

POST-NEWTONIAN ARRIVAL-TIME ANALYSIS FOR A PULSAR IN A BINARY SYSTEM

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ABSTRACT

We present the post-Newtonian timing model for a binary pulsar that was used by Weisberg and Taylor to make the first observations of post-Newtonian timing effects in PSR 1913+16. Our analysis corrects problems with the earlier model of Epstein, which prevented consistent fittings to binary pulsar timing data.

Detection of post-Newtonian timing effects in PSR 1913+16 consistent with the relativistic two-point-mass model of the system provides powerful, independent evidence that the observed decrease of the pulsar's orbital period is due to the emission of gravitational radiation.

Subject headings: gravitation — pulsars — relativity — stars: binaries

1. INTRODUCTION

The observation of post-Newtonian effects in the timing data for the binary pulsar PSR 1913+16 by Weisberg and Taylor (1984) marks an abrupt improvement in what has been, since its discovery by Hulse and Taylor (1975), a steadily growing knowledge of the dynamics of this binary system. The steady increase in the quality of timing data gathered by observers has reached a point where, suddenly, qualitatively new information about the pulsar system is available. That this information is consistent with the relativistic two-point-mass model of the binary is most significant. This new evidence greatly enhances our confidence that the observed decrease of the orbital period of the pulsar is due to the emission of gravitational radiation from the system.

In this paper we present the post-Newtonian timing model used by Weisberg and Taylor (1984) in making their measurements. The model corrects problems with the earlier model of Epstein (1977 and erratum 1979) to obtain a timing equation for a binary pulsar that is fully consistent through post-Newtonian order.

To put the post-Newtonian model and these new observations in context, recall that the dynamics of the pulsar binary is encoded, by the Doppler shift and other relativistic effects, in the pattern of arrival times of the pulses we receive from the pulsar. Blandford and Teukolsky (1976, hereafter BT) developed what has become the standard method for extracting information from the pulsar timing data. The procedure is to fit the arrival times predicted by a parameterized model of the pulsar system to the actual timing data. The parameters of the model are adjusted until the predictions best fit the data in a least-squares sense. The parameter values are then read off and interpreted.

The model defined by BT treats the pulsar's orbit as an osculating Newtonian ellipse. This provides a completely general kinematic description of the orbit. Specific dynamical models of the pulsar system will predict different time dependence of the osculating orbital elements, for example, of the rate of periastron advance or the rate of change of the orbital period. Knowledge of the dynamics of the pulsar system is gleaned from the comparison of such predictions with the elements inferred from the pulsar timing measurements. In practice, the BT model restricts the osculating orbital elements to be linear functions of the time in order to represent only the dominant secular variations. The model does include timing effects of the transverse Doppler shift and the gravitational redshift. Before Weisberg and Taylor obtained their post-Newtonian results, our knowledge of the dynamics of the binary pulsar came entirely from the interpretation of fittings of the BT model to the pulsar timing data.

The simplest dynamical model of the pulsar system consists of a pair of point masses interacting gravitationally as predicted by general relativity. The dynamics of this relativistic two-body model is entirely consistent with the observed pattern of pulse arrival times, a statement that remains true in the light of the post-Newtonian results, as mentioned above. It is, however, a nontrivial statement even in the context of the BT model. It has been possible to measure eight parameters of that model, while seven completely determine the relativistic two-body model. The relativistic model, therefore, makes a definite prediction for the value of the redundant eighth parameter, the rate of change of the orbital period of the pulsar. The prediction agrees well with the measured value. This is exciting because the basis for the prediction of the relativistic model is a presumed steady increase of the binding energy of the pulsar system due to the emission of energy from the system in the form of gravitational waves (Wagoner 1975). These measurements (Taylor, Fowler, and McCulloch 1979; Weisberg and Taylor 1982) are, though indirect, the first evidence for the existence of gravitational radiation.

Much of the effort to learn more about the pulsar system is motivated by the desire to assess the quality of that evidence. Is the pulsar system "clean" enough to be a reliable testing ground for relativistic gravity? The relativistic two-body model provides a natural explanation for the observed pulse arrival times. However, a variety of effects not included in that model could possibly contribute to the variation of the osculating BT orbital elements, thus invalidating that explanation. A tidally or rotationally distorted stellar companion to the pulsar could, for example, produce such effects (see Smarr and Blandford 1976). The weakness of such alternatives is the conspiracy of effects they require in order to mimic the relativistic model well enough to be consistent with observations. They become less likely possibilities as observations improve and reveal closer agreement with the predictions of the relativistic model. The most recent measurements (Weisberg and Taylor 1984) establish the consistency of the predicted and observed rate of change of the pulsar's orbital period to the 4% level. This is very impressive evidence supporting the relativistic model.

Continuing observation promises more precise consistency checks, but, as the quality of observations improves, another avenue for testing the validity of the relativistic two-body model presents itself. It becomes reasonable to consider searching the pulsar data for tiny timing effects that were neglected in the BT model. The search for post-Newtonian timing effects is a step in this new direction. Since the relativistic model is completely determined by the measured BT parameters, the relativistic model makes a definite prediction for these post-Newtonian effects. Observation of timing effects consistent with these predictions by Weisberg and Taylor (1984) provides powerful, independent evidence for the validity of the relativistic model of the binary pulsar. It should also be possible to search for detailed timing effects that would be produced by possible nonrelativistic perturbations of the pulsar system. Observational limits on the magnitude of such effects promise additional, direct evidence of the “cleanliness” of the binary and on the nature of the pulsar’s companion. In this paper we will deal exclusively with post-Newtonian timing effects.

In practice, it is simpler to develop a new timing model based on an osculating post-Newtonian orbit than to calculate the relativistically induced variation of the osculating Newtonian elements of the BT timing equation. Epstein (1977 and erratum 1979) developed such a post-Newtonian timing model based on a solution of the post-Newtonian two-body problem by Wagoner and Will (1976, hereafter WW). In this model, the relativistic timing effects that would be represented by periodic variation of the BT longitude of periastron and other elements are present explicitly in the timing equation. The timing effects of nonrelativistic perturbations are taken into account by osculating post-Newtonian orbit parameters.

The present work is motivated by difficulties encountered in attempts to fit the Epstein model to timing data from the binary pulsar (J. H. Taylor 1981, private communication). In this paper we report the additions and changes to the Epstein model that have been found necessary in the course of a complete review of the foundations of the post-Newtonian timing model. The amended model has been used successfully by Weisberg and Taylor (1984) to fit data from PSR 1913 + 16. We repeat that the consistency of their observations and the post-Newtonian predictions of the relativistic two-body model of the binary pulsar provides striking new evidence in support of the conclusion that we see the effects of gravitational radiation reaction in this system.

To summarize briefly the remainder of this paper: In § II we provide some necessary background and review results from BT which reveal the most direct route to the timing equation. In §§ IIIa and IIIb we present an integration of the post-Newtonian two-body equations of motion which yields directly a uniformly convergent expression for the orbit, and we relate our result to the orbit derived by WW. In §§ IIIc we derive the post-Newtonian Kepler equation. It differs slightly from Epstein’s because of the way in which we handle the possibility of nonrelativistic perturbations to the orbit. In § IV we derive our timing equations and the differential timing coefficients. The origin of the terms that must be added to Epstein’s timing equation in order to make it consistent through post-Newtonian order is discussed. Finally, in § V we present a brief discussion of our results and possible directions for future developments of the theory and observations of PSR 1913 + 16.

II. BACKGROUND AND SOME PARTIAL RESULTS

Introduce a quasi-Lorentzian coordinate system $(t, \mathbf{x})$ in which to analyze the motion of the pulsar and its companion and the propagation of pulses from the pulsar to an observer. Center the coordinates on the barycenter of the pulsar system, and impose standard post-Newtonian gauge conditions. Recall that the post-Newtonian expansion for a gravitationally bound system like the pulsar binary is an expansion in a single parameter ϵ^2 which is of the order of the dimensionless Newtonian potential $U \approx m/p$ = total system mass/typical separation, and also, by the virial theorem, of the order of v^2 = typical velocity squared. We will use the symbols m_1 , $\mathbf{x}_1$, and $\mathbf{v}_1$ to refer to the mass, position, and velocity of the pulsar. We use m_2 , $\mathbf{x}_2$, and $\mathbf{v}_2$ to represent the corresponding quantities for the pulsar’s companion.

BT carried out the analysis of pulse propagation from the pulsar to an observer on the Earth with sufficient accuracy for a post-Newtonian timing analysis. Their timing model is of lower accuracy only because the motion of the pulsar was treated at Newtonian order. In the next section, we analyze the post-Newtonian orbit of the pulsar. Here we highlight the BT analysis of pulse propagation and state the results we will use in the derivation of our post-Newtonian timing equation.

BT introduce a “proper” time, T_p , which is, up to a constant scale factor, equal to the proper time measured by a clock on the pulsar. This time is the natural one to use when discussing timing effects intrinsic to the pulsar. The time of emission of the N th pulse is given implicitly by

$$N = N_0 + \nu T_p + \frac{1}{2} \dot{\nu} T_p^2 + \frac{1}{6} \ddot{\nu} T_p^3, \quad (1)$$

where N_0 is an arbitrary constant, ν is the rotation frequency of the pulsar, and $\dot{\nu}$ and $\ddot{\nu}$ are the first and second derivatives of the frequency. Higher derivatives are not needed to describe the intrinsic timing of PSR 1913 + 16.

The differential equation

$$\frac{dT_p}{dt_{em}} = 1 + \Phi(\mathbf{x}_1) - \frac{1}{2} v_1^2 + O(\epsilon^4) \quad (2)$$

expresses the relationship between T_p and the coordinate time of the emission of the N th pulse. This relationship reflects the gravitational redshift and transverse Doppler shift of the pulsar’s pulse rate.

Finally, BT analyze the propagation of pulses from the pulsar to a distant observer fixed in our quasi-Lorentzian coordinate system in order to determine the coordinate time of arrival, t , of the N th pulse. The distant observer measures coordinate times, which correspond to the infinite-frequency barycentric arrival times introduced by BT. These times are obtained by correcting for the motion of a real observer about the solar system barycenter, for the gravitational effects of the solar system, and for plasma dispersion of the pulses. The final result is

$$t = t_{em} - \mathbf{n} \cdot \mathbf{x}_1(t_{em}) + 2m_2 \ln \left[\frac{2r_b}{x_1(t_{em}) + \mathbf{n} \cdot \mathbf{x}_1(t_{em})} \right], \quad (3)$$

where $\mathbf{n}$ is a unit vector directed toward the solar system barycenter and r_b is the constant coordinate distance to the solar system barycenter, which will not enter our timing formula. The term $\mathbf{n} \cdot \mathbf{x}_1$ represents the integrated effect of the first-order Doppler shift. Using the Newtonian expression for $\mathbf{x}_1$, the result is of the order of the orbital period of the pulsar times ϵ . We will use the post-Newtonian expression for $\mathbf{x}_1$ and obtain new timing effects of the order of the period times ϵ^3 .

Our final timing equation will have the form

$$N = N(t), \quad (4)$$

expressing the infinite-frequency barycentric arrival time of the N th pulse. It is derived through use of equations (1), (2), and (3) and of the post-Newtonian solution for the location $\mathbf{x}_1$ of the pulsar.

III. THE POST-NEWTONIAN ORBIT

a) Derivation and Integration of the Two-Body Equations of Motion

We begin our derivation with the Lagrangian which yields the post-Newtonian equations governing the motion of a pair of point masses through our $(t, \mathbf{x})$ coordinates (Haugan 1979),

$$L = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2) + \frac{m_1 m_2}{|\mathbf{x}_{12}|} + \left[\frac{1}{8} (m_1 v_1^4 + m_2 v_2^4) - \frac{1}{2} \frac{m_1 m_2^2 + m_1^2 m_2}{|\mathbf{x}_{12}|^2} + \frac{3}{2} \frac{m_1 m_2}{|\mathbf{x}_{12}|} (v_1^2 + v_2^2) - \frac{7}{2} m_1 m_2 \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{x}_{12}|} - \frac{1}{2} m_1 m_2 \frac{(\mathbf{x}_{12} \cdot \mathbf{v}_1) \mathbf{x}_{12} \cdot \mathbf{v}_2}{|\mathbf{x}_{12}|^3} \right], \quad (5)$$

where $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$. The Newtonian, $O(\epsilon^2)$, plus post-Newtonian, $O(\epsilon^4)$, structure of the Lagrangian is apparent.

To reduce the problem of determining the dynamics of the binary to an effective one-body one, change generalized coordinates to those of the post-Newtonian barycenter $\mathbf{X}$, and of the relative coordinate $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. Introducing

$$m_i^* = m_i + \frac{1}{2} m_i v_i^2 - \frac{1}{2} \frac{m_1 m_2}{|\mathbf{x}_{12}|}, \quad (6)$$

where $i = 1$ or 2 , we have

$$\mathbf{X} = (m_1^* \mathbf{x}_1 + m_2^* \mathbf{x}_2) / (m_1^* + m_2^*). \quad (7)$$

The Lagrangian (5), when expressed in terms of $\mathbf{X}$, $\mathbf{V} = \dot{\mathbf{X}}$, $\mathbf{r}$ and $\mathbf{v} = \dot{\mathbf{r}}$, implies the equation of motion

$$\ddot{\mathbf{X}} = 0 \quad (8)$$

for the barycenter. The barycenter remains forever at rest at the origin of our coordinate system, as we supposed, and we consider it no further. The effective one-body Lagrangian determining the equation of relative motion is

$$L = \frac{1}{2} \mu v^2 + \frac{m\mu}{r} + \left[\frac{\mu}{8} \left(1 - 3 \frac{\mu}{m} \right) v^4 - \frac{1}{2} \frac{\mu m^2}{r^2} + \left(\frac{3}{2} + \frac{\mu}{2m} \right) \frac{m\mu}{r} v^2 + \frac{\mu^2}{2} \frac{(\mathbf{r} \cdot \mathbf{v})^2}{r^3} \right], \quad (9)$$

where $r = |\mathbf{r}|$, $m = m_1 + m_2$, and $\mu = m_1 m_2 / m$. Equation (44) of WW is the relative equation of motion.

It is convenient to introduce a generalized relative coordinate $\mathbf{r}'$ such that

$$\mathbf{r} = \mathbf{r}' \left(1 + \alpha \frac{m}{r'} \right), \quad (10)$$

with α a constant. The relation between $\mathbf{v}' = \dot{\mathbf{r}}'$ and $\mathbf{v}$ is

$$\mathbf{v} = \mathbf{v}' + \alpha \frac{m \mathbf{v}'}{r'} - \alpha \frac{m \mathbf{r}' (\mathbf{r}' \cdot \mathbf{v}')}{r'^3}. \quad (11)$$

Using the fact that $m/r' \approx O(\epsilon^2)$ and working to post-Newtonian order, we find that in terms of $\mathbf{r}'$ and $\mathbf{v}'$ (for the remainder of this section we drop the prime) the Lagrangian (9) becomes

$$L = \frac{1}{2} \mu v^2 + \frac{m\mu}{r} + \left[\frac{\mu}{8} \left(1 - 3 \frac{\mu}{m} \right) v^4 - \left(\frac{1}{2} + \alpha \right) \frac{m^2 \mu}{r^2} + \left(\frac{3}{2} + \frac{\mu}{2m} + \alpha \right) \frac{m\mu}{r} v^2 + \left(\frac{\mu}{2m} - \alpha \right) m\mu \frac{(\mathbf{v} \cdot \mathbf{r})^2}{r^3} \right]. \quad (12)$$

The "form" of the Lagrangian is unchanged, but the coefficients now depend on the parameter α . We will be able to control the form of our post-Newtonian orbit equation by selecting a suitable value for α .

Finally, introduce polar coordinates in the usual way. We will use the result that the post-Newtonian orbit lies in a plane to avoid dealing explicitly with a second angular coordinate. We suppose that the orbit lies in the x - y plane and introduce r and ϕ such that

$$\mathbf{r} = r[(\cos \phi) \mathbf{e}_x + (\sin \phi) \mathbf{e}_y]. \quad (13)$$

The Lagrangian (12) becomes

$$L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{m\mu}{r} + \left[\frac{\mu}{8} \left(1 - 3 \frac{\mu}{m} \right) (\dot{r}^4 + 2r^2 \dot{r}^2 \dot{\phi}^2 + r^4 \dot{\phi}^4) \right. \\ \left. + \left(\frac{3}{2} + \frac{\mu}{2m} + \alpha \right) m\mu \frac{\dot{r}^2 + r^2 \dot{\phi}^2}{r} - \left(\frac{1}{2} - \alpha \right) \frac{m^2 \mu}{r^2} + \left(\frac{\mu}{2m} - \alpha \right) m\mu \frac{\dot{r}^2}{r} \right]. \quad (14)$$

There are two obvious first integrals:

$$\tilde{E} = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{m}{r} + \left[\frac{3}{8} \left(1 - 3 \frac{\mu}{m} \right) (\dot{r}^4 + 2r^2 \dot{r}^2 \dot{\phi}^2 + r^4 \dot{\phi}^4) + \left(\frac{3}{2} + \frac{\mu}{2m} + \alpha \right) \frac{m}{r} (\dot{r}^2 + r^2 \dot{\phi}^2) + \left(\frac{1}{2} + \alpha \right) \frac{m^2}{r^2} + \left(\frac{\mu}{2m} - \alpha \right) \frac{m \dot{r}^2}{r} \right] \quad (15)$$

and

$$\tilde{J} = r^2 \dot{\phi} \left\{ 1 + \left[\frac{1}{2} \left(1 - 3 \frac{\mu}{m} \right) (\dot{r}^2 + r^2 \dot{\phi}^2) + \left(3 + \frac{\mu}{m} + 2\alpha \right) \frac{m}{r} \right] \right\}. \quad (16)$$

The latter is the post-Newtonian second law of Kepler, for our choice of coordinates.

The radial equation of motion is

$$\ddot{r} - r \dot{\phi}^2 + \frac{m}{r^2} = \left(4 + \frac{2\mu}{m} + 2\alpha \right) \frac{m^2}{r^3} + \left(3 - \frac{7\mu}{2m} \right) \frac{m \dot{r}^2}{r^2} - \left(1 + 3 \frac{\mu}{m} - \alpha \right) m \dot{\phi}^2. \quad (17)$$

Derive an orbit equation in the usual way: introduce $u = r^{-1}$ into the radial equation and substitute for $\dot{\phi}$ in the radial equation from Kepler's second law. The result is

$$u'' + \left[1 - \left(4 - 6 \frac{\mu}{m} + 2\alpha \right) \frac{m^2}{\tilde{J}^2} \right] u - \frac{m}{\tilde{J}^2} \left[1 + 2 \left(1 - 3 \frac{\mu}{m} \right) \tilde{E} \right] = \left(1 + \frac{3\mu}{2m} + 2\alpha \right) m (u')^2 + \left(1 + 3 \frac{\mu}{m} - \alpha \right) m u^2, \quad (18)$$

where the prime denotes $d/d\phi$.

Were the right-hand side of this equation identically zero, the solution would be a precessing ellipse,

$$u = u_0 = \frac{1 + e \cos(k\phi - \omega_0)}{p}, \quad (19)$$

where the coefficient of u in equation (18) is k^2 and the constant term in equation (18) is $-k^2/p$. Since the right-hand side of the orbit equation (18) is of post-Newtonian order, we seek a perturbative solution to the equation which has the ellipse (19) as its lowest order piece,

$$u = u_0 + u_1 + O(\epsilon^4/p). \quad (20)$$

Notice that the coefficient k which determines the precession of the zeroth-order orbit, $u = u_0$, depends on the parameter α . Selecting a suitable value for α , we are able to account fully for the secular advance of the periastron of the post-Newtonian orbit at zeroth order, thus ensuring that the orbit perturbation u_1 is uniformly $O(\epsilon^2/p)$. In fact, we must choose

$$\alpha = 1 + 3 \frac{\mu}{m}, \quad (21)$$

which annihilates the perturbing term in equation (18) proportional to u^2 . With this choice, the equation determining u_1 becomes

$$u_1'' + \left(1 - \frac{6m^2}{\tilde{J}^2} \right) u_1 = \left(3 + \frac{15\mu}{2m} \right) m (u_0')^2. \quad (22)$$

We find

$$p u_1 = \left(\frac{3}{2} + \frac{15\mu}{4m} \right) \frac{m e^2}{p} + \left(\frac{1}{2} + \frac{5\mu}{4m} \right) \frac{m e^2}{p} \cos 2(k\phi - \omega_0). \quad (23)$$

So the full post-Newtonian solution to our orbit equation is

$$p u = 1 + e \cos(k\phi - \omega_0) + \left(\frac{3}{2} + \frac{15\mu}{4m} \right) \frac{m e^2}{p} + \left(\frac{1}{2} + \frac{5\mu}{4m} \right) \frac{m e^2}{p} \cos 2(k\phi - \omega_0), \quad (24)$$

with

$$k = 1 - 3 \frac{m}{p} \quad (25)$$

and

$$\tilde{J} = (mp)^{1/2} \left\{ 1 + \frac{m}{2p} \left[\left(5 + 3 \frac{\mu}{m} \right) + \left(1 - 3 \frac{\mu}{m} \right) e^2 \right] \right\}, \quad (26)$$

where we have used

$$\tilde{E} = -\frac{m}{2p} (1 - e^2) + O(\epsilon^4). \quad (27)$$

The solution to the orbit equation (24) and Kepler's second law (16) represent complete knowledge of the post-Newtonian orbit. The former determines its shape, while the latter determines the rate at which it will be traversed. For substitution in equations (2) and (3), when we come to derive the timing equation, we require the motion of the pulsar through the original quasi-Lorentzian coordinate system. Retracing the transformations made en route to the orbit equation, we find

$$\mathbf{x}_1(\phi) = \frac{m_2^*}{m_1^* + m_2^*} u^{-1}(\phi) \left[1 + \left(1 + 3 \frac{\mu}{m} \right) mu(\phi) \right] [(\cos \phi) \mathbf{e}_x + (\sin \phi) \mathbf{e}_y]. \quad (28)$$

b) Relation to the Post-Newtonian Orbit of Wagoner and Will

The relationship between our result for the post-Newtonian orbit, equations (16), (24), and (26), and that of WW, their equations (61), (58), and (67), is not immediately clear. We have expressed our result in terms of the radial coordinate r' rather than r and have defined elements e and p via an approximate elliptical orbit that differs at post-Newtonian order from the one used by WW. In this section we demonstrate that the WW orbit is equivalent to an approximation, which treats periastron precession via secular perturbations, to our result. This verification, by independent methods, of the WW expression for the post-Newtonian two-body orbit forms a significant part of our review of the foundations of the theory of the binary pulsar and of our effort to isolate problems in the Epstein timing model. The simplicity of our method for determining the orbit and the ease with which it can be modified to handle additional dynamical effects, e.g., effects caused by a tidally or rotationally induced distortion of the pulsar's companion, suggests it will be a useful tool in the effort to extract more information about the binary pulsar and its companion from timing data.

Equations (10) and (21) define the relationship between the WW radial coordinate r and our r' ,

$$r = r' \left[1 + \left(1 + 3 \frac{\mu}{m} \right) \frac{m}{r'} \right], \quad (29)$$

which implies

$$u_{\text{WW}} = r^{-1} = u \left[1 - \left(1 + 3 \frac{\mu}{m} \right) mu \right], \quad (30)$$

where u is given by equation (24). We find

$$pu_{\text{WW}} = 1 + e \cos(k\phi - \omega_0) - \left(1 + 3 \frac{\mu}{m} \right) \frac{m}{p} + \left(1 + \frac{9\mu}{4m} \right) \frac{me^2}{p} - \left(2 + 6 \frac{\mu}{m} \right) \frac{me}{p} \cos(k\phi - \omega_0) - \left(\frac{\mu}{4m} \right) \frac{me^2}{p} \cos 2(k\phi - \omega_0). \quad (31)$$

We must establish the relationship between our orbital elements p and e and those of WW in order to complete our comparison of results. Equating our expression for $\tilde{J}$, equation (26), and that of WW, their equation (67), we infer that

$$p_{\text{WW}} = p \left[1 - \left(2 - 4 \frac{\mu}{m} \right) \frac{m}{p} \right]. \quad (32)$$

Rewrite equation (31) in terms of p_{WW} and substitute the expansion

$$\cos n(k\phi - \omega_0) \approx \cos n(\phi - \omega_0) - (k-1)\phi \sin n(\phi - \omega_0) + O(\epsilon^4), \quad (33)$$

valid for $(k-1)\phi \ll 1$. Comparing the result with WW's equation (61), conclude that

$$e_{\text{WW}} = e \left[1 - \left(\frac{15}{2} + \frac{\mu}{m} \right) \frac{m}{p} \right]. \quad (34)$$

We have, thus, established the equivalence of our expression for the post-Newtonian orbit and that of WW. The same orbit may be parameterized either in terms of the elements (p, e) or the elements $(p_{\text{WW}}, e_{\text{WW}})$.

In what follows, we will use the WW elements in order to maintain contact with the notation of Epstein and to express the modifications of his timing equation in a form that necessitates a minimum of modification to existing software for the fitting of

timing data. Dropping the WW subscript and summarizing results,

$$pu = 1 + e \cos \eta + \frac{m}{p} \left[\left(-3 + \frac{\mu}{m} \right) + \left(1 + \frac{9\mu}{4m} \right) e^2 + \frac{1}{2} \left(7 - 2 \frac{\mu}{m} \right) e \cos \eta - \left(\frac{\mu}{4m} \right) e^2 \cos 2\eta \right], \quad (35)$$

with

$$\eta = \left(1 - 3 \frac{m}{p} \right) \phi - \omega_0, \quad (36)$$

$$\mathbf{x}_1 = x_1 [(\cos \phi) \mathbf{e}_x + (\sin \phi) \mathbf{e}_y], \quad (37)$$

$$x_1 = \frac{m_2^*}{m_1^* + m_2^*} r = \left[\frac{m_2}{m} + \frac{\mu(m_1 - m_2)}{2m^2} \left(v^2 - \frac{m}{r} \right) \right] r, \quad (38)$$

and

$$r^2 \dot{\phi} = (mp)^{1/2} \left[1 - \left(\frac{me}{p} \right) \left(4 - 2 \frac{\mu}{m} \right) \cos \eta \right], \quad (39)$$

where $r = u^{-1}$. These equations are equivalent to equations (A4), (A5), and (A6) of Epstein (1977, as corrected in erratum 1979). Equivalent, that is, until Epstein replaces our equation (36) with

$$\eta = \phi - \omega_0 - \dot{\omega}(t - T_0), \quad (40)$$

where $\dot{\omega}$ is a constant. By this replacement, Epstein seeks to allow for nonrelativistic contributions to periastron precession. The formula is, however, not sufficiently flexible to fully represent the post-Newtonian precession of the periastron and is therefore the source of errors in Epstein's timing equation.

c) The Eccentric Anomaly and Kepler's Equation

The geometry of an ellipse,

$$\frac{p}{r} = 1 + e \cos \eta, \quad (41)$$

provides an alternative to the true anomaly η , called the eccentric anomaly, E , for specifying location on the ellipse. The derivation of the relations

$$\sin \eta = \frac{(1 - e^2)^{1/2} \sin E}{1 - e \cos E}, \quad \cos \eta = \frac{\cos E - e}{1 - e \cos E}, \quad (42)$$

which imply

$$r = a(1 - e \cos E), \quad (43)$$

where $p = a(1 - e^2)$, can be found in any number of texts (see, e.g., Smart 1977).

We use the geometry of the ellipse which is the lowest order approximation to our post-Newtonian orbit (19) to relate the orbit's true anomaly η to an eccentric anomaly E , which therefore satisfies relations (42). We introduce the eccentric anomaly for the same reason it was introduced classically. Kepler's equation, which yields $E = E(t)$ implicitly, is far more convenient than Kepler's second law for use in determining location on the orbit at any time.

Using relations (42) and (43) to carry out the tedious algebraic process of expressing the pulsar's orbit as a function of E , we find a result in agreement with Epstein's equations (A8) and (A9). We have

$$x_1 = \left\{ \frac{m_2}{m} + \frac{\mu \delta m}{2m^2} \left[\frac{me \cos E}{a(1 - e \cos E)} \right] \right\} r, \quad (44)$$

where $\delta m = m_1 - m_2$, with

$$\begin{aligned} r = & a(1 - e \cos E) \\ & + \frac{m}{(1 - e^2)^2} \left[\left(3 + \frac{23}{4} e^2 - \frac{1}{2} e^4 \right) - \left(1 + \frac{17}{4} e^2 + \frac{3}{4} e^4 \right) \left(\frac{\mu}{m} \right) \right] \\ & - \frac{me}{(1 - e^2)^2} \left[\frac{1}{2} (19 + 3e^2) - (3 + 5e^2) \left(\frac{\mu}{m} \right) \right] \cos E \\ & + \frac{me^2}{(1 - e^2)^2} \left[\left(\frac{13}{4} - \frac{1}{2} e^2 \right) - \left(\frac{3}{4} + \frac{5}{4} e^2 \right) \left(\frac{\mu}{m} \right) \right] \cos 2E. \end{aligned} \quad (45)$$

Epstein expresses the result as follows:

$$x_1 = a_1(1 - e' \cos E - f \cos 2E) + O(ae^4), \quad (46)$$

where

$$a_1 = \frac{m_2 a}{m} \left\{ 1 + \frac{m}{4a(1-e^2)^2} \left[(12 + 23e^2 - 2e^4) - (4 + 17e^2 + 3e^4) \left(\frac{\mu}{m} \right) \right] \right\}, \quad (47)$$

$$e' = e + \Delta e, \quad (48)$$

$$\Delta e = \frac{me}{4a(1-e^2)^2} \left[(26 - 17e^2 + 2e^4) - (8 + 3e^2 - 3e^4) \left(\frac{\mu}{m} \right) \right] - \frac{m_1 \delta m}{2ma}, \quad (49)$$

and

$$f = \frac{e^2 m}{4a(1-e^2)^2} \left[(-13 + 2e^2) + (3 + 5e^2) \left(\frac{\mu}{m} \right) \right]. \quad (50)$$

We now derive the post-Newtonian Kepler equation. Note that we do not attempt, as Epstein did, to allow for the possibility of perturbations of the post-Newtonian orbit at this point. Later, to handle possible perturbations, we will describe the actual orbit of the pulsar in terms of the orbital elements of a sequence of instantaneously tangential post-Newtonian orbits, the osculating post-Newtonian orbit. Motion along the instantaneously tangential orbits is governed by the unperturbed post-Newtonian Kepler equation.

The derivation of the Kepler equation begins from Kepler's second law,

$$r^2 \dot{\phi} = (mp)^{1/2} \left[1 - \left(\frac{me}{p} \right) \left(4 - 2 \frac{\mu}{m} \right) \cos \eta \right], \quad (51)$$

and the relations among ϕ , η , and E , which imply

$$\dot{\phi} = \left(1 + 3 \frac{m}{p} \right) \dot{\eta}, \quad \dot{\eta} = \frac{(1-e^2)^{1/2}}{1-e \cos E} \dot{E}. \quad (52)$$

Using these relations and inserting $r = r(E)$ from equation (45) or equation (46) in Kepler's second law, we obtain an equation of the form

$$\begin{aligned} \left(1 - 3 \frac{m}{p} \right) (mp)^{1/2} &= \left[1 + \frac{me}{p} \left(4 - 2 \frac{\mu}{m} \right) \left(\frac{\cos E - e}{1 - e \cos E} \right) \right] r^2(E) \frac{(1-e^2)^{1/2} \dot{E}}{1 - e \cos E} \\ &= \mathcal{F}(E) \dot{E}. \end{aligned} \quad (53)$$

Simply integrate with respect to time in order to obtain the Kepler equation. The result agrees with Epstein's equations (A10)–(A13) (1977, as amended in the erratum 1979), with the exception that his quantity $\dot{\omega} P_0 / 2\pi$ is replaced by its purely relativistic value, $3m/p$. We have

$$\frac{t}{\mathcal{P}} + \sigma = E - g \sin E - h \sin 2E + O(\epsilon^4), \quad (54)$$

where

$$g = e \left\{ 1 + \frac{m}{2a(1-e^2)^2} \left[(18 - e^2 - 6e^4) - (4 + 11e^2 - 7e^4) \left(\frac{\mu}{m} \right) \right] \right\}, \quad (55)$$

$$h = \frac{e^2 m}{4a(1-e^2)^2} \left[(-13 + 2e^2) + (3 + 5e^2) \left(\frac{\mu}{m} \right) \right], \quad (56)$$

and

$$\mathcal{P} = \left(\frac{P_0}{2\pi} \right) \left\{ 1 + \frac{3m}{p} + \frac{m}{2a(1-e^2)^2} \left[(12 + 15e^2 + 6e^4) - (4 + 13e^2 + 7e^4) \left(\frac{\mu}{m} \right) \right] \right\}, \quad (57)$$

where

$$P_0 = 2\pi(a^3/m)^{1/2}. \quad (58)$$

IV. THE PULSE-TIMING EQUATION AND DIFFERENTIAL TIMING COEFFICIENTS

Armed with our knowledge of the post-Newtonian orbit of the pulsar, we are prepared to return to the equations of § II which determine the relationships among the proper time of emission of a pulse, T_p , the coordinate time of emission, t_{em} , and the infinite-frequency barycentric arrival time of the pulse, t .

Begin by examing equation (2) We must integrate this equation to obtain the relationship between T_p and t_{em} . Because we may neglect terms of $O(\epsilon^4)$, the necessary integration is precisely that performed by BT. They found

$$T_p = t_{\text{em}} - \gamma \sin E(t_{\text{em}}) + O(P_0 \epsilon^4), \quad (59)$$

where

$$\gamma = \frac{P_0 e m_2 (m_1 + 2m_2)}{2\pi a m}. \quad (60)$$

The evaluation of equation (3), which relates t_{em} and t , rests on the result

$$\mathbf{x}_1 \cdot \mathbf{n} = -x_1 \sin i \sin \phi, \quad (61)$$

where ϕ is the angular coordinate in the orbit plane measured from the line of nodes. Inserting our expressions for x_1 and ϕ , equations (38) and (36), in equation (3), we find

$$\begin{aligned} t_{\text{em}} = t - a_1 \sin i (1 - e' \cos E - f \cos 2E) \sin \left[\left(1 + 3 \frac{m}{p} \right) \eta + \omega_0 \right] \\ - 2m_2 \ln \left\{ \frac{1 + e \cos \eta}{1 - \sin i \sin [(1 + 3m/p)\eta + \omega_0]} \right\} \\ + O(a_1 \epsilon^4), \end{aligned} \quad (62)$$

where we have dropped a constant difference between t and t_{em} .

This equation is a modification of Epstein's equation (A14). The difference lies in our use of

$$\phi = \left(1 + 3 \frac{m}{p} \right) \eta + \omega_0, \quad (63)$$

rather than Epstein's

$$\phi = \eta + \omega = \eta + \dot{\omega}(t - T_0) + \omega_0, \quad (64)$$

with $\dot{\omega}$ constant. To show that Epstein's parameterization cannot fully represent our post-Newtonian expression, write our expression (63) as

$$\phi = \eta + 3 \frac{m}{p} \left(\eta - \frac{2\pi}{P_0} t - \sigma \right) + \omega_0 + 3 \left(\frac{m}{p} \right) \left(\frac{2\pi}{P_0} t + \sigma \right), \quad (65)$$

that is,

$$\phi = \eta + 3 \frac{m}{p} (\eta - M) + \omega_0 + \dot{\omega}_{\text{rel}}(t - T_0). \quad (66)$$

We have introduced the mean anomaly of the orbit, M , and constant parameter $\dot{\omega}_{\text{rel}}$. The term $3(m/p)(\eta - M)$, absent from Epstein's parameterization, represents, in the language of the BT timing model, a periodic variation of the osculating Newtonian longitude of periastron. This new term depends on the eccentricity of the orbit and is uniformly post-Newtonian in time.

Casting our equation (62) in a form easily compared with Epstein's equation (A14), we have

$$\begin{aligned} t_{\text{em}} = t - a_1 \sin i (1 - e' \cos E - f \cos 2E) \sin (\eta + \omega_{\text{rel}}) \\ - a_1 \sin i (1 - e \cos E) \cos (\eta + \omega_{\text{rel}}) \left[3 \frac{m}{p} (\eta - M) \right] \\ - 2m_2 \ln \left[\frac{1 + e \cos \eta}{1 - \sin i \sin (\eta + \omega_{\text{rel}})} \right] + O(a_1 \epsilon^4), \end{aligned} \quad (67)$$

where $\omega_{\text{rel}} = \omega_0 + 3(m/p)(2\pi/P_0)t$. The term proportional to $\cos (\eta + \omega_{\text{rel}})$ results from the expansion of $\sin [\eta + 3(m/p)(\eta - M) + \omega_{\text{rel}}]$ and is the source of post-Newtonian timing effects not included in Epstein's timing model. This explains the difficulties encountered in attempts to use that model.

Note that equation (67) does not express $t_{\text{em}} = t_{\text{em}}(t)$ explicitly. The eccentric anomaly E in that equation determines the location of the pulsar in its orbit at the coordinate time of pulse emission, i.e., $E = E(t_{\text{em}})$. We require the explicit relationship $t_{\text{em}} = t_{\text{em}}(t)$ in order to substitute in equation (59) and finally in equation (1) to obtain our final pulse timing equation.

We solve equation (67) iteratively for $t_{\text{em}} = t_{\text{em}}(t)$, since the difference between t and t_{em} is small. Expand

$$E(t_{\text{em}}) = E(t) + \dot{E}(t)(t_{\text{em}} - t) + \frac{1}{2}\ddot{E}(t)(t_{\text{em}} - t)^2 + \dots \quad (68)$$

The function $E(t)$ determines the location of the pulsar in its orbit as function of pulse arrival time. It satisfies a Kepler equation of the same form as that satisfied by $E(t_{\text{em}})$ because t advances as coordinate time.

Taking the result for $t_{\text{em}} = t_{\text{em}}(t)$ and substituting in equation (59), we find

$$\begin{aligned}
 T_p = & t - \alpha(\cos E - e) - (\beta + \gamma) \sin E \\
 & - 3\left(\frac{m}{p}\right)(\eta - M)x[(\cos E - e) \cos \omega - (1 - e^2)^{1/2} \sin E \sin \omega] \\
 & + 2m_2 \ln \{1 - e \cos E - \sin i[(1 - e^2)^{1/2} \sin E \cos \omega + (\cos E - e) \sin \omega]\} \\
 & + \frac{\alpha(\cos E - e) + \beta \sin E}{(1 - e \cos E)} \left[\left(\frac{\beta + \gamma}{\mathcal{P}} + \Delta e \right) \cos E - \frac{\alpha \sin E}{\mathcal{P}} + f \cos 2E \right. \\
 & + \frac{1}{2\mathcal{P}^2(1 - e \cos E)} \left\{ (\alpha \cos E + \beta \sin E)[\alpha(\cos E - e) + \beta \sin E] \right. \\
 & - (\alpha \sin E - \beta \cos E) \left\{ 2(\alpha \sin E - \beta \cos E) \right. \\
 & \left. \left. + \frac{e \sin E}{1 - e \cos E} [\alpha(\cos E - e) + \beta \sin E] \right\} \right\} \left. \right], \quad (69)
 \end{aligned}$$

where we have dropped the subscript on ω_{rel} and use $x = a_1 \sin i$, $\alpha = x \sin \omega$ and $\beta = (1 - e^2)^{1/2} x \cos \omega$. In this equation, E is the function $E(t)$. Substitution of expression (69) in equation (1) and use of the Kepler equation, which yields $E = E(t)$ implicitly, provides the final timing-model equation,

$$N = N(t; N_0, v, \dot{v}, \ddot{v}, x, \omega, e, \mathcal{P}, \sigma, \gamma, \sin i). \quad (70)$$

Equation (69) is identical with Epstein's equation (A22), except for the addition of the new term proportional to $(\eta - M)$. Note that we may express this factor in terms of the eccentric anomaly E through use of equations (42) and of the Kepler equation, in the form

$$M = E - e \sin E + O(\epsilon^2). \quad (71)$$

The timing equation (70) supposes that the pulsar follows precisely the post-Newtonian orbit we have analyzed. In general, it will not. A variety of effects not included in our relativistic two-body model of the pulsar system could be present and produce deviations from that orbit. If, however, we describe the actual orbit of the pulsar in terms of the elements of a sequence of instantaneously tangential post-Newtonian orbits, i.e., in terms of an osculating post-Newtonian orbit, it is a simple thing to set up a general timing equation.

Equation (69) and the Kepler equation (54) are expressed in a form where simply allowing the orbital elements to vary in time correctly implements the model based on an osculating orbit. If we restrict attention to a secular variation of the parameters that is linear in time, a discussion paralleling that of BT applies. Note that when we write

$$\omega = \omega_0 + \dot{\omega}(t - T_0), \quad (72)$$

$\dot{\omega}$ includes a relativistic contribution $3(m/p)(2\pi/P_0)$ plus any nonrelativistic contribution to periastron advance. Although the consideration of secular effects alone is no longer particularly natural, since we have included corresponding nonsecular effects of relativistic origin, the timing model including only secular variation of orbital elements represents the system currently in use for the reduction of timing data. We are considering the extension of the timing model to include periodic variation of the osculating elements caused by several possible nonrelativistic effects.

The iterative procedure for fitting a timing model to arrival-time data for the binary pulsar is carefully described both in BT and in Epstein (1977). To complete this section, we simply state our results for the differential timing coefficients which relate the computed timing residuals to the corrections to the parameters of the timing model that one makes at each step of the fitting procedure.

$$\begin{aligned}
 R(t) = & -\frac{\delta N_0}{v} - t \frac{\delta v}{v} - t^2 \frac{\delta \dot{v}}{2v} - t^3 \frac{\delta \ddot{v}}{6v} + V \left[\left(\frac{t}{\mathcal{P}} \right) \frac{\delta \mathcal{P}}{\mathcal{P}} + \frac{1}{2} \left(\frac{t}{\mathcal{P}} \right)^2 \delta \dot{\mathcal{P}} - \delta \sigma \right] \\
 & + \frac{1}{x} [\alpha(\cos E - e) + \beta \sin E](\delta x + \delta \dot{x}) + (1 - e^2)^{1/2} \left[\frac{\beta(\cos E - e)}{1 - e^2} - \alpha \sin E \right] (\delta \omega_0 + t \delta \dot{\omega}) \\
 & - \left[\alpha + \left(V + \frac{e\beta}{1 - e^2} \right) \sin E \right] (\delta e + t \delta \dot{e}) + \sin E \delta \gamma - \left\{ 2 \left(\frac{\partial m_2}{\partial \sin i} \right)_\gamma \ln \left[(1 - e \cos E) \left(1 - \sin i \frac{W}{x} \right) \right] \right. \\
 & - \frac{2m_2 W}{x - W \sin i} + V \left[\left(\frac{\partial g}{\partial \sin i} \right)_\gamma \sin E + \left(\frac{\partial h}{\partial \sin i} \right)_\gamma \sin 2E \right] + W \left[\left(\frac{\partial \Delta e}{\partial \sin i} \right)_\gamma \cos E + \left(\frac{\partial f}{\partial \sin i} \right)_\gamma \cos 2E \right] \\
 & \left. - \frac{3}{(1 - e^2)^2} \left[m_2 + \left(\frac{\partial m_2}{\partial \sin i} \right)_\gamma \sin i \right] [(\cos E - e) \cos \omega - (1 - e^2)^{1/2} \sin E \sin \omega](\eta - M) \right\} \delta \sin i, \quad (73)
 \end{aligned}$$

where

$$V = \frac{\alpha \sin E - \beta \cos E}{1 - e \cos E}, \quad (74)$$

$$W = \frac{\alpha(\cos E - e) + \beta \sin E}{1 - e \cos E}. \quad (75)$$

To evaluate the partial derivatives in equation (73), we use equations (49), (50), (55), and (56) along with Epstein's identities

$$m_2 = \frac{[(\gamma \sin^2 i)/e - x^2/\mathcal{P}]^2}{x \sin^3 i} + O(m_2 \epsilon^2), \quad (76)$$

$$\frac{m}{a} = \frac{m_2 \sin i}{x} + O(\epsilon^4), \quad (77)$$

$$\frac{\mu}{a} = \frac{1}{\mathcal{P}} \left(\frac{\gamma}{e} - \frac{2x^2}{\mathcal{P} \sin^2 i} \right) + O(\epsilon^4), \quad (78)$$

and

$$\frac{m_1 \delta m}{ma} = \frac{[(\gamma \sin^2 i)/e - 3x^2/\mathcal{P}][(\gamma \sin^2 i)/e - 2x^2/\mathcal{P}]}{x^2 \sin^2 i} + O(\epsilon^4). \quad (79)$$

Expression (73) is in agreement with Epstein's equation (A26), except for the new term proportional to $(\eta - M)$ in the differential timing coefficient for $\delta \sin i$. Remember that because fitting to timing data is an iterative procedure, it is only necessary to keep these leading terms in the expressions for the differential timing coefficients.

V. SUMMARY AND DISCUSSION

This paper reports the results of a complete review of the foundations of the post-Newtonian timing model for a binary pulsar. We have corrected the handling of the periastron advance of the pulsar's orbit and of the possibility of nonrelativistic orbit perturbations in the earlier model of Epstein (1977 and erratum 1979) to obtain a timing model for a binary pulsar consistent through post-Newtonian order. This new model was used by Weisberg and Taylor (1984) in the first successful attempt to observe post-Newtonian timing effects in PSR 1913+16. This observation has significance for both our immediate and our future understanding of the dynamics and nature of this binary system. It provides new evidence supporting the validity of the relativistic two-point-mass model of the system. It also, by demonstrating that tiny timing effects neglected in the earlier timing model of Blandford and Teukolsky (1976) can be detectable, provides motivation to search for detailed timing effects that further test the relativistic model and reveal more about the nature of the pulsar's companion.

The general relativistic two-body model of the binary pulsar provides a natural explanation for all features of the observed pattern of pulse arrival times, including those attributed to the decrease in the orbital period of the pulsar, which, in this model, is due to loss of energy from the system in the form of gravitational waves. To appreciate the significance of the post-Newtonian observation and future observations that will be made to test this model of the pulsar system, one must realize that the accurate determination of the parameters of the relativistic model depends upon the measured value of the rate of advance of the periastron of the pulsar's orbit. The inference of gravitational radiation effects rests on this determination of the model and on the measured value of the rate of decrease of the orbital period of the pulsar. The problem is that both of these secular effects are susceptible to influence by possible nonrelativistic perturbations of the pulsar system. If we are to be confident of the correctness of the important conclusions that follow from the relativistic model of the binary pulsar, we need additional, independent evidence, less susceptible to possible perturbations, of the consistency of this model and evidence that directly limits the possible role of such perturbations.

The post-Newtonian observations provide such evidence for the relativistic model. The complicated dependence of these timing effects on the pulsar's orbit phase is not easily mimicked by alternative models of the pulsar system. They provide an independent test because, in principle, the measured post-Newtonian timing effects, i.e., the measured value of $\sin i$, can replace use of the rate of periastron advance in determining the parameters of the relativistic model. In practice, this does not determine the masses of the pulsar and its companion very accurately because it is difficult to measure the tiny post-Newtonian effects accurately. However, it is highly significant that the masses determined in this way are consistent with those derived using the measured periastron advance. This is essentially the point made in the discussion of Weisberg and Taylor (1984) and which can be gleaned immediately from their figure, which they have kindly allowed me to reproduce, (see Fig. 1). The measurements of $\dot{\omega}$, γ , and $\sin i$ are all consistent with a relativistic two-body model of the pulsar system in which the pulsar and companion are each roughly $1.4 M_\odot$ objects.

It is also interesting to note that, as one can see from Figure 1, the measurements of the relativistic parameters γ and $\sin i$ alone become inconsistent with very low mass alternative models of the pulsar system. Smarr and Blandford (1976) cataloged a variety of possible perturbations of the pulsar's orbit and alternative models that incorporate them. Included in their survey was a class of low-mass models in which the pulsar's companion is a helium star. In such models, the tidal distortion of the companion produces a substantial quadrupole component of the companion's Newtonian gravitational field, which in turn causes the observed advance of the periastron of the pulsar's orbit. For a pulsar mass of the order of $0.25 M_\odot$ and a companion mass of the order of $0.85 M_\odot$, such a model is consistent with the measured values of γ and $\dot{\omega}$. Without addressing the question of an explanation for the observed

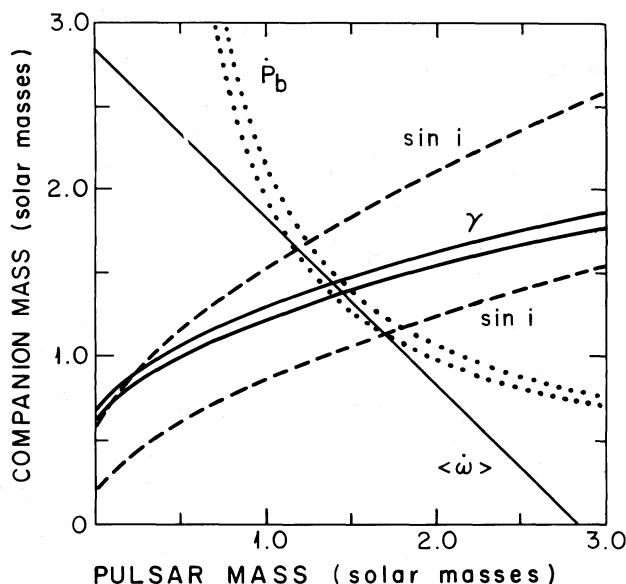


FIG. 1.—Families of curves showing how the stellar masses in the PSR system are constrained by the measured values (and estimated errors) of the parameters $\langle\dot{\omega}\rangle$, γ , and $\sin i$. (The uncertainty in $\langle\dot{\omega}\rangle$ is less than the width of the sloping straight line.) The fourth pair of curves brackets those mass values that would, according to general relativity, cause the system to emit enough gravitational radiation to explain the observed orbital period derivative $\dot{P}_b$. (In this paper, we use the symbols $\dot{\omega}$ and $\dot{P}$ for the $\langle\dot{\omega}\rangle$ and $\dot{P}_b$ of Weisberg and Taylor.) Reprinted by permission of the American Physical Society, from J. M. Weisberg and J. H. Taylor, *Phys. Rev. Letters*, **52**, 1348 (1984).

decrease of the orbital period of the pulsar in this model, it is satisfying to see the first measurement of $\sin i$ on the verge of ruling out such low-mass models.

Further study of the post-Newtonian timing effects promises additional tests of the relativistic model of the binary pulsar. In our timing model all post-Newtonian timing effects are combined, and their contribution to the timing equation is parameterized by a single parameter, $\sin i$. Weisberg and Taylor (1984) argue that the Shapiro (1964) propagation delay across the pulsar system dominates these effects and that their measurement is a measurement of this delay. It is true, however, that it was not possible to obtain consistent fits to the binary pulsar timing data without the post-Newtonian terms we added to the Epstein model. It seems reasonable, then, to argue that we have seen some evidence of the other contributions to the post-Newtonian “signal” from the pulsar, relativistic effects never observed before. For the future, it is desirable to examine these additional timing effects in detail; to introduce additional parameters into the timing model to parameterize these different contributions to the post-Newtonian “signal.” Measurement of the amplitudes and relativistic orbit phases of these contributions, if they agree with the definite predictions of the relativistic model, could provide strong new evidence of this model’s consistency.

The relativistic model of the binary pulsar is not alone in predicting tiny, periodic timing effects neglected in the original Blandford and Teukolsky (1976) timing model. The successful observation of the relativistic effects suggests that it is reasonable to search for timing effects predicted by alternatives to the relativistic model. As for the relativistic effects, the detailed dependence of such timing effects on orbit phase should be quite characteristic. Observational limits on the presence of specific timing effects might easily yield sharp limits to the possibility of specific nonrelativistic perturbations of the pulsar system. For example, one might directly limit the degree of possible tidally or rotationally induced distortion of the pulsar’s companion and, thus, sharpen our knowledge of the nature of this star. Derivation of the periodic timing effects to be expected in alternative models of the pulsar system goes beyond the scope of the work by Smarr and Blandford (1976) on such models. They worked in the context of the Blandford and Teukolsky model, considering only secular effects. The methods used in this paper to analyze the pulsar orbit equation should be useful tools for the prediction of periodic timing effects in alternative models of the binary pulsar. The derivation of the orbit equation is easily extended to include general perturbing terms, secular perturbations are safely handled, and the method of integration is quite general.

To conclude, timing observations strongly establish the consistency of the relativistic two-body model of the binary pulsar. One should feel quite confident that the observed decrease of the orbital period of the pulsar is due to the emission of gravitational radiation. Continued improvement in the quality of observations and searches for new timing effects promise precise new tests of the validity of these conclusions. As study of the binary pulsar continues, it is intriguing to wonder at what level of accuracy we might observe some deviation from the behavior expected of a general relativistic two-body system.

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REFERENCES

- Blandford, R. D., and Teukolsky, S. A. 1976, *Ap. J.*, **205**, 580 (BT).
Epstein, R. 1977, *Ap. J.*, **219**, 92; erratum 1979, **231**, 644.
Haugan, M. P. 1979, *Ann. Phys. (NY)*, **118**, 156.
Hulse, R. A., and Taylor, J. H. 1975, *Ap. J. (Letters)*, **195**, L51.
Shapiro, I. I. 1964, *Phys. Rev. Letters*, **13**, 789.
Smarr, L. L., and Blandford, R. D. 1976, *Ap. J.*, **207**, 574.
Smart, W. M. 1977, *Textbook on Spherical Astronomy* (6th ed., rev. R. M. Green; Cambridge: Cambridge University Press).
Taylor, J. H., Fowler, L. A., and McCulloch, P. M. 1979, *Nature*, **277**, 437.
Wagoner, R. V. 1975, *Ap. J. (Letters)*, **196**, L63.
Wagoner, R. V., and Will, C. M. 1976, *Ap. J.*, **210**, 764 (WW).
Weisberg, J. M., and Taylor J. H. 1982, *Ap. J.*, **253**, 908.
———. 1984, *Phys. Rev. Letters*, **52**, 1348.

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