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# Collective relaxation of stellar systems

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**Summary.** The problem of relaxation of stellar systems is investigated from the point of view of ergodic theory. It is shown that an exponential instability peculiar to Kolmogorov K-systems and leading to equilibrium exists in general stellar systems. *The relaxation time for real stellar systems is calculated.* That time is substantially smaller than the binary relaxation time. The physical difference between the two relaxation times is discussed.

**Key words:** stellar dynamics – stellar systems – relaxation – Kolmogorov K-systems

## 1. Introduction

Real stellar systems – globular clusters and galaxies – are known to be generally in equilibrium. This is reflected in the high degree of regularity of some basic physical characteristics of the stellar systems, i.e. surface luminosity, dispersion of velocities, geometric shapes, etc.

Jeans, Schwarzschild, Eddington, Ambartzumian, Chandrasekhar, Spitzer and others have applied many fundamental principles of statistical mechanics to stellar systems. Thus, Chandrasekhar (1942) has considered in detail a relaxation mechanism, based on the most natural process-stellar binary encounters. However the magnitude of the relaxation time of real stellar systems (especially elliptical galaxies), due only to binary encounters, turned out to be more than $10^{13}$ yr, i.e. it exceeded the Hubble time.

An important step to eliminate this paradox was Lynden-Bell's paper (Lynden-Bell, 1967) in which the theory of the collisionless violent relaxation was developed. This theory, although it has a great heuristic advantage and has stimulated a lot of papers, nevertheless could not avoid certain difficulties. Being a theory describing essentially the non-equilibrium phase of evolution of stellar systems, it cannot describe their quasi-equilibrium phase. Among many further attempts to understand the mechanics of collisionless stellar systems, it is worth mentioning the paper by Severne and Luwel (1980), wherein the contribution of fluctuations of the self-consistent field to the relaxation process has been considered.

The interest in the relaxation problem and in the dynamical evolution of stellar systems has grown sharply due to a number of recently obtained interesting observational data. Thus, at the centre of the globular cluster M 15, an anomalous excess of brightness is observed, whose explanation by the presence of a central black hole (Newell et al, 1976) encounters certain difficulties (Gurzadyan, 1982; see also Illingworth and King, 1977). The existence of a "rapid" mechanism of relaxation could be the possible explanation of this fact. The problem of the shape of elliptical galaxies has also acquired a new content (Binney, 1982).

The difficulties for a satisfactory understanding of the dynamics of stellar systems are due to the well-known fact that in a system of N gravitationally interacting stars Debye screening, as distinct from plasma, is absent. This circumstance makes the statistical description of gravitational systems more complicated and requires special methods.

All that points out to the crucial role of collective effects in the process of relaxation of stellar systems.

The present study is aimed at the investigation of this problem from the viewpoint of the ergodic theory which in a way takes into account the collective nature of interaction.

In the ergodic theory (Hopf, 1937; Halmos, 1953; Arnold, 1979; Kornfeld et al, 1980) a great progress is achieved in the investigation of the statistical properties of dynamical systems described by differential equations. A classification of nonintegrable dynamical systems by increasing the degree of their statistical properties is obtained. The K-systems (Kolmogorov, 1958) possess the strongest statistical properties. These systems tend to equilibrium with an exponential rate. In this paper we shall inquire into the relation of K-systems with gravitating ones.

Originally the methods used below had found their application in N.S. Krylov's (1950) outstanding investigation on gas relaxation. Recently by means of these methods one of the authors (Savvidy, 1983, 1984) has studied the statistical properties of a non-Abelian Yang-Mills gauge field. The latter, was shown to be Kolmogorov K-system.

The content of the paper is as follows. In Sect. 2 the problem of $N$ gravitating bodies is reduced by means of the Maupertuis principle to the problem of the geodesics of a Reimannian manifold and its main geometrical characteristics are represented. It is shown why the negativity of *the two-dimensional curvature* is a sufficient condition for an exponential divergence of the geodesics.

In Sect. 3 the relation between the exponential divergence of geodesics and the statistical properties of an $N$-body system is discussed. It is shown that it is possible to define a collective relaxation time as the index of the exponential deviation. In Sects. 4 and 5 after the calculation of the scalar curvature of the

*Send offprint requests to:* V.G. Gurzadyan manifold, *the collective relaxation time is estimated.* Its difference from the binary relaxation time is discussed and the existence of *three scales of length and time for stellar systems is demonstrated.*

In Sect. 6 the sign-definitness of the two-dimensional curvature is investigated. It is shown that in general this curvature can be *both negative and positive*, however in spherical and related systems it is strongly *negative*. Therefore the latter systems possess properties similar to those of K-systems.

A discussion of our results is presented in Sect. 7 and the *calculation of the two-dimensional curvature as an effective method for the numerical investigation of stochasticity of dynamical* systems is proposed.

## 2. Reduction of the $N$-body Problem to the Study of a Geodesic Flow in a $3N$-Dimensional Riemannian manifold.

Denote by $\boldsymbol{r}_a$ ($a = 1, \ldots, N$) the coordinates of stars. The potential of interaction is

$$U = \sum_{a<b} U(\boldsymbol{r}_a - \boldsymbol{r}_b) = -G \sum_{a<b} \frac{M_a M_b}{r_{ab}}; \quad \boldsymbol{r}_{ab} = \boldsymbol{r}_a - \boldsymbol{r}_b \tag{1}$$

where $M_a$ is the stellar mass.

The equations of motion in Hamiltonian form are

$$\dot{\boldsymbol{p}}_a = -\frac{\partial H}{\partial \boldsymbol{r}_a}, \quad \boldsymbol{r}_a = \frac{\partial H}{\partial \boldsymbol{p}_a}, \tag{2}$$

where $H$ is the complete Hamiltonian of the system and $\boldsymbol{p}_a$ is the star's momentum. As $H$ is explicitly time-independent, $H(\boldsymbol{p}, \boldsymbol{r})$ is an integral of motion, hence the equation $H(\boldsymbol{p}, \boldsymbol{r}) = E = \text{const}$ determines a $6N$-1-dimensional energy hypersurface in the $6N$-dimensional phase space.

By means of the variational principle of Maupertuis the trajectories of the system (2) may be presented as geodesics of some Riemann metric given in a region of the configurational space $(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \in Q$ defined by the inequality $U(\boldsymbol{r}_a) < E$. The line element in this metric is given by (Arnold, 1979)

$$ds^2 = (E - U)d\rho^2 = W \sum_{\alpha=1}^{3N} (dq^\alpha)^2, \tag{3}$$

where

$$W = E - U$$

and $\{q^\alpha\}$ is defined as

$$\{q^\alpha\} = \{M_1^{1/2}\boldsymbol{r}_1, \ldots, M_N^{1/2}\boldsymbol{r}_N\}. \ \alpha = 1, \ldots, 3N \tag{4}$$

The main idea is that the study of the behaviour of the geodesics, and hence of the trajectories of the system (2) reduces to the investigation of the geometrical properties of the Riemann manifold prescribed by the metric (3).

The equation of the geodesics on the Riemann manifold (3) is

$$\frac{d^2 q^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dq^\beta}{ds} \frac{dq^\gamma}{ds} = 0, \tag{5}$$

($\Gamma^\alpha_{\beta\gamma}$ are the Cristoffel symbols). If we set

$$g_{\alpha\beta} = W\delta_{\alpha\beta}, \quad g^{\alpha\beta} = \frac{1}{W}\delta_{\alpha\beta} \tag{6}$$

we can write Eq. (5) in the form

$$\frac{d^2 q^\alpha}{ds^2} + \frac{1}{2W}\left[2\frac{\partial W}{\partial q^\gamma}\frac{dq^\gamma}{ds}\frac{dq^\alpha}{ds} - g^{\alpha\gamma}\frac{\partial W}{\partial q^\gamma} g_{\beta\delta}\frac{dq^\beta}{ds}\frac{dq^\delta}{ds}\right] = 0. \tag{7}$$

Equation (7) coincides with equations of motion (2) if the proper time $ds$ is replaced by $\sqrt{2}Wdt$.

The global properties of the geodesics are defined by the linear deviation $\delta q$ between close by geodesics. This deviation satisfies the equation

$$\frac{D^2 \delta q^\alpha}{Ds^2} = -R^\alpha_{\beta\gamma\delta}(q)\frac{dq^\beta}{ds}\delta q^\gamma \frac{dq^\delta}{ds}, \tag{8}$$

where $R^\alpha_{\beta\gamma\delta}$ is the Riemann tensor and $D/Ds$ denotes the covariant derivative.

The measure of the deviation is given by the relation

$$|\delta q|^2 = g_{\alpha\beta}\delta q^\alpha \delta q^\beta \tag{9}$$

and satisfies the following equation

$$\frac{d^2|\delta q|^2}{ds^2} = -2R_{\alpha\beta\gamma\delta}(q)\delta q^\alpha \frac{dq^\beta}{ds}\delta q^\gamma \frac{dq^\delta}{ds} + 2\left|\frac{D\delta q}{Ds}\right|^2. \tag{10}$$

From Eq. (10), one can see that the linear deviation of the geodesics depends on the geometry of the Riemann manifold through the Riemann tensor.

Indeed, the curvature $K$ of the Riemann manifold along the directions $\delta q$ and $dq/ds$ is defined by (Arnold, 1979)

$$K(\delta q, dq/ds) \cdot |\delta q \Lambda dq/ds|^2 = R_{\alpha\beta\gamma\delta}\delta q^\alpha \frac{dq^\beta}{ds}\delta q^\gamma \frac{dq^\delta}{ds}, \tag{11}$$

where

$$|\delta q \Lambda dq/ds|^2 = (g_{\alpha\beta}\delta q^\alpha \delta q^\beta)\left(g_{\delta\gamma}\frac{dq^\delta}{ds}\frac{dq^\gamma}{ds}\right) - \left(g_{\alpha\beta}\delta q^\alpha \frac{dq^\beta}{ds}\right)^2. \tag{11a}$$

If $K$ is negative in all directions $\delta q$ and $dq/ds$ then the linear deviation will change by an exponential rate.

Let us present the linear divergence of the geodesics as the geometric sum of the longitudinal and normal components to the velocity vector

$$\delta q = \delta q_\perp + \delta q_\parallel, \tag{12}$$

where

$$|\delta q_\perp \dot{q}| = 0. \tag{13}$$

Then, one can see from (8) that the longitudinal component satisfies the trivial equation

$$\frac{D^2 \delta q_\parallel^2}{Ds^2} = 0$$

and the normal component the same Eq. (8), where $\delta q$ is replaced by $\delta q_\perp$. Thus now in (11a) we have

$$\left|\delta q \Lambda \frac{dq}{ds}\right|^2 = |\delta q|^2 \left|\frac{dq}{ds}\right|^2. \tag{14}$$

Equation (3) can be written in the form

$$\left|\frac{dq}{ds}\right|^2 = W\left(\frac{dq}{ds}\right)^2 = 1 \tag{15}$$

therefore

$$\left|\delta q \Lambda \frac{dq}{ds}\right|^2 = |\delta q|^2 \tag{16}$$

Now, if $K$ in Eq. (10) is always negative, this equation gives

$$\frac{d^2|\delta q|^2}{ds^2} \geq 2k|\delta q|^2 , \tag{17}$$

where

$$k = \min_{(q,\,\delta q,\,dq/ds)} |K(\delta q, dq/ds)| . \tag{18}$$

Thus we have

$$\begin{aligned} &|\delta q(s)| \gtrsim |\delta q(0)| e^{\sqrt{2k}s} \quad \text{if} \quad \left.\frac{d|\delta q|}{ds}\right|_{s=0} > 0 , \\ &|\delta q(s)| \lesssim |\delta q(0)| e^{-\sqrt{2k}s} \quad \text{if} \quad \left.\frac{d|\delta q|}{ds}\right|_{s=0} < 0 . \end{aligned} \tag{19}$$

Analogous relations can be found for the deviations in the momentum $(dq^\alpha/ds) = p_\alpha$.

Hence one can define a relaxation time

$$\tau = (2k)^{-1/2} , \tag{20}$$

which coincides with the Kolmogorov entropy (Kolmogorov, 1958). Thus, *the negativity of the two-dimensional curvature of the Riemann manifold defined by the metric* (3) *is a sufficient condition for the exponential instability of a stellar system.*

The next section deals with the statistical properties of the dynamical systems having an exponential instability (19), from the viewpoint of the ergodic theory, and their relation with the relaxation time.

## 3. Statistical properties of the dynamical systems: definition of the relaxation time

The dynamical systems can be divided into two classes-integrable ones, i.e. when the number of conserved integrals is equal to the number of degrees of freedom, and the phase trajectories lie on $N$-dimensional tori, and non-integrable ones. The classification of non-integrable dynamical systems is given in the ergodic theory by increasing the degree of their statistical properties. Those are systems with divided phase space (i.e. containing both motion on $N$-dimensional tori and chaotic motions), ergodic systems, systems with weak mixing, with $n$-fold mixing, and finally K-systems. The physical aspects of the classification are treated in more details in Savvidy (1984).

K-system possess maximally strong statistical properties. One of their main properties is the decay of trajectories in the phase space into beams of exponentially approaching and expanding trajectories (transversal fibers) (Anosov, 1967; Sinai 1970). Therefore K-systems tend to an equilibrium state (microcanonical) with an exponential rate.

The question is what is the rate at which an initial cell of phase space will tend to cover uniformly the energy hypersurface $H = \text{const}$. In the ergodic theory it is shown that in mixing systems an initial cell complicates its shape so much (preserving its volume) so that it covers uniformly the hypersurface $H = \text{const}$ as $t \to \infty$. In this sense, a mixing system in non-equilibrium tends to equilibrium after infinite time. However, if we require that the mixing occurs with a prescribed accuracy $\varepsilon$ connected with the accuracy of physical measurements, then this time $\tau_\varepsilon$ will be finite.

The systems with exponential instability, i.e. those with K-mixing, tend to the microcanonical equilibrium state with the exponential rate, i.e. the deviations from equilibriums decrease proportionally to $e^{s/\tau}$, where $s$ is the proper time, and $\tau$ is the characteristic time (20). Therefore, if we adopt a certain accuracy $\varepsilon$, of the equilibrium state then the relaxation time $\tau_\varepsilon$ will be expressed via a characteristic time $\tau$:

$$\tau_\varepsilon = \mathcal{N}(\varepsilon)\tau , \tag{21}$$

where $\mathcal{N}(\varepsilon)$ is the number that depends only on $\varepsilon$, for K-systems it is

$$\mathcal{N}(\varepsilon) \propto \ln \varepsilon^{-1} . \tag{22}$$

It is important to establish to what class of non-integrable systems belong the stellar systems in general.

In (Hadamard, 1901; Hopf, 1939; Hedlund, 1939; Krylov, 1950; Anosov, 1967; Sinai, 1970) and other studies rather general criteria were obtained answering the question to what class belong the systems with given Hamiltonian. As it was shown in Sect. 2 if a stellar system with variable negative curvature is a K-system, the minimum curvature $k$ determines the relaxation time.

## 4. Sign of the scalar curvature

First, let us discuss the sign-definiteness of the scalar curvature $R$. The Riemann tensor $R_{\alpha\beta\gamma\delta}$ for the metric (2) has the form

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = &\frac{1}{2W}\left[W_{\beta\gamma}g_{\alpha\delta} - W_{\alpha\gamma}g_{\beta\delta} - W_{\beta\delta}g_{\alpha\gamma} + W_{\alpha\delta}g_{\beta\gamma}\right] \\ &- \frac{3}{4W^2}\left[W_\beta W_\gamma g_{\alpha\delta} - W_\alpha W_\gamma g_{\beta\delta} - W_\beta W_\delta g_{\alpha\gamma} + W_\alpha W_\delta g_{\beta\gamma}\right] \\ &+ \frac{1}{4W^2}\left[g_{\beta\gamma}g_{\alpha\delta} - g_{\alpha\gamma}g_{\beta\delta}\right]W_\sigma W^\sigma \end{aligned} \tag{23}$$

and the scalar curvature is

$$R = 3N(3N-1)\left[-\frac{\Delta W}{3NW^2} - \left(\frac{1}{4} - \frac{1}{2N}\right)\frac{(\nabla W)^2}{W^3}\right], \tag{24}$$

where

$$W_\alpha = \frac{\partial W}{\partial q^\alpha}, \quad \Delta W = \frac{\partial^2 W}{\partial q^\alpha \partial q^\alpha}, \quad (\nabla W)^2 = \frac{\partial W}{\partial q^\alpha}\frac{\partial W}{\partial q^\alpha} .$$

Let us calculate the quantities in (24) for the scalar curvature. Taking into account Eq. (4) and the relation

$$\Delta_{\mathbf{r}}\left(\frac{1}{|\mathrm{r} - \mathrm{a}|}\right) = -4\pi\delta(\mathrm{r} - \mathrm{a}) \tag{25}$$

for the term with $\Delta W$ in eq. (24) we obtain for this term the expression

$$\frac{3N(3N-1)}{3NW^2} 4\pi G \sum_{a \neq b}^{N} M_a M_b \delta(\mathrm{r}_a - \mathrm{r}_b) . \tag{26}$$

The expression (26) is evidently equal to zero in all cases except those when any two or more stars undergo direct impact. Since for real stellar systems the time interval between two impacts is very large, one can neglect direct impacts. Therefore, for $R$ we arrive at the expression

$$R = -\frac{3N(3N-1)}{W^3}\left(\frac{1}{4}-\frac{1}{2N}\right)(\nabla W)^2. \tag{27}$$

Note that neglecting the first term in (24) and speaking on the absence of direct impacts, we nevertheless do not ignore collisions in the Chandrasekhar sense, i.e. close encounters between stars, during which the direction of motion changes by a finite angle. Thus in (27) the effect of binary stellar encounters as well as possible collective effects is taken into account.

One can see from (27) that for $N = 2$, $R = 0$ and this reflects the fact that the problem of two bodies is integrable. For $N \geq 3$ ($W > 0$) we come to the important conclusion that $R$ is negative and the system may have exponential instability and is a candidate as a $K$-system. More precise statements will be made by analyzing the sign of the two-dimensional curvature (11) in Sect. 6. But now already we can claim that stellar systems may belong to the class of systems with strongly developed statistical properties.

## 5. Estimation of the collective relaxation time

According to the results of the previous section, stellar systems may possess exponential instability, and the relaxation time is determined by the minimum value of the curvature along all two-dimensional directions (19).

It is helpful, first, to estimate the mean scalar curvature (27). One can readily see that

$$(\nabla W)^2 = \sum_{a=1}^{N}\frac{1}{M_a}\left(\frac{\partial U}{\partial r_a}\right)^2 = \sum_{a=1}^{N} M_a\varepsilon_a^2\,, \tag{28}$$

where $\varepsilon_a$ is the field strength.

To estimate the quantities in $R - W$ and $\Delta W$ we shall proceed from the fact that $W$ is the kinetic energy

$$W = T = \sum_{a=1}^{N}\left(\frac{M_a^2 v_a^2}{2}\right) = N\langle M\rangle\langle v^2\rangle/2\,. \tag{29}$$

We assume that the force follows a Holtsmark distribution (Holtsmark, 1924; Chandrasekhar, 1943; Chandrasekhar and von Neumann, 1943)[1].

Using the Holtsmark distribution $\mathscr{H}(\varepsilon)$ we find the mean square force affecting a single star

$$\langle\varepsilon^2\rangle = \int \varepsilon^2\mathscr{H}(\varepsilon)d\varepsilon = a^{4/3}\int_0^y y^2dy\,\frac{2}{\pi y}\int_0^\infty e^{-(x/y)^{3/2}}x\sin x dx$$

$$= a^{4/3}\int_0^\infty H(y)y^2dy = ca^{4/3}\,, \tag{30}$$

where

$$a = \tfrac{4}{15}(2\pi G)^{3/2}\langle M\rangle^{3/2}n\,, \tag{31}$$

$n$ is the stellar density and $\langle M\rangle$ is the mean star mass. From (28–31) we have

$$(\nabla W)^2 = Nca^{4/3}\langle M\rangle\,. \tag{32}$$

[1] The fact that stellar systems posses exponential instability may justify the Holtsmark-Chandrasekhar-von Neumann probability approach.

Substituting (29), (32) into (27) we get the desired expression for $R$, namely

$$R = -\frac{3N^2(3N-1)}{T^3}\langle M\rangle ca^{4/3}\left(\frac{1}{4}-\frac{1}{2N}\right) \simeq -\frac{9N^3ca^{4/3}\langle M\rangle}{4T^3}. \tag{33}$$

Note that calculating the mean square force using the Holtsmark distribution, we obtain a divergent quantity, because the constant $c$ in (30–33) is formally equal to infinity. This is due to the fact that the Holtsmark distribution predicts too high probabilities for $\varepsilon^2$ at $|\varepsilon| \to \infty$. This fact in turn is connected with the long-range Coulomb character of the interaction.

In order to avoid the divergence of $c$ we introduce a cutoff[2] for the forces of the order of Chandrasekhar (1943)

$$|\varepsilon_{\text{cutoff}}| \sim \frac{GM}{r_{\text{cutoff}}^2}, \tag{34}$$

where

$$r_{\text{cutoff}} = \frac{2G(M_a + M_b)}{\langle v^2\rangle}$$

is the distance at which the escape velocity (from a star) equals the average velocity $\langle v\rangle$. Thus we find a finite value for $c \simeq 1$.

Now using the above-obtained result for $R$ we can estimate roughly the index of the exponent in Eq. (19).

Using (11) we estimate the mean curvature over the two-dimensional directions

$$\bar{k}(q, \delta q, dq/ds) \sim \frac{R}{(3N)^2} \tag{35}$$

and from (3), (15) we have

$$k\cdot\left|\delta q\Lambda\frac{dq}{ds}\right|^2 \sim \frac{R}{(3N)^2}|\delta q|^2; \tag{36}$$

for details see Sect. 6.

Having in mind Eq. (36) and that $ds = \sqrt{2}Wdt$ we rewrite (17) in the form

$$\frac{d^2}{dt^2}|\delta q|^2 \approx \frac{4RW^2}{(3N)^2}|\delta q|^2 \tag{37}$$

and for the relaxation time we have

$$\tau \simeq \frac{1}{2}\frac{3N}{(RW^2)^{1/2}}. \tag{38}$$

Using the previously obtained expression of the mean curvature radius (33) for the relaxation time we finally have (see also, Gurzadyan and Savvidy, 1984)

$$\tau \simeq \left(\frac{T}{a^{4/3}\langle M\rangle N}\right)^{1/2} = \left(\frac{15}{4}\right)^{2/3}\frac{1}{2\pi\sqrt{2}}\frac{\langle v\rangle}{G\langle M\rangle n^{2/3}}. \tag{39}$$

The relaxation time (39) normalized by using characteristic values of the parameters of stellar systems like globular clusters and galaxies is

$$\tau \simeq 10^8\,\text{yr}\left(\frac{\langle v\rangle}{10\frac{\text{km}}{\text{s}}}\right)\left(\frac{n}{1\,\text{pc}^{-3}}\right)^{-2/3}\left(\frac{\langle M\rangle}{M_\odot}\right)^{-1}. \tag{40}$$

[2] A similar divergence takes place in electrodynamics, which in particular arises when calculating the Lamb shift and is overcome, as it is well known, by a cutoff of the contributions at small distances.

For clusters of galaxies this formula yields $10^{10}$–$10^{12}$ yr.

Compare the relaxation time $\tau$ with $\tau_b$ taking into account the binary encounters only:

$$\tau_b = \frac{\sqrt{2}\langle v^3\rangle}{\pi G^2\langle M^2\rangle n \ln(N/2)}. \tag{41}$$

The ratio of these relaxation times is

$$\frac{\tau_b}{\tau} \sim \frac{\langle v^2\rangle/GM}{n^{1/3}} \frac{1}{\ln N} \simeq \frac{d}{r_*}\frac{1}{\ln N}, \tag{42}$$

where $r_* = GM/\langle v^2\rangle$ is the radius of the gravitational influence of the star and $d$ is the mean distance between stars.

The expression (42) reflects the main physical difference between the relaxation times $\tau$ and $\tau_b$ which is the following: as we have already mentioned, the relaxation time $\tau$ is defined by the curvature of the Riemann manifold, the contribution to which is determined by the presence of neighbours at mean distances $d$, whereas the contribution to $\tau_b$ is determined by binary encounters only, characterized by the effective radius $r_*$. Since for the real stellar systems $d \gg r_*$ we have

$$\tau \ll \tau_b. \tag{43}$$

Note that in the mechanism of collective relaxation discussed above we take essentially into account the multiple mutual scattering of all $N$ bodies. With increasing density, $d$ decreases and approaches $r_*$, so that the binary encounters become dominating in the relaxation mechanism.

The times $\tau$ and $\tau_b$ are related to the dynamical time $\tau_{\text{dyn}} = D^{3/2}/(GMN)^{1/2}$ by the relations

$$\tau \simeq \frac{D}{d}\tau_{\text{dyn}}, \qquad \tau_b \simeq \frac{D}{r_*}\tau_{\text{dyn}}, \tag{44}$$

where $D$ is the characteristic size of the system. The relations (44) reflect the fact that *there are three scales of time and length for stellar systems*:

$$\left\{\frac{D}{\tau_{\text{dyn}}};\ \frac{d}{\tau};\ \frac{r_*}{\tau_b}\right\}.$$

In the following chapter the curvature over two-dimensional directions is investigated and the relations (35), (36) are derived. Several physical consequences are discussed.

## 6. The mean radius of curvature along two-dimensional directions

In the previous two sections we have dealt with the scalar curvature $R$, whereas actually in the equation for the geodesic divergence we need the two-dimensional curvature $K$ given by (11). Below, we give the calculation of the two-dimensional curvature.

Let us calculate the right-hand side of (11) using the expression for the Riemann tensor (23). After substitution we obtain

$$\begin{aligned} &R_{\alpha\beta\gamma\delta}\delta q^\alpha \dot q^\beta \delta q^\gamma \dot q^\delta \\ &= \frac{1}{2W}[2|\dot q W''\delta q|\,|\delta q \dot q| - |\delta q W''\delta q|\,|\dot q\dot q| - |\dot q W''\dot q|\,|\delta q\delta q|] \\ &\quad - \frac{3}{4W^2}[2|\dot q W'|\,|W'\delta q|\,|\delta q\dot q| - |\delta q W'|\,|\delta q W'|\,|\dot q\dot q| \\ &\quad - |\dot q W'|\,|W'\dot q|\,|\delta q\delta q|] + \frac{1}{4W^2}[|\dot q\delta q|^2 - |\dot q\dot q|\,|\delta q\delta q|]\cdot|W'W'|, \end{aligned} \tag{45}$$

where the dot over $q(\dot q)$ denotes differentiation with respect to $s$, a dash over $W(W')$ differentiation with respect to $q$, and the vertical bars denote

$$|\dot q W''\delta q| = \frac{dq^\alpha}{ds}\frac{\partial^2 W}{\partial q^\alpha \partial q^\beta}\delta q^\beta; \qquad |\delta q W'| = \delta q^\alpha \frac{\partial W}{\partial q^\alpha};$$

$$|W'W'| = \frac{\partial W}{\partial q^\alpha}\frac{\partial W}{\partial q^\beta} g^{\alpha\beta}.$$

As it was shown in Sect. 2 only the normal component of $\delta q$ (12–14) has a physical interest. Hence, in Eq. (45) $\delta q$ may be treated only as the normal component, and therefore, using (13) we arrive at

$$\begin{aligned} R_{\alpha\beta\gamma\delta}\delta q^\alpha \dot q^\beta \delta q^\alpha \dot q^\delta &= -\frac{1}{2W}[|\delta q W''\delta q|\,|\dot q\dot q| + |\dot q W''\dot q|\,|\delta q\delta q|] \\ &\quad + \frac{3}{4W^2}[|\delta q W'|^2|\dot q\dot q| + |\dot q W'|^2|\delta q\delta q|] \\ &\quad - \frac{1}{4W}[|W'W'|\,|\delta q\delta q|\,|\dot q\dot q|]. \end{aligned} \tag{46}$$

The first two terms of (46) are of the same nature. They contain the second derivatives of the potential and they correspond to the first term of (24) with the Laplacian $\Delta W$. The third, fourth and fifth terms of (46) have the first derivatives of the potential and correspond to the second term of (24), with the gradient $\Delta W$.

We calculate explicitly the expressions

$$\begin{aligned} -|\delta q W''\delta q| &= \sum_{a,b}\sum_{i,k}\delta r_a^i \frac{\partial^2 U}{\partial r_a^i \partial r_b^k}\delta r_b^k \\ &= G\sum_{a<b}\left\{\frac{M_a M_b}{r_{ab}^3}\left[\delta r_{ab}\delta r_{ab} - \frac{3(\delta r_{ab} r_{ab})^2}{r_{ab}^2}\right]\right. \\ &\quad \left. -\frac{4\pi}{3}M_a M_b(\delta r_{ab}\delta r_{ab})\delta^{(3)}(r_{ab})\right\}, \end{aligned} \tag{47}$$

and

$$|\dot q W''\dot q| \to |\delta q W''\delta q|$$

(using the substitution $\dot q \to \delta q$), where the second-derivative matrix is defined by

$$\begin{aligned} \frac{\partial^2 U}{\partial r_a^i \partial r_b^k} &= -GM_a M_b\left\{\frac{1}{r_{ab}^3}\left(\delta^{ik} - \frac{3r_{ab}^i r_{ab}^k}{r_{ab}^2}\right)\right. \\ &\quad \left. + \frac{4\pi}{3}\delta_{ik}\delta^{(3)}(r_{ab})\right\}, \quad a \neq b \end{aligned} \tag{48}$$

and

$$\begin{aligned} \frac{\partial^2 U}{\partial r_a^i \partial r_b^k} &= G\sum_{c\neq a} M_a M_c\left\{\frac{1}{r_{ac}^3}\left(\delta^{ik} - \frac{3r_{ac}^i r_{ac}^k}{r_{ac}^2}\right)\right. \\ &\quad \left. + \frac{4\pi}{3}\delta_{ik}\delta^{(3)}(r_{ac})\right\}; \quad a = b \end{aligned} \tag{49}$$

has been used.

Note that the term without the $\delta^{(3)}(r)$ function corresponds to the quadrupolar moment of the gravitational system whose trace is zero, so that the total trace of this matrix is

$$\Delta U = -\Delta W = 4\pi G\sum_{a\neq b} M_a M_b \delta^{(3)}(r_{ab}). \tag{50}$$

(see (24)–(27)). The last three terms can be obtained by means of the first-derivative vector.

$$\frac{\partial U}{\partial r_a^i} = \sum_{b,b\neq a} U'_{ab}\frac{r^i_{ab}}{r_{ab}} = \sum_{b,b\neq a} G\frac{M_a M_b}{r^3_{ab}} r^i_{ab}\,. \tag{51}$$

Let us introduce the notation

$$\frac{\partial^2 U}{\partial q^\alpha \partial q^\beta} = A_{\alpha\beta}\,; \quad \frac{\partial U}{\partial q^\alpha} = B_\alpha\,. \tag{52}$$

Consider the case when $r_{ab} \neq 0$. Then the singular part of $A_{\alpha\beta}$ turns into zero, while the remaining real part is symmetric and by virtue of (48), (49) has a zero trace. Among its eigenvalues $\lambda_\alpha$ there are both positive and negative terms, the difference between their numbers being the invariant of this matrix by virtue of the law of inertia. In fact, we are interested in the sign-definiteness of the quadratic forms $|\delta q A \delta q|$ and $|\dot{q} A \dot{q}|$ composed by means of $A_{\alpha\beta}$ in the 3N-dimensional spaces $\delta q^\alpha$ and $\dot{q}^\alpha$. This matrix has eigenvalues of different signs. Therefore the forms $|\delta q A \delta q|$ and $|\dot{q} A \dot{q}|$ are sign-indefinite, so that a surface where such a quadratic form is zero is a hypersurface given by an expression of the form

$$\lambda_1 \delta q_1^2 + \dots + \lambda_M \delta q_M^2 - \lambda_{M+1}\delta q_{M+1}^2 - \dots - \lambda_{3N}\delta q_{3N}^2 = 0\,. \tag{53}$$

For $N = 1$ this equations represents a conical surface (Fig. 1). At certain instants of time $r_{ab}$ may become zero. Then singular terms will also contribute to (46). However, as was mentioned before, such events take place extremely rarely.

To study the sign-definiteness of the last three terms of Eq. (46) we write down the scalar products in the $3N$-dimensional spaces $\delta q^\alpha$ and $\dot{q}^\alpha$ in the form

$$\delta q^\alpha B_\alpha = |\delta q \delta q|^{1/2} |BB|^{1/2} \cos\vartheta_{\delta q}\,, \tag{54}$$

$$\dot{q}^\alpha B_\alpha = |\dot{q}\dot{q}|^{1/2} |BB|^{1/2} \cos\vartheta_{q}\,, \tag{55}$$

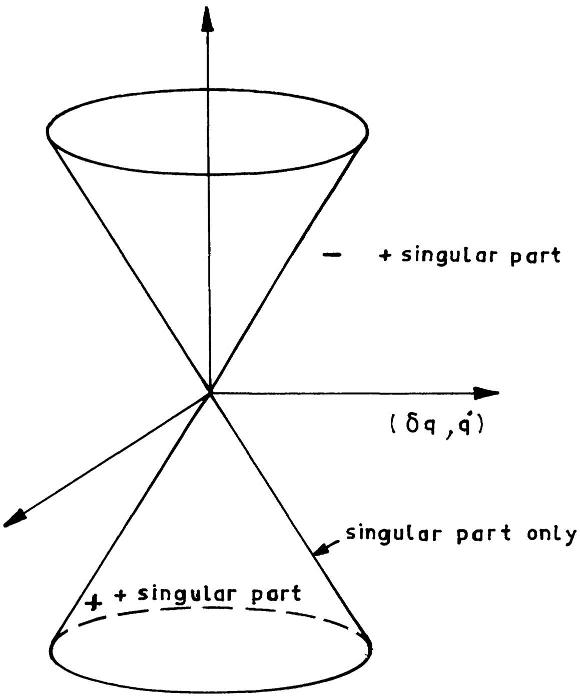


**Fig. 1.** Schematic representation of the regions of positive and negative sign of the first two terms from Eq. (46)

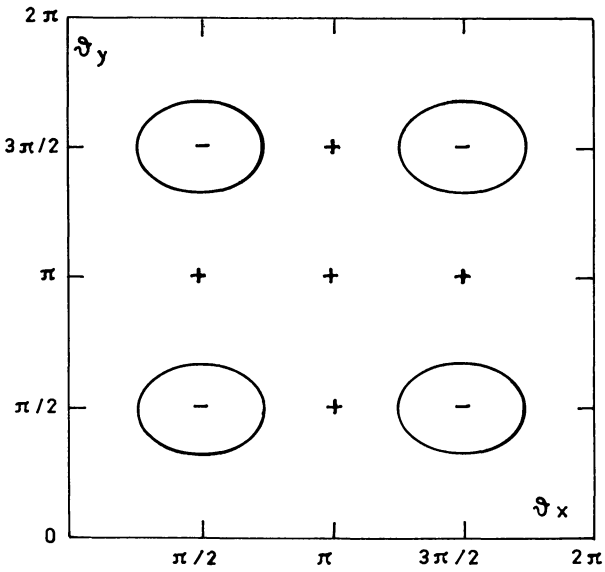


**Fig. 2.** Regions of positive and negative sign of the last three terms from Eq. (46)

where $\vartheta_{\delta q}$ and $\vartheta_{\dot{q}}$ are the angles between the vectors $\delta q^\alpha$, $B_\alpha$ and $\dot{q}^\alpha$, $B_\alpha$ respectively. Then these terms can be rewritten in the form

$$\frac{3}{4W^2}|\delta q \delta q||BB|\left[\cos^2\vartheta_{\delta q} + \cos^2\vartheta_{\dot{q}} - \frac{1}{3}\right]. \tag{56}$$

Figure 2 shows the positive and negative regions of the expression in the square bracket of (56). The null line is determined by the equation

$$\cos^2\vartheta_{\delta q} + \cos^2\vartheta_{\dot{q}} = \tfrac{1}{3} \tag{57}$$

and the maximum and minimum are achieved at the points

$$\vartheta^{\max}_{\delta q} = \vartheta^{\max}_{\dot{q}} = 0, \pi, 2\pi, \dots, \tag{58a}$$

$$\vartheta^{\min}_{\delta q} = \vartheta^{\min}_{\dot{q}} = \frac{\pi}{2}, \frac{3\pi}{2} \dots. \tag{58b}$$

The maximum and minimum equal to

$$+\frac{5}{4} W^2 |BB|\,|\delta q \delta q|\,|\dot{q}\dot{q}|$$

and

$$-\frac{1}{4W^2}|BB|\,|\delta q \delta q|\,|\dot{q}\dot{q}|\,, \tag{59}$$

respectively.

Thus, during the evolution of the system, at different times the form (46) can take both positive and negative values.

However with increasing $N$ for spherically symmetric systems, the time the system spends in the region of negative values starts to prevail strongly. In order to make sure of that let us choose an assembly of systems with spherically symmetric initial velocities and shifts and average (46) over them. We use here the following relations:

$$\overline{\delta q^\alpha \delta q^\beta} = \frac{1}{3N} g^{\alpha\beta}|\delta q \delta q|\,,$$

$$\overline{\dot{q}^\alpha \dot{q}^\beta} = \frac{1}{3N} g^{\alpha\beta}|\dot{q}\dot{q}|\,. \tag{60}$$

As a result of such an averaging, the first two terms in (46) transform into the expression

$$-\frac{\Delta W}{3NW^2}|\dot{q}\dot{q}|\,|\delta q\delta q|\,. \tag{61}$$

The last three terms in (46) after averaging are equal to

$$\left(\frac{1}{2N}-\frac{1}{4}\right)\frac{(\nabla W)^2}{W^3}|\delta q\delta q|\cdot|\dot{q}\dot{q}|\,. \tag{62}$$

One can see that, with increasing $N$, the system spends most of the time near the negative values (58a) equal to $-\frac{1}{4}B^2|\delta q\delta q|$.

After averaging, the expression (48) becomes

$$\begin{aligned} &R_{\alpha\beta\gamma\delta}\delta q^\alpha \dot{q}^\beta \delta q^\gamma \dot{q}^\delta \\ &= +\left\{-\frac{1}{3N}\frac{\Delta W}{W^2}+\left(\frac{1}{2N}-\frac{1}{4}\right)\frac{(\nabla W)^2}{W^3}\right\}\cdot|\delta q\delta q| \\ &= \frac{R}{3N(3N-1)}|\delta q\delta q| = K\cdot|\delta q\delta q| \end{aligned} \tag{63}$$

wherein we have used the relations (15), (24) and (50). Thus, the validity of the relation (35) is shown.

Thus, accurately speaking, the dynamics of the system is not determined by the scalar curvature $R$ only. However, as is seen from (63), the averaged curvature over two dimensional directions $K$ is proportional to the scalar curvature and turns out negative in view of our analysis above. On the other hand the dynamics of non-spherical systems is determined by the two-dimensional curvature $K$ which may be both positive and negative.

## 7. Discussion and conclusion

The statistical properties of stellar systems have been investigated from the point of view of the ergodic theory. The first step was to reduce the $N$-body problem to the investigation of the behaviour of the geodesic flow on a Riemann manifold. An exponential divergence of the geodesics was formed with an exponent determined by the curvature of this manifold. We use the index of this exponent as the relaxation time.

We compare this definition of the relaxation time with that of Chandrasekhar (1942). Our definition depends on a trajectory instability which implies that a change of the initial data by $\delta q(0)$ develops in time at an exponential rate: $\delta q(s) \sim \delta q(0)e^{s/\tau}$. As an example let us consider the binary encounters of 2 pairs of stars with initial scattering angles $\varphi_{\text{in}}$ and $\varphi_{\text{in}}+\Delta\varphi_{\text{in}}$. Then the exponential instability means that the difference between the scattering angles after one encounter is

$$\Delta\varphi_{\text{out}} = k\Delta\varphi_{\text{in}} \tag{64}$$

and after $n$ encounters the angle divergence is

$$\Delta\varphi^{(n)}_{\text{out}} = k^n\Delta\varphi_{\text{in}} = \Delta\varphi_{\text{in}}e^{n\ln k}\,. \tag{65}$$

Therefore the relaxation time is:

$$\tau = 1/\ln k\,. \tag{66}$$

On the other hand Chandrasekhar (1942) considers the difference

$$\Delta\varphi = \varphi_{\text{out}} - \varphi_{\text{in}}\,, \tag{67}$$

and he defines the relaxation time as the time after which

$$\Delta\varphi^{(h)} \sim \pi/2$$

owing to binary encounters, i.e. when the change of the angle of a single star will become of the order of $\pi/2$, so that $|\Delta\varphi|_{\text{Chandrasekhar}}$ increases linearly for $n$ and not exponentially!

In conclusion we mention several additional astrophysical consequences. As is well known, if the velocity distribution of stars is maxwellian, some stars evaporate from the system (Ambartzumian, 1938). As the time is much smaller than the Chandrasekhar relaxation time, the role of the evaporation process in the evolution of the stellar system increases sharply. The results obtained can also serve as a foundation to the hypothesis of the local equilibrium of the stellar systems (Gurzadyan and Kechek, 1979).

The recent progress of computer techniques (Hamann, 1983) greatly increases the possibility of numerical investigation of the gravitational $N$-body problem. Two main difficulties arise here. The first is the necessity of integration of too many differential equations. The second occurs even we partially overcome of the first one: it is the problem of understanding the basic meaning of the numerical information. The formalism described here allows one, using the numerical information, to compute the two-dimensional curvature as an effective criterion of stochasticity. The latter difficulty is also overcome in a number of papers (Hénon and Heiles, 1964; Miller, 1964; Benettin et al. 1976; Contopoulos et al. 1978; Zaslavsky and Chirikov 1971; Pesin, 1977; Contopoulos 1983; and others), where effective numerical criteria are found by means of the Liapunov characteristic numbers. Both our method, and the use of the Liapunov characteristic numbers give estimates of the experimental deviations of the orbits. However, in spite of the similarity of the aims of the two methods as regards the study of the stability of the systems, our scheme, as it is noted in the Introduction, has a clear geometrical interpretation.

The results of the numerical experiments will be published elsewhere.

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