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GRAVITATIONAL LENSING EFFECTS OF VACUUM STRINGS: EXACT SOLUTIONS

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ABSTRACT

Exact interior and exterior solutions to Einstein's field equations are obtained for vacuum strings. For a uniform density vacuum string the interior solution (in cross section) is a spherical cap while the exterior solution is conical. If the mass per unit length in the string μ is expressed in units of Planck masses per Planck length, then for $0 < \mu < \frac{1}{4}$ the exterior metric $ds^2 = -dt^2 + dr^2 + (1 - 4\mu)^2 r^2 d\phi^2 + dz^2$. A maximum mass per unit length for a string is found: $\mu_{\max} = 6.73 \times 10^{27}$ g cm^{-1}. Grand unified vacuum strings with $\mu \sim 10^{24}$ g cm^{-1}, consistent with the observed isotropy of the microwave background and large enough to promote galaxy formation, would produce equal brightness double images of QSOs with separations of up to 6′. Formulae for lensing probabilities, image splittings, time delays, etc., are derived for strings in a reasonable cosmological setting. String searches using ST, the VLA, and the *COBE* satellite are discussed.

Subject headings: gravitation — quasars — relativity

I. INTRODUCTION

Zeldovich (1980) proposed that vacuum strings produced in the early universe could provide the fluctuations necessary to produce galaxies. Kibble, Lazarides, and Shafi (1982), have shown that in the symmetry breaking of SO(10) via SU(5) stable strings can appear which survive subsequent transitions. In a vacuum string the only component of pressure $P_z = -\rho$ is along the direction of the string. The string is characterized by a mass per unit length μ and a tension $t = \mu$ produced by the negative pressure. (We use units where $G = c = \hbar = 1$.) In general relativity mass has units of length, so mass per unit length is a dimensionless quantity; $\mu = 1$ corresponds to one Planck mass per Planck length $= 1.35 \times 10^{28}$ g cm^{-1}. For grand unified strings we expect $\mu \sim \alpha^{-1} m^2$, where $\alpha \sim 10^{-2}$ is the coupling constant and m is the typical boson mass in units of the Planck mass; the diameter of the string is of the order m^{-1} (cf. Vilenkin 1981*a*). For $m \sim 10^{16}$ GeV, $\mu \sim 10^{-4}$. With $P_z = -\rho$ strings just satisfy the weak energy condition (cf. Hawking and Ellis 1973). Strings stretch during the cosmological expansion preserving μ since the PdV work done by the expansion against the negative $P_z = -\rho$ pressure is exactly that required to produce the new mass in the additional length of string. On scales larger than the horizon, strings are conformally stretched (cf. Turok 1983). Within the horizon, if the string is curved the tension will bring about relativistic motion straightening out any kinks. Closed loops will contract relativistically. A perfectly circular loop would contract to form a black hole, but a general loop may cross itself in a figure eight pattern. If the loop elements interchange upon crossing as envisaged by Zeldovich (1980) then two smaller loops will be formed with the process leading to a cascade of loops and formation of black holes. Thus in the Zeldovich scenario loops coming within the horizon are rapidly destroyed and we should expect to find within the horizon today of order one string, reasonably straight. (For convenience we shall adopt $\Omega_0 = 8\pi\rho_0/3H_0^2 = 1$ and $H_0 = 50$ km s^{-1} Mpc^{-1} in what follows.) The mass currently within the horizon is $M_H = \rho_0(4\pi/3)R_H^3 = (3H_0^2/8\pi) \times (4\pi/3)(2H_0^{-1})^3 = 4H_0^{-1}$. A straight string within the horizon can have a maximum length of $2R_H = 4H_0^{-1}$ and mass $M_S = 4\mu H_0^{-1}$. Thus in the Zeldovich scenario $M_S/M_H \sim \mu$. Such a relativistically moving string can produce observable fluctuations in the microwave background of order $(\delta T/T) \sim \mu$ so the observed isotropy of the microwave background to one part in 10^4 sets the limit $\mu \leq 10^{-4}$. Random fluctuations in the mass within the horizon of order $\mu = 10^{-4}$ can be sufficient form galaxies (cf. Zeldovich 1972; Gott and Rees 1975; Gott 1977). However the dynamics of loops is such that in general they are not expected to cross themselves in a figure eight pattern (cf. Kibble and Turok 1982) so that as Vilenkin (1981*b*) and Turok (1983) have argued the loops within the horizon will continue to oscillate until they radiate away their energy by gravitational radiation. Hogan and Rees (1984) have discussed the possibility of detecting such gravitational radiation using the millisecond pulsar. In this long-lived loop scenario the total mass of loops within the horizon is of the order $M_L/M_H \sim -\mu \ln \mu$. Curvature fluctuations on the scale of the horizon are dominated by the largest loop size $l \sim R_H$. The smallest surviving loops are of size $l \sim \mu H_0^{-1}$ and mass $M \sim \mu^2 H_0^{-1}$. A final possibility is that strings can pass through each other without inter-commuting. In this case (cf. Vilenkin 1984) the universe eventually becomes string dominated: $M_S/M_H \sim 1$, with μ^{-1} approximately straight strings within the horizon. In this scenario Vilenkin finds exceedingly small values of $\mu \lesssim 10^{-20}$ are plausible.

Vilenkin (1981*a*) has calculated the exterior gravitational field of a vacuum string in the weak field limit and finds that it corresponds to a conical space. He notes that this can produce double images of objects behind the string and that this may be relevant to the double quasar. In this paper (§ II) we shall present exact solutions to Einstein's field equations for both the interior and exterior geometry. We shall show how the solutions look in both the limit of large and small μ. In § III we will examine in more detail the gravitational lensing properties of these solutions in a realistic cosmological setting.

II. EXACT SOLUTIONS

These are solutions of Einstein's field equations $R_\nu{}^\mu - \frac{1}{2}\delta_\nu{}^\mu R = 8\pi T_\nu{}^\mu$ with the appropriate values of $T_\nu{}^\mu$ for a string in the interior solution and $T_\nu{}^\mu = 0$ for the exterior solution. The interior solution is:

$$ds^2 = -dt^2 + r_0^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2 \tag{1}$$

with $-\infty < t < \infty$, $0 \le \theta \le \theta_M$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$, and $r_0 = \text{const}$. For this metric the only nonzero connection coefficients are $\Gamma^\theta{}_{\phi\phi} = \sin\theta\cos\theta$, and $\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$; the only nonzero components of the Ricci tensor are $R_\theta{}^\theta = R_\phi{}^\phi = r_0{}^{-2}$. Thus $R = 2r_0{}^{-2}$, and the only nonzero components of the energy momentum tensor are $T_t{}^t = T_z{}^z = -(1/8\pi r_0{}^2)$. Since $T_t{}^t = -\rho$ and $T_z{}^z = P_z$ this corresponds to $\rho = (1/8\pi r_0{}^2) = \text{const}$ and $P_z = -\rho$ as desired. The geometry of a section $t = \text{const}$, $z = \text{const}$ of this solution is that of a spherical cap with a radius of curvature of r_0. The circumference of the string (measured around its waist) is $C = 2\pi r_b = 2\pi r_0 \sin\theta_M$.

The exterior solution is

$$ds^2 = -dt^2 + B_0\,dr^2 + r^2 d\phi^2 + dz^2\,, \tag{2}$$

where $B_0{}^{-1/2} = \cos\,\theta_M$ and $-\infty < t < \infty$, $-\infty < z < \infty$, $0 \le \phi < 2\pi$ (the coordinates t, z, and ϕ correspond exactly to those in the interior metric) and $r_0 \sin\,\theta_M = r_b < r < \infty$ if $\theta_M < \pi/2$. We can adopt new coordinates $r' = (B_0)^{1/2} r$ and $\phi' = (B_0)^{-1/2}\phi$. In terms of these new conditions the exterior metric is:

$$ds^2 = -dt^2 + dr'^2 + r'^2 d\phi'^2 + dz^2 \tag{3}$$

with $0 \le \phi' \le (B_0)^{-1/2} 2\pi$. Equation (3) is just the metric of Minkowski space in cylindrical coordinates. Thus $R^\alpha{}_{\beta\gamma\delta} = 0$ for the exterior solution (it is locally flat) so $R_\mu{}^\nu = 0$, $R = 0$, and $T_\mu{}^\nu = 0$.

If we set $z = \text{const}$, $t = \text{const}$ we may visualize the two surfaces defined by the interior $[ds^2 = r_0{}^2(d\theta^2 + \sin^2\,\theta d\phi^2)]$ and exterior $(ds^2 = B_0\,dr^2 + r^2 d\phi^2)$ metrics by embedding them in a Euclidean 3-space with $ds^2 = dw^2 + dr^2 + r^2 d\phi^2$. We may embed the exterior solution as the surface $w(r) = (B_0 - 1)^{1/2} r$. This is the surface of a cone which if extended would have its vertex at $r = 0$. The interior solution is a spherical cap of radius r_0 with $0 \le \theta < \theta_M$. The entire embedding diagram for $\theta_M < \pi/2$ is shown in Figure 1. The appropriate matching conditions at the boundary (cone tangent to sphere) are $r_b = r_0 \sin\,\theta_M$, $B_0{}^{-1/2} = \cos\,\theta_M$. The metric (3) corresponds to cutting the cone and laying it flat. Now $0 \le \phi' < B_0{}^{-1/2} 2\pi$ so the cone has an angle deficit $D = 2\pi(1 - B_0{}^{-1/2}) = 2\pi(1 - \cos\,\theta_M)$. The mass per unit length in the string is

$$\mu = \rho r_0{}^2 \int_0^{2\pi} d\phi \int_0^{\theta_M} \sin\,\theta d\theta = \tfrac{1}{4}(1 - \cos\,\theta_M)\,, \tag{4}$$

using the value of $\rho = (1/8\pi r_0{}^2) = \text{const}$ from the interior solution. Note that the value of μ is independent of r_0 and depends only on θ_M. The angle deficit in the cone is

$$D = 8\pi\mu \text{ for } \mu < \tfrac{1}{4}\,. \tag{5}$$

We will now show that the O'Brien-Synge-Lichnerowicz jump conditions are satisfied at the boundary between the interior and exterior solutions. With these we can guarantee that the boundary does not contain a surface layer but is just a boundary surface. The criterion is that the extrinsic curvature of the boundary should be the same whether measured in the interior or exterior solution. Geometrically it is clear that the cone tangent to sphere condition adopted above does exactly this, and we can prove it by adopting "natural" cylindrical coordinates with $g_{rr} = 1$ for both the interior and exterior metrics as described by Israel (1966). Then the desired jump conditions are that $\partial g_{\mu\nu}/\partial r|^- = \partial g_{\mu\nu}/\partial r|^+$, i.e., that these derivatives of the interior and exterior metrics match as the boundary is approached from each side. To rewrite metric (1) in "natural" coordinates adopt a new coordinate $r = r_0\,\theta$:

$$ds^2 = -dt^2 + dr^2 + r_0{}^2 \sin^2\,(r/r_0) d\phi^2 + dz^2 \tag{6}$$

at the boundary $r = r_0\,\theta_M$, and $\partial g_{\phi\phi}/\partial r = 2r_0 \sin\,\theta_M \cos\,\theta_M$, all other $\partial g_{\mu\nu}/\partial r$ terms are zero. Now from equation (4) we find $(1 - 4\mu) = \cos\,\theta_M = B_0{}^{-1/2}$ so for the exterior solution (2) we adopt a new coordinate $r'' = (1 - 4\mu)^{-1} r$. Substituting this and for convenience dropping the double primes makes equation (2) become:

$$ds^2 = -dt^2 + dr^2 + (1 - 4\mu)^2 r^2 d\phi^2 + dz^2\,. \tag{7}$$

Note that this exact solution applies for $0 \le \mu < \frac{1}{4}$ and represents a conical space with angle deficit $D = 8\pi\mu$. [Setting $\phi' = (1 - 4\mu)\phi$ brings us directly to the Minkowskian form of eq. (3); since $0 \le \phi < 2\pi$, $0 \le \phi' < 2\pi(1 - 4\mu)$, giving $D = 8\pi\mu$ as expected.] Now the circumference at the boundary must be $2\pi r_0 \sin\,\theta_M$ so as to agree with the value from the interior solution. So in the exterior coordinate system of equation (7), the value of r at the boundary is $r = r_0(1 - 4\mu)^{-1} \sin\,\theta_M$. Now at the boundary all components $\partial g_{\mu\nu}/\partial r$ are zero except $\partial g_{\phi\phi}/\partial r = 2(1 - 4\mu)^2 r = 2r_0(1 - 4\mu) \sin\,\theta_M = 2r_0 \cos\,\theta_M \sin\,\theta_M$. So the values of $\partial g_{\phi\phi}/\partial r$ agree on both sides of the boundary, and the jump conditions (for a boundary surface with no surface layer) are satisfied. As this example should illustrate the jump conditions will be satisfied in the $\mu \ge \frac{1}{4}$ cases we discuss below as well.

In the limit $\mu \ll 1$ equation (7) reduces to the metric

$$ds^2 = -dt^2 + dr^2 + (1 - 8\mu) r^2 d\phi^2 + dz^2 \tag{8}$$

found by Vilenkin (1981*a*) in a weak field limit.

For $0 < \mu < \frac{1}{4}$, $\theta_M < \pi/2$ and the spherical cap is less than a hemisphere and the external cone opens out extending to infinity with an angle deficit $0 < D < 2\pi$. For $\mu = \frac{1}{4}$, $\theta_M = \pi/2$; in this limiting case the embedding diagram for the interior solution is a hemisphere and the external solution is a cylinder of radius r_0. The external metric in this case is $ds^2 = -dt^2 + dw^2 + r_0{}^2 d\phi^2 + dz^2$. One can think of this as a cone with an angle deficit of $D = 2\pi$. For $\frac{1}{4} < \mu < \frac{1}{2}$, $\pi/2 < \theta_M < \pi$; the spherical cap is more than a hemisphere and the external conical solution is like a dunce cap sitting on top of the sphere. The exterior metric 2 still applies with $B_0 = (\cos\,\theta_M)^{-2}$ still positive. But now r starts at $r_b = r_0 \sin\,\theta_M$ and decreases as one moves away from the string. The coordinate r reaches 0 at the apex of the dunce cap. The line singularity at $r = 0$ contains nonregular points (the circumference C of a small circle drawn around such a point does not approach $2\pi r$ as the radius $r \to 0$), but the Riemann curvature tensor is bounded as one approaches arbitrarily close to the singularity. Thus it is a quasi-regular singularity (cf. Ellis and Schmitt 1980). Now quasi-regular singularities are produced by cutting up perfectly

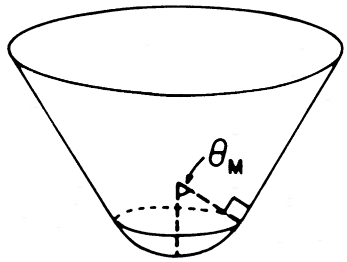

FIG. 1.—Embedding diagram for the cross sectional geometry of a uniform density vacuum string showing the interior and exterior solutions.

regular spacetimes, and no previous examples of physically reasonable quasi-regular singularities had been found. In this case the singularity may be considered as a singular vacuum string with zero radius, infinite density, and a mass per unit length of $\mu_s = \frac{1}{2} - \mu$. If $\mu > \frac{1}{4}$, the string and its associated singularity together have a mass per unit length which add up to $\frac{1}{2}$. The external cone (dunce cap) has an angle deficit

$$D = 8\pi(\tfrac{1}{2} - \mu) \text{ for } \tfrac{1}{4} < \mu < \tfrac{1}{2} . \tag{9}$$

If we want to avoid the presence of a singularity we can always do this by capping the dunce cap with a small spherical cap with $r_0' < r_0$, and $\theta_M' = \pi - \theta_M$. Then we would have two parallel strings with $\mu_2 + \mu_1 = \frac{1}{2}$ and some empty space between them if $r_0' < r_0$. If $\mu = \frac{1}{2}$ then $\theta_M = \pi$ and the interior solution closes completely on itself so that the cross section is a complete sphere. In this case the string occupies all of spacetime and there is no exterior solution.

Thus there is a maximum mass per unit length that a string can have $\mu_{\max} = \frac{1}{2} = 6.73 \times 10^{27}$ g cm^{-1}. If we demand that the string exist in a large external spacetime (as our observations suggest), then $\mu < \mu_{\text{crit}} = \frac{1}{4} = 3.37 \times 10^{27}$ g cm^{-1}.

Since $g_{tt} = -1$ in both the interior and exterior solutions one can show that $t(\tau) = \tau$, $r(\tau) = r_1 =$ const [or $\theta(\tau) = \theta_1 =$ const], $\phi(\tau) = \phi_1 =$ const, $z(\tau) = z_1 =$ const, is a geodesic. Thus the string exerts no Newtonian attraction on a particle that is at rest with respect to it. Two strings may be placed parallel at rest with respect to each other and they will remain where they are without any force needed to keep them apart. That is why no radial pressure terms P_θ are needed in the interior solution to produce a static solution. In fact a string of mass per unit length μ can be composed of a set of parallel mini strings with $\mu = \Sigma\mu_i$. In this case the spherical cap interior solution is approximated by a convex polyhedron where each ministring is at a vertex and the angle deficit at each vertex is $D_i = 8\pi\mu_i$. The faces and edges are locally flat, therefore solving Einstein's vacuum field equations. (For example the case $\mu = \frac{1}{2}$, $\theta_M = \pi$ [complete sphere], may be approximated by a cube. Three squares meet at each vertex giving each an angle deficit of $D = \pi/2$. Each vertex is a ministring with $\mu_i = 1/16$. The eight together add to give $\mu_{\text{tot}} = \frac{1}{2}$.) This makes it clear that we can find general solutions for strings that need not have cylindrical symmetry or uniform density, but which have $P_z = -\rho$ at each point (we assume $\rho \geq 0$ everywhere). Such models have metrics of the form $ds^2 = -dt^2 + ds'^2 + dz^2$, where $ds'^2 = g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2$ is the metric of an arbitrary spacelike two-surface σ_2, where the Gaussian curvature K can vary with position but is never negative. The density at each point is $\rho = K/8\pi$ and $P_z = -\rho$. The form of the metric guarantees that the only nonzero components of $R_\mu{}^\nu$ are $R_x{}^x = R_y{}^y = \frac{1}{2}R = K$. Substituting in Einstein's field equations gives $T_t{}^t = T_z{}^z = -K/8\pi$ or $\rho = +K/8\pi = -P_z$. If σ_2 is compact the Gauss-Bonnet Theorem guarantees that $\mu_{\text{tot}} = (1 - g)/2$, where g is the genus of σ_2. If $\rho \geq 0$ everywhere and singularities with negative mass are not allowed, $\mu_{\text{tot}} \geq 0$; thus $g > 1$ is not allowed. There is one trivial case with $\mu = 0$, $g = 1$, where $\sigma_2 = T^2$ (torus) and $\rho = 0$ everywhere. This is a flat geometry with a complex topology; it is square with opposite sides identified. The general compact solution σ_2 has $g = 0$ and $\mu_{\text{tot}} = \frac{1}{2}$, equal to the maximum allowed valve of μ obtained in the cylindrically symmetric case (eq. [1]), $\theta_M = \pi$. Solutions with $0 < \mu < \frac{1}{4}$ and having a large external space of infinite extent may be constructed by cutting out part (having $\mu < \frac{1}{4}$) of a compact solution and attaching to it a conical exterior solution.

It is interesting to note that in cross section the interior and exterior solutions for vacuum strings are exactly equal to the solutions for extended masses in a $(2+1)$–dimensional spacetime (flatland) previously found by Gott and Alpert (1984). In a $(2+1)$–dimensional spacetime, curvature has dimensions of (length)$^{-2}$ so density must also have units of (length)$^{-2}$, but it also has units of mass × (length)$^{-2}$ so mass must be dimensionless. The external field of a point mass is a cone with an angle deficit proportional to the mass. In a $(3+1)$–dimensional spacetime, the mass per unit length in a vacuum string is also a dimensionless quantity. The string is invariant with respect to velocity boosts in the z-direction. A simple rod of mass per unit length μ but $P_z > 0$ would not be. So it is the vacuum string that is the higher dimensional analog of a mass in flatland rather than a massive rod.

III. GRAVITATIONAL LENSING PROPERTIES

On a cone with angle deficit D, two geodesics that are originally parallel but pass on opposite sides of the vertex will eventually meet at an angle D. This can be shown simply by noting that geodesics are straight lines on the flattened-out cone which does not cover the plane but which has a wedge of angle D missing. This focusing effect was noted by Gott and Alpert (1984) in the $(2+1)$–dimensional case and by Vilenkin (1981*a*) for strings. He noted that this would cause observers to see double images of objects behind the string. He calculated the bend angle of each image to be $\delta\phi = 4\pi\mu$ and noted that this can give rise to double images of objects situated behind the string within the angle of order $\delta\phi$ from the string. He further noted that this might be relevant to the double quasar. We shall examine the gravitational lensing properties in more detail.

First, consider the situation with observer, QSO, and string at rest in an otherwise empty spacetime. (We shall assume that the string is of negligible width so that eq. [3] applies down to $r' = 0$ and $\mu < \frac{1}{4}$.) Let the observer and QSO be on opposite sides of the string at $z = 0$, and let the proper distance from the observer to the string be r_s and the proper distance from the QSO to the string be r_q. If the observer is at $\phi' = \pi - (D/2)$ then the QSO is at $\phi' = 0$ and $\phi' = 2\pi - D$ (recall that the range of ϕ' is 0 to $2\pi - D$). The observer sees two QSO images, at an angle θ_1 on each side of the string.

$$\sin \theta_1 = r_q \sin\left(\frac{D}{2}\right) \times \left(r_s{}^2 + r_q{}^2 + 2r_s r_q \cos\frac{D}{2}\right)^{-1/2} \tag{10}$$

in the relevant limit where D is small, $\theta_1 = (\frac{1}{2})Dr_q(r_s + r_q)^{-1}$. The two images are separated by $\Delta\theta = 2\theta_1$ in the sky. $\Delta\theta \leq D$ with maximum separation occurring when $r_s \ll r_q$.

To examine the gravitational lensing properties in the cosmological context we shall utilize formulae derived by Turner, Ostriker, and Gott (1984). We shall consider first the scenario proposed by Zeldovich where there is one relatively straight string within the current particle horizon. The cosmological metric (for simplicity adopting $\Omega_0 = 1$, matter-dominated) is

$$ds^2 = -dt^2 + (t/t_0)^{4/3}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] , \tag{11}$$

where $t_0 = \frac{2}{3}H_0{}^{-1}$ is the current age of the universe. If we are located at $r = 0$ an object with comoving radius r appears at redshift z where

$$r = 2H_0^{-1}[1 - (1 + z)^{-1/2}] . \tag{12}$$

The particle horizon ($z = \infty$) is at $r_H = 2H_0^{-1}$. We take the string to lie on a random geodesic which crosses within the horizon. Since the geometry of a spacelike slice in this cosmology is Euclidean, this is a straight line. Let r_s (with corresponding z_s given by eq. [12]) be the minimum distance r between us and the string, i.e., its point of closest approach. Then the probability that $r_s < r$ is $P(r_s < r) = r^2/r_H^2$. Thus the probability that z_s (the minimum redshift of the string) is less than z is

$$P(z_s < z) = 1 + (1 + z)^{-1} - 2(1 + z)^{-1/2} . \tag{13}$$

The string acts like a gravitational lens producing a deflection of $\pm(\frac{1}{2})D \cos \alpha$ in its rest frame, where α is the angle between the string and the plane of the sky (which is perpendicular to the line of sight). The string shows up in the sky as a sequence of double QSO images stretching along part of a great circle in the sky. Let α measure the angle in the sky as measured along the great circle and let $\alpha = 0$ denote the closest point in the string at $z = z_s$. Then (assuming $D \ll 1$) we show after some algebra using filled beam formulas in Turner, Ostriker, and Gott (1984) that QSO double images with separation in the sky of $\Delta\theta$ will be formed where

$$\Delta\theta = D\left[\cos \alpha - \frac{1 - (1 + z_s)^{-1/2}}{1 - (1 + z_q)^{-1/2}}\right] . \tag{14}$$

The formula applies and double lens images will be formed only for values of z_q (QSO redshift), z_s, and α such that the term in brackets in the equation above is positive. Let the first image be at an angle θ_1 to one side of the great circle and the second image at an angle θ_2 to the other side. Then $\theta_1 + \theta_2 = \Delta\theta(\alpha)$. If a QSO has one image which lies at a distance $\theta > \Delta\theta(\alpha)$ from the great circle, then no secondary image will be formed.

The most favorable case for lensing occurs when $z_s \ll 1$. In this limit $\Delta\theta(\alpha) = D[\cos \alpha]$ and $-\pi/2 < \alpha < \pi/2$. Choose spherical coordinates (θ, ϕ) in the sky such that the line of sight from us to the pole ($\theta = 0$) is parallel to the string, and the point of nearest approach to the string at z_s is at $\theta = \pi/2$, $\phi = 0$. Thus $\alpha = \theta - \pi/2$. The string itself, if we could see it, would be the half great circle $\phi = 0$, $0 < \theta < \pi$. All QSO images in the wedge $0 < \theta < \pi$, $0 < \phi < D$ are double lensed with second images in the wedge $0 < \theta < \pi$, $-D < \phi < 0$. The same wedge of sky simply appears twice. With the value of $\mu = 10^{-4}$ (consistent with fluctuations in the microwave background) $D = 8\pi\mu = 8'.6$. So double lens with separations of up to $8'.6$ could be seen. The optical depth for double lensing is $\tau = D/2\pi = 4\mu = 4 \times 10^{-4}$ (equal to the fraction of QSOs which are double images). Turner, Ostriker, and Gott (1984) calculate that a fraction of between 10^{-2} and 10^{-3} of randomly selected QSOs ($z_q > 1$) are multiply lensed $> 0''.1$ by ordinary galaxies and clusters. In a flux-limited sample the fraction is considerably higher because the magnification produced by galactic lenses brings fainter QSOs into the sample which would have otherwise been unobservable. They estimate that this enhancement may be as much a factor of 25 for reasonable luminosity functions. Thus in a flux-limited sample of QSOs perhaps as many as 2.5% are multiply lensed ($> 0''.1$) by ordinary galaxies and clusters. Now for QSOs doubly lensed by the string there is no magnification (the conical space is locally flat) so both images are of equal brightness $I_1 = I_2 =$ (original brightness), and there is no magnification selection effect aiding their inclusion in a flux-limited sample. However if $\mu = 10^{-4}$, the images are widely enough separated so that each image could be detected individually so that there are twice as many to be detected. Thus if we pick out individual QSOs from a flux-limited sample and then check each one for multiple imaging we expect to turn up $\gtrsim 30$ multiply lensed by galaxies and clusters for every one lensed by the string. Thus we would not expect a string-lensed case to be among the first discovered. Put another way, for $\mu = 10^{-4}$, we would have to find ~ 1250 QSOs placed randomly over the sky and search each for twin images out to $8'.6$ before we would expect to find one double lensed by the string. This does not look too bad. We have already discovered over 2000 QSOs (although a number of these are in selected fields rather than being randomly scattered over the sky). About 700 QSOs are found in VLA surveys, so a detailed study of VLA QSOs for double images at minutes of arc would be interesting.

Because $g_{tt} = -1$ throughout the string solution, it is easy to compute the time delay between the two images; there is no potential delay, only a geometric delay. Using the law of cosines we compute the difference δr in the comoving distances to the two images. A given event at the QSO will be observed with a time delay $t_2 - t_1 = \delta r$. We find

$$t_2 - t_1 = H_0^{-1}(\theta_1 - \theta_2)D[1 - (1 + z_s)^{-1/2}] . \tag{15}$$

As an example note that the absolute maximum time delay $(t_2 - t_1)_{\max} = \frac{1}{4}H_0^{-1}D^2$ occurs when $\alpha = 0$, $\theta_1 = \Delta\theta$, $\theta_2 = 0$, $z_s = 3$, $z_q = \infty$. For $\mu = 10^{-4}$ this is 3.1×10^4 yr. Over such a period of time the QSO properties, line strengths, brightness, etc. may change appreciably, so we must be tolerant of some differences in the images. Since we are now seeing the QSO as it appeared at two different times in the past and the universe is decelerating, the two QSO images will show differing redshifts with δz_q related to δr through equation (12). Thus $\delta z_q = (\theta_1 - \theta_2)D(1 + z_q)^{3/2}[1 - (1 + z_s)^{-1/2}] \sim O(D^2)$. However, this is calculated in the approximation that the string is at rest with respect to the comoving coordinate system, if the string is moving relativistically $\delta z_q \sim O(D)$. In the rest frame of the string, comoving geodesics having relativistic velocities $v \sim c$ with respect to the string that are parallel but on opposite sides of the string will have a relative velocity $\delta v \sim Dc$ after passage of the string. This effect will cause an observed temperature shift in the microwave background $\delta T/T \sim D$ from one side of a nearby string to the other (this point has recently been noted independently by Kaiser and Stebbins 1984). Thus a single string within the horizon moving relativistically (as expected) should be detected by the *COBE* satellite, provided $\mu \gtrsim 10^{-5}$. Since $P(z_s < 4) = 0.31$, there is an appreciable chance that the string is simply more distance than any QSO and therefore no double QSO images will be seen. However, $P(z_s < 10^3) = 0.94$ so the string will very likely be seen in front of the microwave background radiation.

Let us consider the long-lived loop scenario. Since curvature fluctuations on the scale of the horizon are dominated by the largest loop of size $l \sim R_H \sim 2H_0^{-1}$ the isotropy of the microwave background demands that $\mu \lesssim 10^{-4}$ just as in the Zeldovich case. However in the long-lived loop case the *COBE* satellite will see fluctuations of order $\delta T/T \sim \mu$ on all angular scales larger than $\theta \sim \mu$. This complex pattern could be difficult to distinguish from a simple Zeldovich (1972) fluctuation spectrum. The smallest loops are of size $l \sim \mu H_0^{-1}$ and mass $M \sim \mu^2 H_0^{-1}$. Thus the typical loop has an angular size in the sky ($\theta \sim \mu$) which is of the same order as the lensing deflections its string is capable of producing ($\Delta\theta \sim \mu$). Thus the lensing may be complicated. No exact solutions are available for such closed relativistically moving loops of string, but in general it is thought (cf. Vilenkin 1981*a*) that the exterior solution when the string is not straight need not be locally flat, and that at large distances from a closed loop the geometry approaches the Schwarzschild geometry. Thus there could be in principle some image amplification. The optical depth for multiple lensing distant QSOs is of order $\tau \sim -\mu \ln \mu \lesssim 10^{-3}$. This is somewhat more optimistic than the Zeldovich case which gives $\tau \sim \mu$.

As an example of the added complications of loops consider a case we can compute, namely that of two parallel strings separated by a distance b and each with an angle deficit $D = 8\pi\mu$. This may be thought of as an infinite elliptical loop with minor axis b and major axis $a = \infty$. The external metric is then of the form $ds^2 = -dt^2 + dz^2 + ds'^2$, where $ds'^2 = dx^2 + dy^2$ is the metric for the surface σ_2, where σ_2 is a plane with two wedge-shaped pieces with opening angle D excluded. String 1 is located at $x = b/2$, $y = 0$, string 2 is located at $x = -b/2$, $y = 0$. A cut in the plane is made along ray 1a: $x = b/2$ with $y > 0$, and a cut is made along ray 1b: $x = b/2 + y \tan D$, with $y > 0$. Ray 1a and 1b are then identified producing an angle deficit D at string 1. Similarly cuts are made along ray 2a: $x = -b/2$, with $y > 0$ and ray 2b: $x = -b/2 - y \tan D$, with $y > 0$, and these two rays are also joined producing an angle deficit at string 2 also. An observer located at $x = 0$, $y = -y_0$, where $y_0 > b/(2 \tan D)$ will see three images of a quasar located at $x = 0$, $y = y_1$ for $y_1 > b/\{2 \tan [D - \tan^{-1}(b/2y_0)]\}$. The observer sees one direct image, one image deflected around the left side of string 2, and one image deflected around the right side of string 1. The observer begins to see triple images of distant QSOs when the angular separation of the two strings in the sky is less than $2D$. As we have noted, in the long-lived loop case, we expect the typical loop angular size to be of order D. Thus we can expect triple or more complex images from loops.

Are any already known or proposed gravitational lens cases actually due to vacuum strings? There are five known gravitational lens cases with image separations ranging up to 7″.3. If any of these were due to strings then $\mu \sim 1.5 \times 10^{-6}$. This is probably too small to be interesting for galaxy formation. Vilenkin (1981*b*) estimates $\mu \sim 10^{-5}$ in the long-lived loop scenario would be sufficient to promote galaxy formation. Of course strings may be present even if they are not of sufficient mass to aid in galaxy formation. Now with μ as small as 1.5×10^{-6} if there were only one string within the horizon the total optical depth to lensing $\tau \lesssim 4\mu \sim 6 \times 10^{-6}$ is so low that it would be unlikely for us to have discovered one after only inspecting 2×10^3 QSOs. Even with the long-lived loop scenario we would expect only $\tau \sim 2 \times 10^{-5}$.

Consider each of the five known cases.

1. 0957+561 (cf. Walsh, Carswell, and Weymann 1979): the two images A and B are not of equal brightness, the luminosity ratio in the emission lines is quite stable with time giving B = 0.75 A, besides a lensing galaxy at appropriate redshift has been found (cf. Young *et al.* 1981).

2. 1115+080 (cf. Weymann *et al.* 1980): four images of widely different brightness.

3. 2345+007 (cf. Weedman *et al.* 1982): a widely separated double (7″.3) with no known lensing galaxy or cluster, but the two images are of rather different brightness.

4. 2016+112 (cf. Lawrence *et al.* 1984): this is a double of relatively equal brightness, but a lensing galaxy is seen.

5. 1635+276 (cf. Djorgovski and Spinrod 1984): here, the two images are of rather unequal brightness.

The burden of proof clearly rests on the side of showing that a vacuum string is more attractive than the more conservative possibilities such as galaxies and clusters.

At present there are no compelling or even telltale signs that these may be due to strings, and in fact in each system there are at least some counter indications. Then there is the proposed gravitational lens case of Paczyński and Gorski (1981). These are three QSOs with redshifts $z_1 = 2.048$, $z_2 = 2.054$, $z_3 = 2.040$ that lie in a somewhat bent line with $\theta_{12} = 2'.2$ and $\theta_{23} = 1'.9$. Now these separations appear quite reasonable for strings with $\mu \sim 2 \times 10^{-5}$, a value of μ that is large enough to be interesting as far as growth of structure in the universe is concerned and not big enough to be inconsistent with the isotropy of the cosmic microwave background. Now the redshifts of these QSOs differ by about 10^3 km s^{-1}, rather too large to be explained by relativistic movement of a $\mu \sim 2 \times 10^{-5}$ string. But as pointed out by Paczyński and Gorski (1981), the redshifts are not too well determined, and the time delays between the images would be of the order of 10^2–10^3 years and the emission lines could have changed over this long a time period. Paczyński and Gorski proposed that the lensing was done by two extremely rich clusters of galaxies. The required line of sight velocity dispersion for each cluster is $\gtrsim 1600$ km s^{-1} if the cosmological constant is zero (as compared to 1000 km s^{-1} for Coma). Thus, this is a system whose separation is so large that we have difficulty explaining it in terms of known objects. Since three images are seen we can not do the lensing with one string and must use a loop. As we have seen, loops are indeed capable of producing a triple image. The three images are not of equal brightness, but this is not a problem since the QSO may vary in brightness over the time delay time scale. The images are approximately equally separated, and the bent line could undoubtedly be explained by some loop geometrics. In this system, however, the most conservative assumption, of course, is that the three images are simply of three different QSOs which lie in the same supercluster. This would explain the differences in redshifts. (Undoubtedly this would be a much stronger lensing candidate if the three redshifts were closer to each other.) Here the Space Telescope (ST) will be most useful in studying this system. If the lensing is done by rich clusters at intermediate redshift ($z < 1$), then ST will certainly be able to see them. If the QSOs are part of a supercluster at $z = 2.047$, then depending on galaxy evolution the ST may even be able to see some of these galaxies. (They would be much fainter and suffering a much larger K-correction than galaxies at $z \sim 1$.) Finally if a string is indeed lensing these QSOs, then ST may detect other faint galaxies ($z \gtrsim z_s$) that are multiply lensed by the string. A number of such cases could really elucidate the nature of the lensing. ST may also uncover other cases like this.

IV. CONCLUSIONS

Vacuum strings may arise naturally as a consequence of symmetry breaking in the very early universe. We have derived exact interior and exterior solutions to Einstein's field equations for vacuum strings. For $0 < \mu < \frac{1}{4}$ the external metric is $ds^2 = -dt^2 + dr^2 + (1 - 4\mu)^2 r^2 d\phi^2 + dz^2$, where μ is the mass per unit length in the string in Planck masses per Planck length. We have examined the gravitational lensing properties of vacuum strings in some detail. Strings can cause temperature fluctuations in the cosmic microwave background which could in principle be detected by *COBE*, and they can produce double QSO images separated by up to several minutes of arc. Thus gravitational lensing offers a promising way to detect vacuum strings.

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REFERENCES

Djorgovski, S., and Spinrad, H. 1984, talk at AAS Meeting, Las Vegas.
Ellis, G. F. R., and Schmidt, B. G. 1980, *Gen. Rel. Grav.*, **2**, 915.
Gott, J. R. 1977, *Ann. Rev. Astr. Ap.*, **15**, 235.
Gott, J. R., and Alpert, M. 1984, *Gen. Rel. Grav.*, **16**, 243.
Gott, J. R., and Rees, M. J. 1975, *Astr. Ap.*, **45**, 365.
Hawking, S. W., and Ellis, G. F. R. 1973, *The Large Scale Structure of Space-time* (London: Cambridge University Press).
Hogan C., and Rees, M. 1984, preprint.
Israel, W. 1966, *Nuovo Cimento*, **44B**, 1.
Kaiser, N., and Stebbins, A. 1984, *Nature*, **310**, 391.
Kibble, T. W. B., Lazarides, G., and Shafi, Q. 1982, *Phys. Letters*, **113B**, 237.
Kibble, T. W. B., and Turok, N. 1982, *Phys. Letters*, **116B**, 141.
Lawrence, C. R., Schneider, D. P., Schmidt, M., Bennett, C. L., Hewitt, J. N., Burke, B. F., Turner, E. L., and Gunn, J. E. 1984, *Science*, **223**, 46.
Paczyński, B., and Gorski, K. 1981, *Ap. J.* (*Letters*), **248**, L101.
Turner, E. L., Ostriker, J. P., and Gott, J. R. 1984, *Ap. J.*, **284**, 1.
Turok, N. 1983, *Phys. Letters*, **123B**, 387.
Vilenkin, A. 1981*a*, *Phys. Rev. D.*, **23**, 852.
———. 1981*b*, *Phys. Rev. Letters*, **46**, 1169.
———. 1984, preprint
Walsh, D., Carswell, R. F., and Weymann, R. J. 1979, *Nature*, **279**, 381.
Weedman, D. W., Weymann, R. J., Green, R. F., and Heckman, T. M. 1982, *Ap. J.* (*Letters*), **255**, L5.
Weymann, R. J., Latham, D., Angel, J. R. P., Green, R. F., Leibert, J. W., Turnshek, D. A., Turnshek, D. E., and Tyson, J. A. 1980, *Nature*, **285**, 641.
Young, P., Deverill, S., Gunn, J. E., Westphal, J. E., and Kristian, J. 1981, *Ap. J.*, **244**, 723.
Zeldovich, Ya. B. 1972, *M.N.R.A.S.*, **192**, 663.
———. 1980, *M.N.R.A.S.*, **192**, 663.

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