentials, the trigonometric techniques would produce excellent results.

The final step in the evaluation of the collision integrals is an integration over the velocity distribution (cf. eq. [18]). This presents no difficulty, and, with the change of variable

$$X = g^2, \quad (63)$$

we obtain

$$\Omega_{st}^{(ij)} = \frac{1}{2} \left(\frac{kT}{2\pi m M_s M_t} \right)^{1/2} \int_0^\infty e^{-X} X^{j+1} \phi_{st}^{(i)} dX. \quad (64)$$

This is evaluated with a 6-point Gauss-Laguerre formula.

We have checked our numerical code by comparing its predictions with exact analytic formulae that could be derived for several types of potentials. For example, we have run exhaustive experiments (by changing $Z_s, Z_t, n_s, n_t, \rho, T$) for the case of the pure Coulomb potential with a cutoff at λ_D . Comparison of the $\phi_{st}^{(i)}$ integrals evaluated numerically with the analytic results of equations (26)–(28) reveals that, in all cases considered, the relative differences are less than $\sim 0.1\%$. Similarly, comparison of the total cross sections for repulsive

TABLE 3
 SPLINE COEFFICIENTS (REPULSIVE POTENTIAL): $j = 3$

n	$c_{1n}^{(3)}$	$c_{2n}^{(3)}$	$c_{3n}^{(3)}$	$c_{4n}^{(3)}$
1	1.46130E-02	-2.92259E-02	-1.93212E+01	-1.85804E+01
2	-2.92259E-02	-1.81472E-02	-1.85804E+01	-1.78467E+01
3	-1.81472E-02	-2.17897E-02	-1.78467E+01	-1.71173E+01
4	-2.17897E-02	-2.14905E-02	-1.71173E+01	-1.63931E+01
5	-2.14905E-02	-2.22463E-02	-1.63931E+01	-1.56741E+01
6	-2.22463E-02	-2.27134E-02	-1.56741E+01	-1.49604E+01
7	-2.27134E-02	-2.32457E-02	-1.49604E+01	-1.42522E+01
8	-2.32457E-02	-2.37412E-02	-1.42522E+01	-1.35496E+01
9	-2.37412E-02	-2.42183E-02	-1.35496E+01	-1.28526E+01
10	-2.42183E-02	-2.46613E-02	-1.28526E+01	-1.21615E+01
11	-2.46613E-02	-2.50615E-02	-1.21615E+01	-1.14763E+01
12	-2.50615E-02	-2.54057E-02	-1.14763E+01	-1.07971E+01
13	-2.54057E-02	-2.56787E-02	-1.07971E+01	-1.01240E+01
14	-2.56787E-02	-2.58623E-02	-1.01240E+01	-9.45709E+00
15	-2.58623E-02	-2.59347E-02	-9.45709E+00	-8.79637E+00
16	-2.59347E-02	-2.58697E-02	-8.79637E+00	-8.14187E+00
17	-2.58697E-02	-2.56362E-02	-8.14187E+00	-7.49358E+00
18	-2.56362E-02	-2.51970E-02	-7.49358E+00	-6.85144E+00
19	-2.51970E-02	-2.45075E-02	-6.85144E+00	-6.21534E+00
20	-2.45075E-02	-2.35146E-02	-6.21534E+00	-5.58513E+00
21	-2.35146E-02	-2.21544E-02	-5.58513E+00	-4.96057E+00
22	-2.21544E-02	-2.03503E-02	-4.96057E+00	-4.34132E+00
23	-2.03503E-02	-1.80104E-02	-4.34132E+00	-3.72695E+00
24	-1.80104E-02	-1.50236E-02	-3.72695E+00	-3.11691E+00
25	-1.50236E-02	-1.12556E-02	-3.11691E+00	-2.51047E+00
26	-1.12556E-02	-6.54378E-03	-2.51047E+00	-1.90674E+00
27	-6.54378E-03	-6.90518E-04	-1.90674E+00	-1.30457E+00
28	-6.90518E-04	6.54438E-03	-1.30457E+00	-7.02574E-01
29	6.54438E-03	1.54523E-02	-7.02574E-01	-9.90038E-02
30	1.54523E-02	2.63848E-02	-9.90038E-02	5.08275E-01
31	2.63848E-02	3.97599E-02	5.08275E-01	1.12189E+00
32	3.97599E-02	5.60625E-02	1.12189E+00	1.74504E+00
33	5.60625E-02	7.58259E-02	1.74504E+00	2.38165E+00
34	7.58259E-02	9.95773E-02	2.38165E+00	3.03645E+00
35	9.95773E-02	1.27708E-01	3.03645E+00	3.71516E+00
36	1.27708E-01	1.60207E-01	3.71516E+00	4.42451E+00
37	1.60207E-01	1.96165E-01	4.42451E+00	5.17232E+00
38	1.96165E-01	2.32939E-01	5.17232E+00	5.96720E+00
39	2.32939E-01	2.64985E-01	5.96720E+00	6.81799E+00
40	2.64985E-01	2.82691E-01	6.81799E+00	7.73238E+00
41	2.82691E-01	2.72481E-01	7.73238E+00	8.71461E+00
42	2.72481E-01	2.20588E-01	8.71461E+00	9.76224E+00
43	2.20588E-01	1.23271E-01	9.76224E+00	1.08628E+01
44	1.23271E-01	5.64105E-05	1.08628E+01	1.19930E+01
45	5.64105E-05	-1.01669E-01	1.19930E+01	1.31231E+01
46	-1.01669E-01	-1.45351E-01	1.31231E+01	1.42289E+01
47	-1.45351E-01	-1.12670E-01	1.42289E+01	1.52998E+01
48	-1.12670E-01	-1.25510E-01	1.52998E+01	1.63436E+01
49	-1.25510E-01	4.14471E-02	1.63436E+01	1.73573E+01
50	4.14471E-02	-2.07236E-02	1.73573E+01	1.83810E+01

potentials of the type $\sim 1/r^4$ and $\sim 1/r^6$ shows a $\sim 0.05\%$ agreement between the numerical results and the analytic values tabulated by Chapman and Cowling (1970, p. 172). We have also considered a 12-4 potential that represents the interaction between a proton and a neutral atom of helium. No analytic solutions exist for this type of potential, but Mason and Schamp (1958) have obtained a numerical solution. A comparison of our results with theirs indicates relative differences less than 0.6%. Finally, some partial numerical results are available for the SSCP itself. In fact, evaluating collision integrals for a Debye-Hückel potential is nothing really new, since Liboff (1959), Mason, Munn, and Smith (1967), and, recently, Muchmore (1984) have all done such calculations. However, the present calculations apply to a much broader range of conditions, and the claim is that they

are more accurate. In the range where our results overlap with those of Mason, Munn, and Smith (1967), the collision integrals have average differences of the order of 0.6%. However, their cross sections show small oscillatory behaviors for the smaller values of λ_D that they have considered (cf. their Tables II and III). Muchmore (1984) has not distinguished between the attractive and repulsive cases and has also experienced some numerical difficulties at low densities, as evidenced by his Figure 1. Liboff (1959) has evaluated the collision integrals through a series expansion and has kept only the first-order term. His results are consequently less accurate than those obtained from a full numerical integration. From this discussion, we conclude that our computer code generates reliable collision integrals, and we may expect our results for the SSCP to be also reliable and useful.

TABLE 4
SPLINE COEFFICIENTS (REPULSIVE POTENTIAL): $l = 1-4$

n	d_{1n}	d_{2n}	d_{3n}	d_{4n}
1	1.18229E-02	-2.36458E-02	-2.55112E+01	-2.47319E+01
2	-2.36458E-02	-1.46794E-02	-2.47319E+01	-2.39583E+01
3	-1.46794E-02	-1.76226E-02	-2.39583E+01	-2.31882E+01
4	-1.76226E-02	-1.73748E-02	-2.31882E+01	-2.24223E+01
5	-1.73748E-02	-1.79780E-02	-2.24223E+01	-2.16606E+01
6	-1.79780E-02	-1.83439E-02	-2.16606E+01	-2.09032E+01
7	-1.83439E-02	-1.87580E-02	-2.09032E+01	-2.01502E+01
8	-1.87580E-02	-1.91359E-02	-2.01502E+01	-1.94017E+01
9	-1.91359E-02	-1.94907E-02	-1.94017E+01	-1.86579E+01
10	-1.94907E-02	-1.98074E-02	-1.86579E+01	-1.79186E+01
11	-1.98074E-02	-2.00758E-02	-1.79186E+01	-1.71842E+01
12	-2.00758E-02	-2.02817E-02	-1.71842E+01	-1.64545E+01
13	-2.02817E-02	-2.04080E-02	-1.64545E+01	-1.57298E+01
14	-2.04080E-02	-2.04344E-02	-1.57298E+01	-1.50099E+01
15	-2.04344E-02	-2.03360E-02	-1.50099E+01	-1.42949E+01
16	-2.03360E-02	-2.00828E-02	-1.42949E+01	-1.35848E+01
17	-2.00828E-02	-1.96388E-02	-1.35848E+01	-1.28796E+01
18	-1.96388E-02	-1.89603E-02	-1.28796E+01	-1.21790E+01
19	-1.89603E-02	-1.79946E-02	-1.21790E+01	-1.14830E+01
20	-1.79946E-02	-1.66781E-02	-1.14830E+01	-1.07913E+01
21	-1.66781E-02	-1.49334E-02	-1.07913E+01	-1.01036E+01
22	-1.49334E-02	-1.26670E-02	-1.01036E+01	-9.41952E+00
23	-1.26670E-02	-9.76488E-03	-9.41952E+00	-8.73845E+00
24	-9.76488E-03	-6.08817E-03	-8.73845E+00	-8.05974E+00
25	-6.08817E-03	-1.46688E-03	-8.05974E+00	-7.38248E+00
26	-1.46688E-03	4.30758E-03	-7.38248E+00	-6.70557E+00
27	4.30758E-03	1.14919E-02	-6.70557E+00	-6.02763E+00
28	1.14919E-02	2.04026E-02	-6.02763E+00	-5.34694E+00
29	2.04026E-02	3.14302E-02	-5.34694E+00	-4.66134E+00
30	3.14302E-02	4.50547E-02	-4.66134E+00	-3.96821E+00
31	4.50547E-02	6.18616E-02	-3.96821E+00	-3.26426E+00
32	6.18616E-02	8.25526E-02	-3.26426E+00	-2.54546E+00
33	8.25526E-02	1.07939E-01	-2.54546E+00	-1.80685E+00
34	1.07939E-01	1.38897E-01	-1.80685E+00	-1.04234E+00
35	1.38897E-01	1.76234E-01	-1.04234E+00	-2.44487E-01
36	1.76234E-01	2.20387E-01	-2.44487E-01	5.95659E-01
37	2.20387E-01	2.70814E-01	5.95659E-01	1.48870E+00
38	2.70814E-01	3.24880E-01	1.48870E+00	2.44673E+00
39	3.24880E-01	3.76096E-01	2.44673E+00	3.48274E+00
40	3.76096E-01	4.11895E-01	3.48274E+00	4.60901E+00
41	4.11895E-01	4.12194E-01	4.60901E+00	5.83413E+00
42	4.12194E-01	3.52176E-01	5.83413E+00	7.15818E+00
43	3.52176E-01	2.14860E-01	7.15818E+00	8.56675E+00
44	2.14860E-01	1.56932E-02	8.56675E+00	1.00269E+01
45	1.56932E-02	-1.84529E-01	1.00269E+01	1.14908E+01
46	-1.84529E-01	-3.17740E-01	1.14908E+01	1.29104E+01
47	-3.17740E-01	-3.51796E-01	1.29104E+01	1.42538E+01
48	-3.51796E-01	-2.86379E-01	1.42538E+01	1.55127E+01
49	-2.86379E-01	-2.68214E-01	1.55127E+01	1.67029E+01
50	-2.68214E-01	1.34107E-01	1.67029E+01	1.78287E+01

c) Analytic Fits

For diffusion applications to stellar plasmas, the collision integrals for the SSCP can be evaluated once and for all. Clearly, an analytic representation of the results would be most useful, and we have, consequently, developed high-accuracy fits for the collision integrals. We define the dimensionless collision integrals

$$F_{st}^{(ij)} = \Omega_{st}^{(ij)} / \epsilon_{st}, \quad (65)$$

where

$$\epsilon_{st} = \pi \left(\frac{Z_s Z_t e^2}{2kT} \right)^2 \left(\frac{kT}{2\pi m M_s M_t} \right)^{1/2}. \quad (66)$$

The values of these integrals depend uniquely upon Z_s , Z_t , λ ,

and T for the SSCP. To make the correspondence with the pure Coulomb potential in the limit of zero density, we choose a combination of these variables such that

$$\gamma_{st} = \frac{4kT\lambda}{Z_s Z_t e^2}. \quad (67)$$

In terms of this dimensionless parameter, the independent variable for the fits to the dimensionless collision integrals is taken as

$$\psi_{st} = \ln [\ln (1 + \gamma_{st}^2)]. \quad (68)$$

This follows from a suggestion of Muchmore (1980). The double logarithmic form allows one to cover a very wide range of stellar conditions against a relatively small range in ψ_{st} .

TABLE 5
SPLINE COEFFICIENTS (ATTRACTIVE POTENTIAL): $j = 1$

n	$c_{1n}^{(1)}$	$c_{2n}^{(1)}$	$c_{3n}^{(1)}$	$c_{4n}^{(1)}$
1	-4.85605E+00	9.71211E+00	-2.44778E+01	-2.50688E+01
2	9.71211E+00	-9.31384E+00	-2.50688E+01	-2.33288E+01
3	-9.31384E+00	9.24600E+00	-2.33288E+01	-2.38242E+01
4	9.24600E+00	-5.09678E+00	-2.38242E+01	-2.21005E+01
5	-5.09678E+00	-3.51130E-01	-2.21005E+01	-2.16001E+01
6	-3.51130E-01	-5.73409E-01	-2.16001E+01	-2.11839E+01
7	-5.73409E-01	1.83302E+00	-2.11839E+01	-2.09054E+01
8	1.83302E+00	-7.68878E-01	-2.09054E+01	-2.01869E+01
9	-7.68878E-01	9.36371E-01	-2.01869E+01	-1.96529E+01
10	9.36371E-01	-8.59172E-01	-1.96529E+01	-1.88943E+01
11	-8.59172E-01	-1.42540E+00	-1.88943E+01	-1.83432E+01
12	-1.42540E+00	4.34705E+00	-1.83432E+01	-1.81343E+01
13	4.34705E+00	3.61550E-01	-1.81343E+01	-1.68820E+01
14	3.61550E-01	-3.50574E+00	-1.68820E+01	-1.55430E+01
15	-3.50574E+00	2.59550E+00	-1.55430E+01	-1.50454E+01
16	2.59550E+00	-1.44563E+00	-1.50454E+01	-1.39249E+01
17	-1.44563E+00	-8.64470E-01	-1.39249E+01	-1.31513E+01
18	-8.64470E-01	8.82476E-01	-1.31513E+01	-1.25851E+01
19	8.82476E-01	1.35231E-01	-1.25851E+01	-1.18072E+01
20	1.35231E-01	8.48986E-02	-1.18072E+01	-1.09968E+01
21	8.48986E-02	-5.89033E-02	-1.09968E+01	-1.01661E+01
22	-5.89033E-02	-1.07275E-01	-1.01661E+01	-9.34947E+00
23	-1.07275E-01	-1.53774E-01	-9.34947E+00	-8.55860E+00
24	-1.53774E-01	-6.39482E-02	-8.55860E+00	-7.80464E+00
25	-6.39482E-02	-1.27604E-01	-7.80464E+00	-7.06602E+00
26	-1.27604E-01	-3.16007E-01	-7.06602E+00	-6.35803E+00
27	-3.16007E-01	-7.55843E-02	-6.35803E+00	-5.72588E+00
28	-7.55843E-02	6.48945E-02	-5.72588E+00	-5.11188E+00
29	6.48945E-02	5.15827E-02	-5.11188E+00	-4.48229E+00
30	5.15827E-02	-4.79565E-01	-4.48229E+00	-3.84033E+00
31	-4.79565E-01	3.59783E-01	-3.84033E+00	-3.31347E+00
32	3.59783E-01	-3.04381E-01	-3.31347E+00	-2.70025E+00
33	-3.04381E-01	1.12514E-01	-2.70025E+00	-2.16009E+00
34	1.12514E-01	1.90268E-01	-2.16009E+00	-1.59292E+00
35	1.90268E-01	1.96365E-01	-1.59292E+00	-9.80089E-01
36	1.96365E-01	1.78507E-01	-9.80089E-01	-3.20131E-01
37	1.78507E-01	1.77190E-01	-3.20131E-01	3.82669E-01
38	1.77190E-01	1.92652E-01	3.82669E-01	1.12800E+00
39	1.92652E-01	2.20472E-01	1.12800E+00	1.91956E+00
40	2.20472E-01	2.52940E-01	1.91956E+00	2.76403E+00
41	2.52940E-01	2.82221E-01	2.76403E+00	3.66921E+00
42	2.82221E-01	3.00417E-01	3.66921E+00	4.64213E+00
43	3.00417E-01	2.99089E-01	4.64213E+00	5.68714E+00
44	2.99089E-01	2.69658E-01	5.68714E+00	6.80394E+00
45	2.69658E-01	2.08129E-01	6.80394E+00	7.98545E+00
46	2.08129E-01	1.12949E-01	7.98545E+00	9.21692E+00
47	1.12949E-01	9.86067E-03	9.21692E+00	1.04755E+01
48	9.86067E-03	-1.12290E-01	1.04755E+01	1.17364E+01
49	-1.12290E-01	-2.95924E-02	1.17364E+01	1.29704E+01
50	-2.95924E-02	1.47962E-02	1.29704E+01	1.41973E+01

Formally, however, our fits are valid in the range $-7.0 \leq \psi_{st} < \infty$, and we have divided this interval into three regions.

$$i) -7.0 \leq \psi_{st} \leq 3.0$$

In this region, the functional dependence is relatively complicated, and we have used cubic spline functions. We have considered 50 equally spaced intervals such that ψ_{st} is in the n th interval when

$$\psi_{st}(n) \leq \psi_{st} \leq \psi_{st}(n+1), \quad (69)$$

where $\psi_{st}(n)$ and $\psi_{st}(n+1)$ are the values of the independent variable at the boundaries of the n th interval. For example, the first interval is bounded by $\psi_{st}(1) = -7.0$ and $\psi_{st}(2) = -6.8$, and the last one is bounded by $\psi_{st}(50) = 2.8$ and $\psi_{st}(51) = 3.0$. The dimensionless collision integrals are given

by

$$\begin{aligned} \ln F_{st}^{(1j)} = & c_{1n}^{(j)} [\psi_{st}(n+1) - \psi_{st}]^3 + c_{2n}^{(j)} [\psi_{st} - \psi_{st}(n)]^3 \\ & + c_{3n}^{(j)} [\psi_{st}(n+1) - \psi_{st}] \\ & + c_{4n}^{(j)} [\psi_{st} - \psi_{st}(n)] \quad (j=1,3) \end{aligned} \quad (70)$$

and

$$\begin{aligned} \ln F_{st}^{(22)} = & d_{1n} [\psi_{st}(n+1) - \psi_{st}]^3 + d_{2n} [\psi_{st} - \psi_{st}(n)]^3 \\ & + d_{3n} [\psi_{st}(n+1) - \psi_{st}] + d_{4n} [\psi_{st} - \psi_{st}(n)]. \end{aligned} \quad (71)$$

The values of the coefficients $c_{ln}^{(j)}$ and d_{ln} ($l=1,4$) are listed in Tables 1–8. A distinction must be made between the case of

TABLE 6
SPLINE COEFFICIENTS (ATTRACTIVE POTENTIAL): $j = 2$

n	$c_{1n}^{(2)}$	$c_{2n}^{(2)}$	$c_{3n}^{(2)}$	$c_{4n}^{(2)}$
1	-3.80453E+00	7.60906E+00	-2.08526E+01	-2.11137E+01
2	7.60906E+00	-1.00577E+01	-2.11137E+01	-1.95486E+01
3	-1.00677E+01	1.09298E+01	-1.95486E+01	-2.03997E+01
4	1.09298E+01	-3.76938E+00	-2.03997E+01	-1.86277E+01
5	-3.76938E+00	-2.50868E+00	-1.86277E+01	-1.77604E+01
6	-2.50868E+00	2.83415E+00	-1.77604E+01	-1.74951E+01
7	2.83415E+00	-1.06745E+00	-1.74951E+01	-1.65496E+01
8	-1.06745E+00	-2.73597E-01	-1.65496E+01	-1.58604E+01
9	-2.73597E-01	5.70462E-01	-1.58604E+01	-1.52368E+01
10	5.70462E-01	1.17072E-01	-1.52368E+01	-1.44762E+01
11	1.17072E-01	-4.26755E-01	-1.44762E+01	-1.36876E+01
12	-4.26755E-01	1.84166E+00	-1.36876E+01	-1.30014E+01
13	1.84166E+00	-4.51420E-02	-1.30014E+01	-1.18732E+01
14	-4.51420E-02	-1.68106E+00	-1.18732E+01	-1.07559E+01
15	-1.68106E+00	7.40637E-01	-1.07559E+01	-1.00419E+01
16	7.40637E-01	-4.15277E-01	-1.00419E+01	-9.15028E+00
17	-4.15277E-01	-5.31841E-01	-9.15028E+00	-8.35829E+00
18	-5.31841E-01	7.54067E-02	-8.35829E+00	-7.69393E+00
19	7.54067E-02	3.64437E-02	-7.69393E+00	-7.01148E+00
20	3.64437E-02	8.52374E-03	-7.01148E+00	-6.32029E+00
21	8.52374E-03	-5.24956E-02	-6.32029E+00	-5.62704E+00
22	-5.24956E-02	-8.76527E-02	-5.62704E+00	-4.94640E+00
23	-8.76527E-02	-1.07693E-01	-4.94640E+00	-4.28679E+00
24	-1.07693E-01	-1.02441E-01	-4.28679E+00	-3.65303E+00
25	-1.02441E-01	-1.04497E-01	-3.65303E+00	-3.04386E+00
26	-1.04497E-01	-1.16430E-01	-3.04386E+00	-2.45976E+00
27	-1.16430E-01	-8.20446E-02	-2.45976E+00	-1.90361E+00
28	-8.20446E-02	-5.53768E-02	-1.90361E+00	-1.36715E+00
29	-5.53768E-02	-4.31036E-02	-1.36715E+00	-8.43977E-01
30	-4.31036E-02	-8.94580E-02	-8.43977E-01	-3.31151E-01
31	-8.94580E-02	1.74820E-02	-3.31151E-01	1.60205E-01
32	1.74820E-02	-4.17248E-02	1.60205E-01	6.55758E-01
33	-4.17248E-02	2.16009E-02	6.55758E-01	1.14130E+00
34	2.16009E-02	5.02965E-02	1.14130E+00	1.63202E+00
35	5.02965E-02	7.34150E-02	1.63202E+00	2.13481E+00
36	7.34150E-02	9.68269E-02	2.13481E+00	2.65522E+00
37	9.68269E-02	1.25372E-01	2.65522E+00	3.19888E+00
38	1.25372E-01	1.59190E-01	3.19888E+00	3.77262E+00
39	1.59190E-01	1.96991E-01	3.77262E+00	4.38456E+00
40	1.96991E-01	2.35595E-01	4.38456E+00	5.04379E+00
41	2.35595E-01	2.69792E-01	5.04379E+00	5.75956E+00
42	2.69792E-01	2.92272E-01	5.75956E+00	6.54007E+00
43	2.92272E-01	2.95067E-01	6.54007E+00	7.39073E+00
44	2.95067E-01	2.70564E-01	7.39073E+00	8.31221E+00
45	2.70564E-01	2.20440E-01	8.31221E+00	9.29863E+00
46	2.20440E-01	1.38511E-01	9.29863E+00	1.03379E+01
47	1.38511E-01	8.80848E-02	1.03379E+01	1.14105E+01
48	8.80848E-02	-7.78941E-02	1.14105E+01	1.25042E+01
49	-7.78941E-02	3.60071E-01	1.25042E+01	1.35792E+01
50	3.60071E-01	-1.80035E-01	1.35792E+01	1.47406E+01

a repulsive SSCP (relevant to ion-ion collisions) and the case of an attractive SSCP (relevant to electron-ion collisions). And, indeed, in the presence of screening (contrary to a pure Coulomb potential), the interaction is felt more strongly by particles that get closer to each other than by particles that are repelled for a given energy and impact parameter. The scattering angle is larger in absolute value (and so is the cross section) for a collision involving two particles with opposite charges than for a collision involving two particles with charges of the same sign.

The lower limit to the range of the independent variable, $\psi_{st} = -7.0$, corresponds to (using eqs. [4], [67], and [68]):

$$\Gamma \approx 132 \frac{Z_i^2}{Z_s Z_t}, \quad (72)$$

i.e., to the case of the strongly coupled plasma. This is already

a very generous lower limit, and there is clearly no point in extending further SSCP collision integral calculations in this regime.

ii) $3.0 < \psi_{st} < 4.0$

For $\psi_{st} \gtrsim 2.7$, the quantities $F_{st}^{(ij)}$ become essentially proportional to $A_{st}^{(1)}$, i.e., to $e^{\psi_{st}}$. In such cases, simple yet accurate analytic formulae can be developed. For the repulsive potential we find

$$F_{st}^{(11)} = 1.00141 e^{\psi_{st}} - 3.18209, \quad (73a)$$

$$F_{st}^{(12)} = 0.99559 e^{\psi_{st}} - 1.29553, \quad (73b)$$

$$F_{st}^{(13)} = 1.99814 e^{\psi_{st}} - 0.64413, \quad (73c)$$

$$F_{st}^{(22)} = 1.99016 e^{\psi_{st}} - 4.56958, \quad (73d)$$

TABLE 7
SPLINE COEFFICIENTS (ATTRACTIVE POTENTIAL): $j = 3$

n	$c_{1n}^{(3)}$	$c_{2n}^{(3)}$	$c_{3n}^{(3)}$	$c_{4n}^{(3)}$
1	-2.06818E+00	4.13636E+00	-1.54059E+01	-1.51150E+01
2	4.13636E+00	-8.93460E+00	-1.51150E+01	-1.38314E+01
3	-8.93460E+00	1.08282E+01	-1.38314E+01	-1.46921E+01
4	1.08282E+01	-2.97377E+00	-1.46921E+01	-1.29540E+01
5	-2.97377E+00	-2.25377E+00	-1.29540E+01	-1.19296E+01
6	-2.25377E+00	2.79041E+00	-1.19296E+01	-1.14462E+01
7	2.79041E+00	-1.37253E+00	-1.14462E+01	-1.02930E+01
8	-1.37253E+00	-3.87183E-01	-1.02930E+01	-9.46928E+00
9	-3.87183E-01	3.36588E-01	-9.46928E+00	-8.73845E+00
10	3.36588E-01	2.05509E-01	-8.73845E+00	-7.92684E+00
11	2.05509E-01	-9.13383E-02	-7.92684E+00	-7.06591E+00
12	-9.13383E-02	5.02385E-01	-7.06591E+00	-6.22691E+00
13	5.02385E-01	-1.25938E-01	-6.22691E+00	-5.26733E+00
14	-1.25938E-01	-6.87925E-01	-5.26733E+00	-4.33797E+00
15	-6.87925E-01	8.12958E-02	-4.33797E+00	-3.57372E+00
16	8.12958E-02	-2.46616E-01	-3.57372E+00	-2.78995E+00
17	-2.46616E-01	-3.03295E-01	-2.78995E+00	-2.06538E+00
18	-3.03295E-01	-9.83392E-02	-2.06538E+00	-1.41359E+00
19	-9.83392E-02	-7.94911E-02	-1.41359E+00	-7.85408E-01
20	-7.94911E-02	-7.66879E-02	-7.85408E-01	-1.76302E-01
21	-7.66879E-02	-8.63431E-02	-1.76302E-01	4.14399E-01
22	-8.63431E-02	-9.05049E-02	4.14399E-01	9.84378E-01
23	-9.05049E-02	-9.00801E-02	9.84378E-01	1.53264E+00
24	-9.00801E-02	-8.47287E-02	1.53264E+00	2.05927E+00
25	-8.47287E-02	-7.78775E-02	2.05927E+00	2.56558E+00
26	-7.78775E-02	-7.05918E-02	2.56558E+00	3.05319E+00
27	-7.05918E-02	-5.90727E-02	3.05319E+00	3.52386E+00
28	-5.90727E-02	-4.74236E-02	3.52386E+00	3.98035E+00
29	-4.74236E-02	-3.62205E-02	3.98035E+00	4.42546E+00
30	-3.62205E-02	-2.90446E-02	4.42546E+00	4.86188E+00
31	-2.90446E-02	-8.69943E-03	4.86188E+00	5.29133E+00
32	-8.69943E-03	-2.75832E-04	5.29133E+00	5.71869E+00
33	-2.75832E-04	1.96726E-02	5.71869E+00	6.14599E+00
34	1.96726E-02	3.91533E-02	6.14599E+00	6.57800E+00
35	3.91533E-02	6.09268E-02	6.57800E+00	7.01941E+00
36	6.09268E-02	8.60213E-02	7.01941E+00	7.47545E+00
37	8.60213E-02	1.15234E-01	7.47545E+00	7.95213E+00
38	1.15234E-01	1.48588E-01	7.95213E+00	8.45646E+00
39	1.48588E-01	1.85161E-01	8.45646E+00	8.99646E+00
40	1.85161E-01	2.22451E-01	8.99646E+00	9.58090E+00
41	2.22451E-01	2.55866E-01	9.58090E+00	1.02187E+01
42	2.55866E-01	2.78501E-01	1.02187E+01	1.09180E+01
43	2.78501E-01	2.83080E-01	1.09180E+01	1.16840E+01
44	2.83080E-01	2.62093E-01	1.16840E+01	1.25180E+01
45	2.62093E-01	2.21026E-01	1.25180E+01	1.34149E+01
46	2.21026E-01	1.44689E-01	1.34149E+01	1.43649E+01
47	1.44689E-01	1.33750E-01	1.43649E+01	1.53496E+01
48	1.33750E-01	-1.00279E-01	1.53496E+01	1.63664E+01
49	-1.00279E-01	7.07988E-01	1.63664E+01	1.73591E+01
50	7.07988E-01	-3.53994E-01	1.73591E+01	1.85217E+01

and for the attractive potential,

$$F_{st}^{(11)} = 1.01101e^{\psi_{st}} - 3.19815, \quad (74a)$$

$$F_{st}^{(12)} = 1.04230e^{\psi_{st}} - 1.89637, \quad (74b)$$

$$F_{st}^{(13)} = 2.15672e^{\psi_{st}} - 2.81038, \quad (74c)$$

$$F_{st}^{(22)} = 2.08699e^{\psi_{st}} - 5.81444. \quad (74d)$$

iii) $\psi_{st} \geq 4.0$

In this region, the plasma is so weakly coupled that screening effects make only small differences between the cross sections for the attractive and the repulsive cases. To a good accuracy, equations (73a)–(73d) can be used in both cases.

The accuracy of the analytic fits as compared with the results of the detailed numerical calculations is very good, especially in region 1, where cubic splines have been used. In this region, which applies to most of the astrophysical problems, the maximum relative errors between the collision integrals evaluated analytically and those evaluated numerically is 0.01%. In regions 2 and 3 this maximum relative error increases to about 0.5%.

IV. SAMPLE RESULTS FOR DIFFUSION COEFFICIENTS

a) Self-diffusion of Helium

As discussed in § II, there exists no exact theory for the transport properties of a plasma in the regime of intermediate coupling. Results are available from kinetic theory for dilute gases and, at the other extreme, from Monte Carlo simulations of very strongly coupled ionic systems. The simple strategy

TABLE 8
 SPLINE COEFFICIENTS (ATTRACTIVE POTENTIAL): $l = 1-4$

n	d_{1n}	d_{2n}	d_{3n}	d_{4n}
1	-1.70341E+00	3.40681E+00	-2.30677E+01	-2.36179E+01
2	3.40681E+00	-1.49538E-01	-2.36179E+01	-2.33505E+01
3	-1.49538E-01	5.36125E+00	-2.33505E+01	-2.31190E+01
4	5.36125E+00	-6.27728E+00	-2.31190E+01	-2.16008E+01
5	-6.27728E+00	7.81096E+00	-2.16008E+01	-2.15892E+01
6	7.81096E+00	-6.95942E-01	-2.15892E+01	-1.97029E+01
7	-6.95942E-01	4.07855E+00	-1.97029E+01	-1.79836E+01
8	4.07855E+00	-2.58381E+00	-1.79836E+01	-1.72432E+01
9	-2.58381E+00	-9.97977E-01	-1.72432E+01	-1.71229E+01
10	-9.97977E-01	5.38341E+00	-1.71229E+01	-1.72422E+01
11	5.38341E+00	-2.04809E+00	-1.72422E+01	-1.60694E+01
12	-2.04809E+00	3.50576E+00	-1.60694E+01	-1.53881E+01
13	3.50576E+00	-4.79859E+00	-1.53881E+01	-1.38655E+01
14	-4.79859E+00	1.16840E+00	-1.38655E+01	-1.34945E+01
15	1.16840E+00	8.19462E-01	-1.34945E+01	-1.28431E+01
16	8.19462E-01	-1.82778E+00	-1.28431E+01	-1.19950E+01
17	-1.82778E+00	-1.65089E-02	-1.19950E+01	-1.15856E+01
18	-1.65089E-02	-4.16448E-01	-1.15856E+01	-1.11802E+01
19	-4.16448E-01	1.88821E+00	-1.11802E+01	-1.08747E+01
20	1.88821E+00	8.26616E-01	-1.08747E+01	-1.01161E+01
21	8.26616E-01	-2.03576E-01	-1.01161E+01	-9.15902E+00
22	-2.03576E-01	-4.49988E-01	-9.15902E+00	-8.25083E+00
23	-4.49988E-01	-3.23824E-01	-8.25083E+00	-7.45064E+00
24	-3.23824E-01	-1.52158E-01	-7.45064E+00	-6.72817E+00
25	-1.52158E-01	-3.19250E-02	-6.72817E+00	-6.04221E+00
26	-3.19250E-02	7.02209E-02	-6.04221E+00	-5.36392E+00
27	7.02209E-02	4.52255E-02	-5.36392E+00	-4.66877E+00
28	4.52255E-02	1.46629E-01	-4.66877E+00	-3.96278E+00
29	1.46629E-01	9.38012E-03	-3.96278E+00	-3.22158E+00
30	9.38012E-03	1.43265E-01	-3.22158E+00	-2.47814E+00
31	1.43265E-01	7.07177E-02	-2.47814E+00	-1.70032E+00
32	7.07177E-02	4.40308E-02	-1.70032E+00	-9.05521E-01
33	4.40308E-02	4.86217E-02	-9.05521E-01	-1.00156E-01
34	4.86217E-02	1.51036E-01	-1.00156E-01	7.16878E-01
35	1.51036E-01	7.56348E-02	7.16878E-01	1.57016E+00
36	7.56348E-02	6.35608E-02	1.57016E+00	2.44160E+00
37	6.35608E-02	1.03660E-01	2.44160E+00	3.32829E+00
38	1.03660E-01	1.48495E-01	3.32829E+00	4.23985E+00
39	1.48495E-01	1.87179E-01	4.23985E+00	5.18706E+00
40	1.87179E-01	2.19549E-01	5.18706E+00	6.17919E+00
41	2.19549E-01	2.34271E-01	6.17919E+00	7.22401E+00
42	2.34271E-01	2.18239E-01	7.22401E+00	8.32506E+00
43	2.18239E-01	1.72842E-01	8.32506E+00	9.47848E+00
44	1.72842E-01	1.06073E-01	9.47848E+00	1.06734E+01
45	1.06073E-01	3.51672E-02	1.06734E+01	1.18938E+01
46	3.51672E-02	-3.76546E-02	1.18938E+01	1.31226E+01
47	-3.76546E-02	-5.81893E-02	1.31226E+01	1.43423E+01
48	-5.81893E-02	-1.82463E-01	1.43423E+01	1.55481E+01
49	-1.82463E-01	2.72096E-01	1.55481E+01	1.67101E+01
50	2.72096E-01	-1.36048E-01	1.67101E+01	1.79374E+01

advocated in this paper for bridging the region of intermediate coupling is to extrapolate the methods of the kinetic theory to higher densities through the use of screened Coulomb potentials. It is therefore of great interest to compare the predictions of this extrapolation with the more exact results of Monte Carlo studies at very high densities.

In stellar astrophysical problems, the most interesting transport properties are usually the diffusion coefficient, D_{st} , and the thermal diffusion coefficient, α_{st} . The one-component plasma studies of Hansen and collaborators (Hansen 1973; Pollock and Hansen 1973; Hansen, McDonald, and Pollock 1975) only provide D_{ss} , the self-diffusion coefficient. Self-diffusion is not, by itself, a very interesting transport property, but it will provide the basis for a comparison with our results. The self-diffusion coefficient for ions of charge Z_1 and mass A_1 , within the framework of the one-component plasma model,

is given by

$$D_{11}^H = 26.2 \frac{Z_1^2}{(A_1 T \Gamma)^{1/2}} D^*, \quad (75)$$

where T is the temperature, Γ the plasma coupling parameter defined by equation (4), and D^* a quantity tabulated as a function of Γ and resulting from detailed Monte Carlo calculations. As an approximation, Hansen gives

$$D^* \approx 2.95 \Gamma^{-1.34}. \quad (76)$$

This analytic formula overestimates D^* by $\sim 50\%$ when $\Gamma = 1.0$ (it should not be used for $\Gamma < 1.0$), but reproduces the exact results to within a few percent for $\Gamma \geq 10$. If, as an example, we consider helium ($Z_1 = 2$, $A_1 = 4$), its self-diffu-

sion coefficient is given by

$$D_{11}^H = 3.98 \times 10^{-9} \frac{T^{1.34}}{\rho^{0.61}}. \quad (77)$$

Stevenson and Salpeter (1977) have provided another means for estimating the self-diffusion of helium at high densities. Their model is strictly limited to perhaps $1 \leq \rho \leq 10$ and $10^3 \leq T \leq 10^5$, when helium is in a *metallic* state, but it provides another point of comparison. It is interesting to note that the interaction model they used is quite different from the one-component plasma model. In their case, the electrons are never so degenerate as to provide a uniform background. On the contrary, they cluster around the ions, and this is approximated by a *hard-sphere* interaction model. From their equations (13) and (16), and taking σ as one-half of the ion-sphere diameter, the helium self-diffusion coefficient is roughly given by

$$D_{11}^{SS} \approx 1.20 \times 10^{-9} \frac{T^{1.50}}{\rho^{0.66}}. \quad (78)$$

In Figure 1 we compare the behavior of the helium self-diffusion coefficient as a function of density for a typical isotherm ($T=10^5$ K) in several theoretical models. First, the dashed curve labeled *H* corresponds to equation (77) and summarizes the results of the Monte Carlo computations of Hansen and collaborators. The curve does not extend to densities such that $\Gamma < 1.0$; the vertical dashed line labeled “ $\Gamma = 1$ ” gives the limiting density. Similarly, the dashed line labeled *SS* corresponds to equation (78), an extension of the results of Stevenson and Salpeter (1977) to both lower and higher densities than their range of interest. These two dashed curves behave approximately the same way and provide the best estimates of the helium self-diffusion coefficient at high densities.

As a comparison, we also show the results of other computations which have been carried out using the methods described in this paper. The plotted results correspond to the first approximation to the self-diffusion coefficient of Chapman and Cowling, as given by equation (5). The continuous curve labeled *PC* shows the self-diffusion coefficient for helium nuclei interacting via a pure Coulomb potential. In these particular computations, a cutoff at λ_D has been introduced (i.e., the first approximation has been made; see § IIIa), but no further approximations have been used. From equation (26), the evaluation of the collision integral has been carried out numerically. As can be seen, $[D_{11}]_1$ becomes constant at very high densities if a pure Coulomb potential is assumed, and this leads to serious overestimates of its true value. This behavior is easily understood because at very high densities $\beta_{st} \ll 1$. Then, from equation (26),

$$\phi_{st}^{(1)} \approx 2\pi \left(\frac{Z_s Z_t e^2}{2kTg^2} \right)^2 \beta_{st}^2 = 2\pi \lambda_D^2. \quad (79)$$

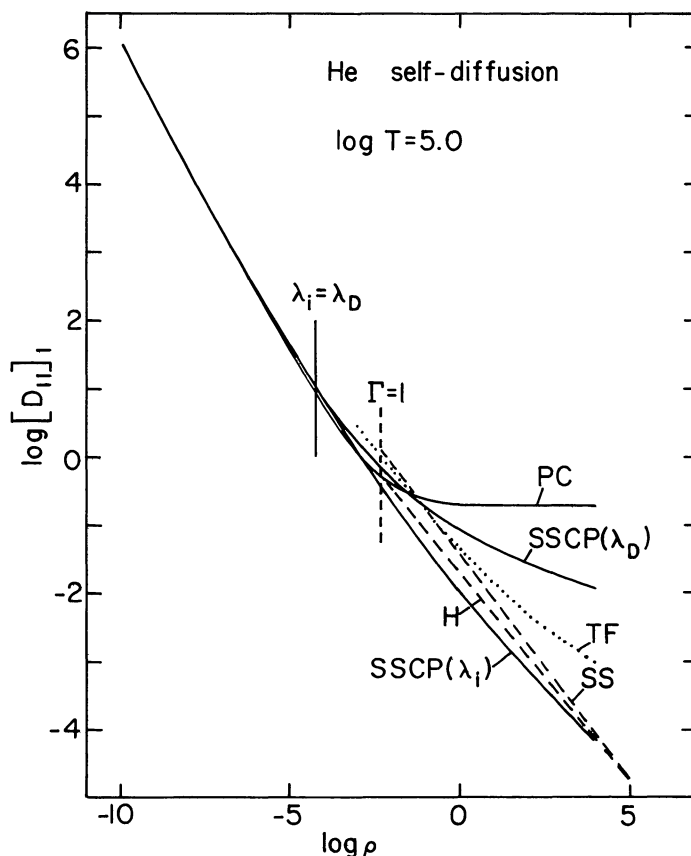


FIG. 1.—Logarithm of the helium self-diffusion coefficient vs. logarithm of the density for the isotherm $T=10^5$ K. The dashed line labeled *H* corresponds to the Monte Carlo studies of Hansen and his collaborators; that labeled *SS* refers to the metallic helium theory of Stevenson and Salpeter (1977). Continuous curves correspond to calculations carried out in the present paper using different interaction potentials. The line labeled *PC* refers to a pure Coulomb potential, that labeled *SSCP* (λ_D) to a Debye-Hückel potential, and that labeled *SSCP* (λ_i) to a Debye-Hückel potential also, but with the screening length taken as the larger of λ_i or λ_D . The dotted line refers to similar calculations with a Thomas-Fermi potential. The vertical lines labeled “ $\lambda_i = \lambda_D$ ” and “ $\Gamma = 1$ ” indicate, respectively, the densities above which $\lambda_i > \lambda_D$ and $\Gamma > 1$.

From equations (5), (17), (18), and (79), we find that

$$[D_{st}]_1 \propto \frac{T^{1/2}}{n\lambda_D^2} \propto \frac{1}{T^{1/2}}, \quad (80)$$

i.e., the diffusion coefficient has lost its density dependence.

By contrast, the continuous curve labeled *SSCP* (λ_i) keeps a density dependence that compares extremely well with the results of high-density models. This curve has been computed with the prescription advocated in this paper, i.e., the use of a Debye-Hückel type potential with the larger of λ_D or λ_i as the screening length. The vertical line labeled “ $\lambda_i = \lambda_D$ ” indicates the density above which $\lambda_D < \lambda_i$. If the Debye length is used above this density, the self-diffusion coefficient behaves as indicated by the continuous curve labeled *SSCP* (λ_D). It is seen that the diffusion coefficient keeps a density dependence,

TABLE 9
DIFFUSION COEFFICIENTS (C VII IN He III, $\log T = 5.0$)

$\log \rho$	$[D_{12}]_1$			$[D_{12}]_2$		
	CC	PC	SSCP	CC	PC	SSCP
-16.....	5.36×10^{10}	5.81×10^{10}	5.98×10^{10}	5.51×10^{10}	5.95×10^{10}	6.13×10^{10}
-14.....	6.31×10^8	6.93×10^8	7.18×10^8	6.49×10^8	7.11×10^8	7.36×10^8
-12.....	7.67×10^6	8.60×10^6	8.99×10^6	7.88×10^6	8.81×10^6	9.20×10^6
-10.....	9.76×10^4	1.13×10^5	1.20×10^5	1.00×10^5	1.16×10^5	1.23×10^5
-8.....	1.34×10^3	1.66×10^3	1.82×10^3	1.38×10^3	1.69×10^3	1.86×10^3
-6.....	2.15×10	3.09×10	3.77×10	2.21×10	3.14×10	3.83×10
-4.....	5.33×10^{-1}	1.11	1.39	5.48×10^{-1}	1.11	1.40
-2.....	8.64×10^{-2}	1.92×10^{-1}	5.78×10^{-2}	8.92×10^{-2}	1.92×10^{-1}	5.79×10^{-2}
0.....	7.85×10^{-2}	1.57×10^{-1}	3.68×10^{-3}	8.10×10^{-2}	1.57×10^{-1}	3.68×10^{-3}
2.....	7.85×10^{-2}	1.57×10^{-1}	3.23×10^{-4}	8.11×10^{-2}	1.58×10^{-1}	3.23×10^{-4}

but its behavior deviates systematically from the results of the high-density models. Both the SSCP curves converge together at the point where $\lambda_i = \lambda_D$, and they also converge to the PC results—as they should—at very low densities. In this limit, screening effects become negligible, and the choice of λ_D as a cutoff distance (first approximation) becomes irrelevant. Finally, Figure 1 also shows (dotted line labeled *TF*) the results of Fontaine and Michaud (1979*b*) for a Thomas-Fermi potential. As compared with the dashed lines, this approach gives intermediate results between the two cases of the SSCP.

The most interesting result is the surprisingly good agreement (within a factor of 2) between the helium self-diffusion coefficient computed according to the extrapolation methods of this paper and the Monte Carlo results of Hansen. We emphasize that there is no theoretical justification for such good agreement. It may simply be that the diffusion coefficient, being a gross transport property (it is related to the collision frequency), is not very sensitive to the exact interaction models being used, as long as the characteristic lengths are comparable. On the other hand, this result gives us rather strong confidence that the intermediate coupling region is probably adequately described. This gives us reason to believe that our estimates of diffusion coefficients under white dwarf conditions are indeed reasonable.

b) Trace C VII in a Background of He III

As another illustrative example, we consider the case of C VII ions of *negligible* abundance diffusing in a background of ionized helium. We wish to compare the Chapman and Cowling binary diffusion coefficients computed with the methods presented in this paper (SSCP) with those obtained using a pure Coulomb potential. Two cases are considered for the latter: the first one is referred to as the PC case and was introduced in the previous subsection. Given a pure Coulomb potential, only the choice of λ_D as the long-range cutoff distance was introduced in the evaluation of the collision integrals (first approximation; cf. § III*a*). This means that the integrations over the velocity distribution were carried out numerically. The second case is referred to as the CC case, and corresponds to the *analytic* formulae of Chapman and Cowling, which were derived after making four approximations as described previously. These formulae are given by equations (40)–(43). Comparing the SSCP results with the PC

TABLE 10
THERMAL DIFFUSION COEFFICIENTS (C VII IN He III, $\log T = 5.0$)

$\log \rho$	$[\alpha_{12}]_1$		
	CC	PC	SSCP
-16.....	1.45×10	1.22×10	1.22×10
-14.....	1.46×10	1.19×10	1.19×10
-12.....	1.47×10	1.14×10	1.14×10
-10.....	1.49×10	1.08×10	1.06×10
-8.....	1.52×10	9.57	9.14
-6.....	1.60×10	7.12	6.11
-4.....	2.00×10	2.52	2.54
-2.....	1.48×10^2	-3.28×10^{-1}	1.00
0.....	1.26×10^4	-4.15	2.96×10^{-1}
2.....	1.25×10^6	-3.45×10^2	3.41×10^{-3}

results shows the effects of screening. Comparing the CC results with the PC results shows the consequences of introducing nonessential approximations in the evaluation of the collision integrals. The various coefficients are compared over a broad range of densities for the isotherm $T = 10^5$ K; this extends and completes the preliminary work presented by Fontaine and Michaud (1979*b*).

Tables 9–12 give, respectively, the diffusion coefficients $[D_{12}]_1$, $[D_{12}]_2$, and the thermal diffusion coefficients $[\alpha_{12}]_1$, $[\alpha_{1e}]_1$, and $[\alpha_{e2}]_1$ for densities in the range $-16 \leq \log \rho \leq 2$. Some of these results have been presented by Fontaine and Michaud (1979*b*), but it is to be noted that they used $\lambda = \lambda_D$ for all densities. For $\log \rho \geq -4$, $\lambda_D < \lambda_i$, and the present SSCP results differ from theirs. We discuss first the case of the diffusion coefficient $[D_{12}]_1$.⁵

The behavior of the diffusion coefficient as a function of the density is identical with that of the self-diffusion coefficient discussed in the previous subsection. In particular, for the two formalisms based on the pure Coulomb potential (CC and PC), the diffusion coefficient becomes a constant at very high densities for a given temperature (see eq. [80]). Taking the SSCP results as reference, Figure 2 shows the differences between the diffusion coefficients of the various formalisms as functions of the density. At low densities ($\log \rho \leq -4$), where

⁵ From Table 9 it is seen that, for all three formalisms, the differences between the first and second approximations to the diffusion coefficient are small, of the order of 2%–3% and less.

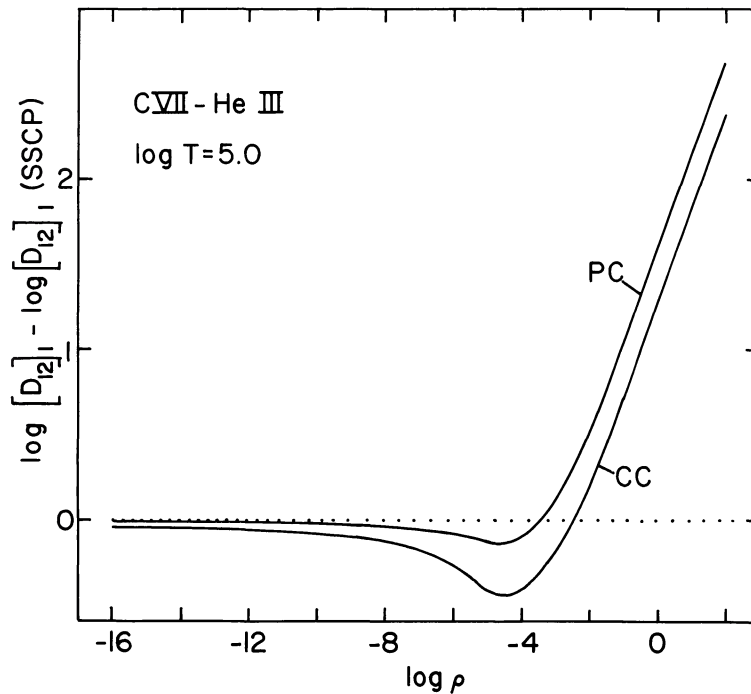


FIG. 2.—Difference between diffusion coefficients as a function of the density for trace C VII ions diffusing in a background of He III at $T=10^5$ K. The dotted horizontal line gives the reference level based on SSCP calculations. The curves labeled PC and CC refer, respectively, to calculations based on the pure Coulomb potential with one and four (analytic results) approximations.

TABLE 11

THERMAL DIFFUSION COEFFICIENTS (C VII IN He III, $\log T = 5.0$)

$\log \rho$	$[\alpha_{1e}]_1$		
	CC	PC	SSCP
-16	-2.86	-2.78	-2.75
-14	-2.86	-2.76	-2.74
-12	-2.86	-2.75	-2.73
-10	-2.86	-2.72	-2.71
-8	-2.87	-2.68	-2.68
-6	-2.89	-2.59	-2.60
-4	-2.92	-2.31	-2.47
-2	-3.07	-1.12	-2.32
0	-3.43	0.60	-1.82
2	-3.46	1.13	-0.36

TABLE 12

THERMAL DIFFUSION COEFFICIENTS (C VII IN He III, $\log T = 5.0$)

$\log \rho$	$[\alpha_{e2}]_1$		
	CC	PC	SSCP
-16	9.86×10	7.99×10	7.96×10
-14	9.91×10	7.75×10	7.72×10
-12	9.99×10	7.42×10	7.36×10
-10	1.01×10^2	6.93×10	6.86×10
-8	1.03×10^2	6.17×10	6.08×10
-6	1.07×10^2	4.76×10	4.91×10
-4	1.23×10^2	2.05×10^{-1}	3.45×10
-2	3.80×10^2	5.02×10^{-1}	2.08×10
0	2.29×10^4	-1.64	7.04
2	2.28×10^6	-4.06 $\times 10$	-2.05

computations based on the pure Coulomb potential are expected to be reasonably accurate, the derived diffusion coefficients are seen to be systematically *smaller* than those obtained using a screened Coulomb potential. Of course, screening effects become very small at very low densities, and this explains the convergence of the PC and SSCP results in this limit. Note, however, that even for $\log \rho = -16$, there still exists a 10% relative difference between the CC and SSCP results. This difference increases to 62% for $\log \rho = -4$, a value of the density that is encountered even in main-sequence star envelopes. Screening effects are responsible for only one-third of this difference, as can be seen by comparing the PC and SSCP results at that density. The remainder of the dif-

ference is due to the additional approximations that have been made to obtain analytic formulae. As mentioned before, the use of a pure Coulomb potential at very high densities produces a theory breakdown, and the diffusion coefficient becomes a constant. Figure 2 shows that, in such cases, the use of the PC or CC formalisms can lead to large *overestimates* of the diffusion coefficient, since the difference with the SSCP value increases roughly as $\rho^{0.55}$.

The thermal diffusion coefficients appear to be much more sensitive than the diffusion coefficient to the approximations that have been made to obtain analytic formulae, and an explanation for this behavior is offered in Pelletier *et al.* (1986). For example, Tables 10–12 indicate that the CC

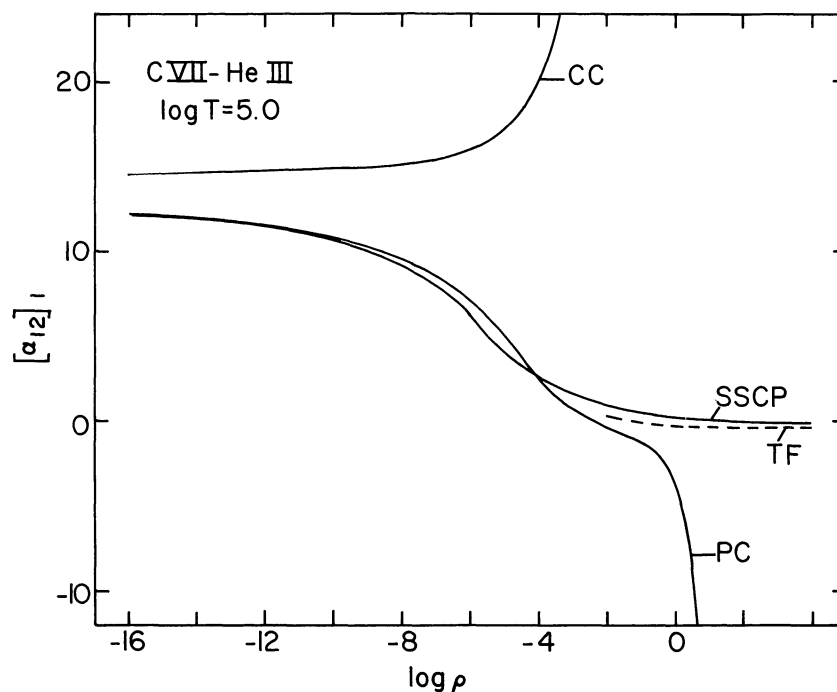


FIG. 3.—Ionic thermal diffusion coefficient as a function of the density for trace C VII ions diffusing in a background of He III at $T = 10^5$ K. The curve labeled SSCP corresponds to calculations carried out according to the methods presented in this paper. The curves labeled PC and CC are as in the previous figure. The dashed line gives the Thomas-Fermi results of Fontaine and Michaud (1979b).

thermal diffusion coefficient is always larger (in absolute value) than the PC and SSCP values and, more important, that it increases (again in absolute value) with increasing density, which is *contrary* to what is observed for the PC and SSCP formalisms. This is illustrated in Figure 3, which shows the behavior of $[\alpha_{12}]_1$ as a function of the density. As expected, in the limit of very low densities, the PC and SSCP results converge. Up to $\log \rho \approx -4$, the two values of the thermal diffusion coefficient remain comparable, but the SSCP values are systematically lower. At higher densities, the PC model breaks down and $[\alpha_{12}]_1$ diverges to large negative values. By contrast, the analytic formula (eq. [41]) makes $[\alpha_{12}]_1$ diverge to large positive values with increasing density. The use of the analytic formula already leads to a relative error of 19% for $\log \rho = -16$, and this error increases very rapidly with density, reaching a value of 690% for $\log \rho = -4$. Such results suggest that the analytic formulae of Chapman and Cowling for the thermal diffusion coefficients give values that can be large *overestimates* (in an absolute sense) even under the physical conditions encountered in the atmospheres of nondegenerate stars.

It is of some interest to discuss the behavior of $[\alpha_{12}]_1$ within the framework of the SSCP model at very high densities. Table 10 and Figure 3 show that $[\alpha_{12}]_1$ assumes only relatively small values for $\log \rho \geq 0$. This is also the case for a Thomas-Fermi (TF) potential, as indicated by the dashed line in the figure (the data were taken from Fontaine and Michaud 1979b). There is a change of sign for the TF case for $\log \rho \geq -1$, but we also observe the same phenomenon as the computation of an additional point (shown in Fig. 3) gives

$[\alpha_{12}]_1(\text{SSCP}) = -0.134$ for $\log \rho = 4$.⁶ The important point, however, is that the absolute value of the coefficient remains small, which suggests that ion-ion thermal diffusion could be negligible in dense plasmas when compared with other phenomena such as gravitational settling or ordinary diffusion, for example. The same could be true for the other two thermal diffusion coefficients ($[\alpha_{1e}]_1$ and $[\alpha_{e2}]_1$), but the latter suffer from additional uncertainties arising from quantum mechanical corrections associated with electron-ion interactions (see Pelletier *et al.* 1986 for a discussion of these points).

Stevenson and Salpeter (1977) have briefly discussed the phenomenon of thermal diffusion for a mixture of hydrogen and helium in the metallic phase within the framework of an ion-ion interaction model based on hard spheres.⁷ They conclude that the thermal diffusion coefficient gives rise only to small effects as compared with ordinary diffusion, and they neglect thermal diffusion in their ensuing discussion. It is also noteworthy to point out that thermal diffusion in a H-He metallic fluid is such that helium will diffuse toward the cooler regions, i.e., the thermal diffusion coefficient has a negative

⁶A negative value of $[\alpha_{12}]_1$ means that a C VII ion in a He III background would move toward the cooler regions in the presence of a temperature gradient.

⁷They refer to thermal diffusion as a "second-order transport coefficient." In the context of the formalism of Chapman and Cowling, this must be understood as meaning that the first approximation to the thermal diffusion coefficient actually comes from using the second term in the expansion of the first-order distribution function on the basis of Sonine polynomials. The first term in this expansion gives a thermal diffusion coefficient that vanishes identically.

sign (in our convention). Thus, the results of Stevenson and Salpeter (1977) give welcome support to our finding that ionic thermal diffusion could be neglected at high densities. This also gives us added confidence that our method for bridging the intermediate-coupling region is adequate, not only for the diffusion coefficient but also for thermal diffusion, although, in the latter case, other uncertainties related to electron-ion collisions remain.

V. CONCLUDING REMARKS

The main results of the present paper are presented in the form of high-accuracy fits to the collision integrals for a screened Coulomb potential of the Debye-Hückel type. This interaction model is applicable to a stellar plasma. The fits are given by equations (69)–(74), in conjunction with Tables 1–8. The derived collision integrals can be used to compute diffusion coefficients for stellar plasmas according to either the Chapman and Cowling approach (eqs. [5]–[17]) or the method of Burgers (eqs. [22]–[25]). For those interested in using and programming our analytic fits, the SSCP data presented in Tables 9–12 provide specific examples of such coefficients and can be used to check the output of their code. The derived diffusion coefficients are very accurate for dilute stellar plasmas such as those encountered in the atmospheres and envelopes of nondegenerate stars. For applications requiring high accuracy, our diffusion coefficients are to be preferred to the analytic expressions of Chapman and Cowling (1970). As discussed in this paper, those expressions have been very widely used in astrophysical problems, but they have been obtained after making nonessential approximations in the evaluation of the collision integrals. For dilute plasmas, we have found that the analytic formulae of Chapman and Cowling systematically underestimate the diffusion coefficient, and systematically overestimate the thermal diffusion coefficients. The differences can become nonnegligible in main-sequence stars. Michaud, Fontaine, and Beaudet (1984) briefly mention

this point in the context of lithium diffusion in solar-type stars.

The main motivation for the present efforts, however, has been the need to obtain more reliable estimates of transport properties for conditions encountered in white dwarf envelopes, and under which the analytic formulae of Chapman and Cowling fail completely. This is part of our long-range efforts to understand better the spectral evolution of white dwarf stars. As we have tried to emphasize throughout the paper, the values of the diffusion coefficients derived for white dwarf conditions can only be regarded as provisional. However, as discussed in § IV, there are reasons to believe that these values indeed provide reasonable estimates of the transport coefficients. Rigorous statistical mechanical treatments are clearly needed to describe the transport and thermodynamic properties of a plasma in the regime of intermediate coupling. It appears, however, that the results of such studies are not likely to become available within several years. Under such circumstances, we have used, without further apology, coefficients computed according to the method presented in this paper to study diffusion in white dwarfs. Specifically, and in addition to the two papers referred to in § I, we have used such coefficients in our recent discussions of the mechanism of diffusion-induced hydrogen burning in white dwarfs (Michaud, Fontaine, and Charland 1984; Michaud and Fontaine 1984). Lacombe *et al.* (1983) have also used similar coefficients in their computations of diffusion time scales in the cool DAZ white dwarf G74–7. Finally, diffusion coefficients computed from the analytic fits presented in this paper have also been used by Lamontagne *et al.* (1985) and Michaud *et al.* (1985) to discuss abundances anomalies in hot B subdwarf stars.

This research has been supported in part by the Natural Sciences and Engineering Research Council of Canada. Computing funds for this project were kindly provided by the Centre de Calcul of the Université de Montréal.

REFERENCES

- Alcock, C., and Illarionov, A. 1980, *Ap. J.*, **235**, 534.
 Burgers, J. M. 1960, in *Plasma Dynamics*, ed. F. H. Clauser (Reading: Addison-Wesley), p. 119.
 ———. 1969, *Flow Equations for Composite Gases* (New York: Academic).
 Chapman, S., and Cowling, T. G. 1970, *The Mathematical Theory of Non-uniform Gases* (3d ed.; Cambridge: Cambridge University Press).
 D'Antona, F., and Mazzitelli, I. 1979, *Astr. Ap.*, **74**, 161.
 DeWitt, H. E. 1961, *J. Nucl. Energy, C*, **2**, 224.
 ———. 1969, in *Low Luminosity Stars*, ed. S. Kumař (New York: Gordon & Breach), p. 211.
 ———. 1976, *Phys. Rev. A*, **14**, 1290.
 DeWitt, H. E., Graboske, H. C., and Cooper, M. S. 1973, *Ap. J.*, **181**, 439.
 Fontaine, G., Graboske, H. C., and Van Horn, H. M. 1977, *Ap. J. Suppl.*, **35**, 293.
 Fontaine, G., and Michaud, G. 1979a, *Ap. J.*, **231**, 826.
 ———. 1979b, in *IAU Colloquium 53, White Dwarfs and Variable Degenerate Stars*, ed. H. M. Van Horn and V. Weidemann (Rochester: University of Rochester), p. 192.
 Fontaine, G., and Van Horn, H. M. 1976, *Ap. J. Suppl.*, **31**, 467.
 Hansen, J. P. 1973, *Phys. Rev. A*, **8**, 3096.
 ———. 1978, in *Strongly Coupled Plasmas*, ed. G. Kalman and P. Carini (New York: Plenum), p. 117.
 Hansen, J. P., McDonald, I. R., and Pollock, E. L. 1975, *Phys. Rev. A*, **11**, 1025.
 Iben, I., Jr., and MacDonald, J. 1985, *Ap. J.*, **296**, 540.
 Lacombe, P., Liebert, J., Wesemael, F., and Fontaine, G. 1983, *Ap. J.*, **272**, 660.
 Lamontagne, R., Wesemael, F., Fontaine, G., and Sion, E. M. 1985, *Ap. J.*, **299**, 496.
 Liboff, R. L. 1959, *Phys. Fluids*, **2**, 40.
 Mason, E. A., Munn, R. J., and Smith, F. J. 1967, *Phys. Fluids*, **10**, 1827.
 Mason, E. A., and Schamp, H. W., Jr. 1958, *Ann. Phys.*, **4**, 233.
 Michaud, G., Bergeron, P., Wesemael, F., and Fontaine, G. 1985, *Ap. J.*, submitted.
 Michaud, G., and Fontaine, G. 1984, *Ap. J.*, **283**, 787.
 Michaud, G., Fontaine, G., and Beaudet, G. 1984, *Ap. J.*, **282**, 206.
 Michaud, G., Fontaine, G., and Charland, Y. 1984, *Ap. J.*, **280**, 247.
 Muchmore, D. O. 1980, Ph.D. thesis, University of Washington, Seattle.
 ———. 1984, *Ap. J.*, **278**, 769.
 Paquette, C., Pelletier, C., Fontaine, G., and Michaud, G. 1986, *Ap. J. Suppl.*, **61**, 197.
 Pelletier, C., Paquette, C., Michaud, G., and Fontaine, G. 1986, in preparation.
 Pollock, E. L., and Hansen, J. P. 1973, *Phys. Rev. A*, **8**, 3110.
 Rogers, F. J., Graboske, H. C., and Harwood, D. J. 1970, *Phys. Rev. A*, **1**, 1577.
 Roussel-Dupré, R. 1981, *Ap. J.*, **243**, 329.
 ———. 1982, *Ap. J.*, **252**, 393.
 Stevenson, D. J., and Salpeter, E. E. 1977, *Ap. J. Suppl.*, **35**, 221.
 Vauclair, G., Vauclair, S., and Greenstein, J. L. 1979, *Astr. Ap.*, **80**, 79.

G. FONTAINE, G. MICHAUD, C. PAQUETTE, and C. PELLETIER: Département de Physique, Université de Montréal, C.P. 6128, Succ. A, Montréal, Québec, Canada H3C 3J7