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IMPERFECT FLUID COSMOLOGIES WITH THERMODYNAMICS: SOME EXACT SOLUTIONS

A. A. COLEY
Department of Applied Mathematics, University of Waterloo
AND
B. O. J. TUPPER
Department of Mathematics and Statistics, University of New Brunswick
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ABSTRACT

The field equations for a viscous magnetohydrodynamic fluid satisfying an appropriate set of thermodynamic relations are investigated. It is shown that zero-curvature Friedmann-Robertson-Walker models can be exact solutions of this system of equations. Two types of solution are discussed: one in which the spacelike component of the tilting velocity vector is axially directed and the other in which the spacelike component is radially directed. The solutions presented satisfy all the necessary conditions for physical acceptability.
Subject headings: cosmology — hydromagnetics — relativity

I. INTRODUCTION

In general relativity theory cosmological models, stellar models, and models of other astrophysical matter distributions are usually constructed under the assumption that the matter content is an idealized perfect fluid. While this assumption may be a good approximation to the actual matter content of the universe at the present epoch, effects such as viscosity, heat conduction, and magnetic fields may not be negligible at earlier epochs of the universe. Such effects should also be considered in any realistic stellar model. Accordingly, we shall investigate the problem of obtaining exact solutions to field equations for a viscous magnetohydrodynamic (VMHD) fluid.

The field equations for a VMHD fluid are

$$G_{\mu\nu} = M_{\mu\nu} \equiv E_{\mu\nu} + (\rho + p - \xi\Theta)u_\mu u_\nu + (p - \xi\Theta)g_{\mu\nu} - 2\eta\sigma_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \,, \quad (1.1)$$

where ρ is the density, p is the thermodynamic pressure, Θ is the expansion of the fluid velocity congruence u^μ, $\sigma_{\mu\nu}$ is the shear tensor, q_μ is the heat conduction vector, ξ is the bulk viscosity coefficient, η is the shear viscosity coefficient, and $E_{\mu\nu}$ is the electromagnetic stress-energy tensor given by

$$E_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \,, \quad (1.2)$$

where $F_{\mu\nu}$ is the Maxwell tensor. In order to be a physically acceptable VMHD model, a solution of the field equations (1.1) must satisfy a required energy condition, such as the dominant energy condition (Hawking and Ellis 1973), the physical quantities occurring in the equations (1.1) must behave in a satisfactory manner, i.e., $\rho > 0$, $p \geq 0$, $\eta \geq 0$, $\xi \geq 0$, and an appropriate set of thermodynamic relations must be satisfied. Furthermore, the Maxwell equations

$$F_{[\mu\nu;\sigma]} = 0 \,, \quad F^{\mu\nu}{}_{;\nu} = J^\mu \,, \quad (1.3)$$

must be satisfied and the 4-current, J^μ, calculated from these equations must be consistent with the generalized Ohm's law expression (Dunn and Tupper 1980), namely

$$(J^\mu - \epsilon u^\mu)(1 + \zeta^2 B^2) = \lambda E^\mu + \lambda\zeta^2 E_\alpha B^\alpha B^\mu + \lambda\zeta S^\mu \,, \quad (1.4)$$

in which J^μ is expressed as the sum of a convection current and a conduction current. In this expression E_μ, B_μ, and S_μ are, respectively, the electric field, the magnetic field, and the Poynting vector as measured by a comoving observer and are defined by

$$E_\mu = F_{\mu\nu}u^\nu \,, \quad B_\mu = \tfrac{1}{2}\eta_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta} \,, \quad S_\mu = \eta_{\nu\mu\alpha\beta}u^\nu E^\alpha B^\beta \,.$$

The quantities ϵ, λ, and $\lambda\zeta$ are, respectively, the charge density, the conductivity, and the transverse conductivity, each of which must be nonnegative.

An appropriate set of thermodynamic relations to be satisfied by the VMHD fluid are those proposed by Eckart (1940), namely:

Baryon conservation equation:

$$N^\mu{}_{;\mu} = 0 \quad (1.5)$$

where $N^\mu = nu^\mu$ is the particle flux and n is the particle density;

Gibbs's relation:

$$Td(S/n) = d(\rho/n) + pd(1/n) \,, \quad (1.6)$$

where T is the temperature and S is the entropy density;

Entropy production:

$$S^\mu{}_{;\mu} \geq 0 \,, \quad (1.7)$$

where S^μ is the entropy flux defined by

$$S^\mu = \frac{S}{n} N^\mu + T^{-1}q^\mu \,; \quad (1.8)$$

Temperature gradient law:

$$q^\mu = -\kappa h^{\mu\nu}(T_{,\nu} + Ta_\nu) \,, \quad \kappa \geq 0 \,, \quad (1.9)$$

where $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor, $a_\nu = u_{\nu;\alpha}u^\alpha$ is the acceleration vector, and κ is the thermal conductivity. Note that the condition (1.7) is automatically satisfied if the condition (1.9) holds, so we need consider only the latter condition. The relations (1.8) and (1.9) are not the most general expressions for S^μ and q^μ (see, for example, Israel and Stewart 1979), but are sufficient for the purposes of this article.

In seeking exact solutions of the field equations (1.1) we shall make use of the fact that the stress-energy tensors of some quite different matter distributions may have precisely the same components (Tupper 1981, 1983*a*, *b*). In particular, the stress-energy tensor of a perfect fluid, namely

$$H_{\mu\nu} = (\bar{\rho} + \bar{p})v_\mu v_\nu + \bar{p}g_{\mu\nu} , \quad (1.10)$$

may be identical to the stress-energy tensor, $M_{\mu\nu}$, of a VMHD fluid. This equality between $H_{\mu\nu}$ and $M_{\mu\nu}$ implies that the spacetime geometry corresponding to a known exact solution of the field equations for a perfect fluid is also the spacetime geometry corresponding to an exact solution of the field equations (1.1). Furthermore, the examples given in one of the above-mentioned articles (Tupper 1983*b*) show that there may be an infinite class of exact VMHD solutions sharing the same spacetime geometry, since the components of the 4-velocity u^μ are not always uniquely determined by the field equations. Note that different observers moving relative to each other will, in general, give different interpretations to the material content of the universe. For example, a perfect fluid solution with respect to a comoving observer will become an imperfect fluid solution with respect to a "tilting" observer. However, the coordinate transformation from the comoving observer to the tilting observer will result in a change in the form of the metric, and the difference in the physical interpretations is due entirely to the different coordinate systems of the two observers. It should be clearly understood that this is *not* the situation that we are discussing here. In our work the interpretation of the material content as a VMHD fluid is given not by another observer but by the same set of hypersurface orthogonal preferred observers who may also interpret the material content as a perfect fluid, so that the spacetime metric is expressed in the same coordinate form for the VMHD interpretation as for the perfect fluid interpretation.

Probably the most simple of the known perfect fluid models are the zero-curvature Friedmann-Robertson-Walker (FRW) models. We have shown (Coley and Tupper 1983*a*) that these models can be exact solutions of the field equations (1.1). The resulting VMHD solution, which we term a *radial solution* since it has a radially directed spacelike velocity component, satisfies all necessary physical conditions, but, apart from the relation (1.9), the thermodynamic conditions were not considered. The physical quantities in this solution depend on the radial coordinate r as well as on t. We have since shown (Coley and Tupper 1983*b*) that the Einstein–de Sitter universe can be the spacetime geometry of a viscous fluid solution which satisfies all the thermodynamic relations (1.5)–(1.9) as well as the other necessary physical conditions, and in which all quantities depend on t alone. We term this solution an *axial solution* since the spacelike velocity component is in the axial z-direction of cylindrical polar coordinates.

In this article we shall generalize the axial solution by considering a general zero-curvature FRW metric and by including a magnetic field. We shall also find radial solutions which satisfy all the thermodynamic relations (1.5)–(1.9). All the solutions presented here are exact solutions of the field equations (1.1) and are physically acceptable in the sense that all the necessary physical requirements are satisfied. We believe that these are the first exact VMHD solutions to be found which satisfy all of the conditions (1.2)–(1.9).

II. THE AXIAL SOLUTION

The zero-curvature FRW model satisfying the perfect fluid field equations with equation of state $p = \gamma\rho$ has, in cylindrical polar coordinates, a metric of the form

$$ds^2 = -dt^2 + t^{2a}(dr^2 + r^2 d\theta^2 + dz^2) , \quad (2.1)$$

where

$$a = \tfrac{2}{3}(1 + \gamma)^{-1} , \quad (2.2)$$

so that, since $0 \leq \gamma \leq 1$, the parameter a satisfies $\frac{2}{3} \geq a \geq \frac{1}{3}$. In order to show that the metric (2.1) satisfies the field equations (1.1), we choose a "tilting" velocity with spacelike component in the z-direction, i.e.

$$u_\mu = (-\alpha, 0, 0, \beta t^a) , \quad (2.3)$$

where $\alpha^2 - \beta^2 = 1$, $\alpha \geq 1$, and α, β are assumed to be functions of t only. We also assume that the electromagnetic field consists of a magnetic field in the z-direction, i.e., the only nonzero component of the Maxwell tensor is F_{12}. The assumption that the magnetic field vector, B_μ, depends only on t, together with equations (1.3) and (1.4), leads to $J^\mu = 0$ and

$$F_{12} = A_0 r , \quad (2.4)$$

where A_0 is a constant. Since $q_\mu u^\mu = 0$, we also assume that q_μ is of the form

$$q_\mu = Q(\beta, 0, 0, -\alpha t^a) , \quad (2.5)$$

where $Q^2 = q_\mu q^\mu$. For simplicity we shall equate the bulk viscosity to zero.

With these assumptions, and using the metric (2.1), the nontrivial components of the field equations are

$$\begin{aligned} 3a^2 t^{-2} &= \tfrac{1}{2}A_0^2 t^{-4a} + \rho\alpha^2 + p\beta^2 - \tfrac{4}{3}\eta\beta^2\dot{\alpha} - 2Q\alpha\beta , \\ a(2 - 3a)t^{-2} &= -\tfrac{1}{2}A_0^2 t^{-4a} + \rho\beta^2 + p\alpha^2 - \tfrac{4}{3}\eta\alpha^2\dot{\alpha} - 2Q\alpha\beta , \\ a(2 - 3a)t^{-2} &= \tfrac{1}{2}A_0^2 t^{-4a} + p + \tfrac{2}{3}\eta\dot{\alpha} , \\ 0 &= \rho + p - \frac{4}{3}\eta\dot{\alpha} - \frac{Q(\alpha^2 + \beta^2)}{\alpha\beta} , \end{aligned} \quad (2.6)$$

where the dot denotes differentiation with respect to t. Solving these equations yields

$$\rho = a(2\beta^2 + 3a)t^{-2} - \tfrac{1}{2}A_0^2 t^{-4a} , \quad (2.7)$$

$$p = \tfrac{1}{3}a(2\beta^2 + 6 - 9a)t^{-2} - \tfrac{1}{6}A_0^2 t^{-4a} , \quad (2.8)$$

$$Q = 2a\alpha\beta t^{-2} , \quad (2.9)$$

$$\eta\dot{\alpha} = -a\beta^2 t^{-2} - \tfrac{1}{2}A_0^2 t^{-4a} . \quad (2.10)$$

Note that the last equation implies that $\dot{\alpha} < 0$.

Before proceeding, we shall expand on some of the comments made in § I. The only observers considered in this article are not observers moving with the viscous fluid, but are the family of observers whose world lines are orthogonal to the $t =$ const. hypersurfaces of the metric (2.1). These observers can interpret the gravitational field, i.e., the metric (2.1), as being due to any member of a one-parameter family of energy-momentum tensors of the form (1.1) with matter 4-velocity given by equation (2.3), where the parameter is the function α.

The standard perfect fluid solution is obtained by considering that member of the family of energy-momentum tensors which corresponds to $\alpha = 1$; we are interested in the case $\alpha \neq 1$.

Integrating equation (1.5), we find that

$$n = n_0 t^{-3a} \alpha^{-1} , \tag{2.11}$$

where n_0 is a constant. Equation (1.6) together with equations (2.7), (2.8), and (2.11) implies that $T = T(t)$ and the temperature gradient law (1.9) reduces to the single scalar equation

$$2a = \kappa \alpha^{-1} t^2 [\dot{T} + T(at^{-1} + \dot{\beta}\beta^{-1})] . \tag{2.12}$$

In order to satisfy this condition we assume that T is of the form

$$T = T_0 + T_1 t^{-m} \alpha^l , \tag{2.13}$$

where T_0 and T_1 are positive constants and m and l numerical parameters with $m > 0$. Equation (2.12) then becomes

$$2a = \kappa\alpha^{-1}t^2 T_0(at^{-1} + \dot{\beta}\beta^{-1}) + \kappa\alpha^{l-1}t^{2-m}T_1[(a - m)t^{-1} + \dot{\beta}\beta^{-1} + l\dot{\alpha}\alpha^{-1}] \tag{2.14}$$

and the condition $\kappa \geq 0$ will be satisfied for all t if

$$at^{-1} + \dot{\beta}\beta^{-1} \geq 0 , \tag{2.15}$$

$$(a - m)t^{-1} + \dot{\beta}\beta^{-1} + l\dot{\alpha}\alpha^{-1} \geq 0 . \tag{2.16}$$

Since $m > 0$ and $\dot{\alpha} < 0$, the condition (2.16) will imply condition (2.15) if $l > 0$.

We now impose the requirement that the electromagnetic and viscous fluid parts of $M_{\mu\nu}$ each, independently, satisfy the dominant energy condition (Hawking and Ellis 1973). It is easily seen that the electromagnetic part, $E_{\mu\nu}$, does satisfy this condition; the field equations (2.6) show that the viscous fluid part will satisfy this condition if the following conditions hold:

$$6a^2 t^{-2} - A_0^2 t^{-4a} > 0 , \tag{2.17}$$

$$2a(3a - 1)t^{-2} - A_0^2 t^{-4a} \geq 0 , \tag{2.18}$$

$$2at^{-2} - A_0^2 t^{-4a} \geq 0 , \tag{2.19}$$

and the allowable interval of values of the parameter a shows that the most stringent of these requirements is (2.18), i.e.,

$$t^{4a-2} \geq [2a(3a - 1)]^{-1} A_0^2 . \tag{2.20}$$

Assuming that the model is valid as $t \to \infty$, the requirement (2.20) implies that $a \geq \frac{1}{2}$. Hence, the model starts at time $t = t_0$, where t_0 is given by

$$t_0^{4a-2} = [2a(3a - 1)]^{-1} A_0^2 , \tag{2.21}$$

and t_0 will be small when the magnetic field is small. When the material content is a viscous fluid only, the initial time is $t_0 = 0$.

Since all the physical quantities in the solution, i.e., ρ, p, Q, η, n, S, T, and κ, are functions of the velocity component α, it follows that to complete the solution we need to specify α subject to the conditions (2.15), (2.16) and $\dot{\alpha} < 0$, corresponding to the conditions $\kappa \geq 0$, $\eta \geq 0$. To simplify the choice of α we impose the following conditions which may be regarded as "boundary conditions" on α:

Condition 1. $\alpha \to 1$ as $t \to \infty$.

This condition implies that as $t \to \infty$, the model approaches the standard perfect fluid solution.

Condition 2. $\alpha \to \infty$ as $t \to 0$.

This condition implies that the fluid velocity becomes large as $t \to 0$, in keeping with the behavior of the other physical quantities as $t \to 0$. Note that this condition is meaningful only when $A_0 = 0$, but we shall assume that the resulting form for α applies even when $A \neq 0$.

From equations (2.7) and (2.8), a consequence of condition 2 is that $p/\rho \to \frac{1}{3}$ as $t \to 0$. Thus the material content changes from an initial radiation state to a final perfect fluid state with $p/\rho \to \gamma$ as $t \to \infty$. Accordingly, we also impose the condition that as $t \to 0$ the density and temperature are related by the Stefan-Boltzmann law, i.e.,

Condition 3. $T^4 \rho^{-1} \to$ constant as $t \to 0$.

From equations (2.7) and (2.13) this condition implies that

$$t^{2-4m} \alpha^{4l-2} \to \text{constant} \quad \text{as} \quad t \to 0 . \tag{2.22}$$

We shall discard the obvious solution $m = l = \frac{1}{2}$ since it implies that $T^4 \rho^{-1} =$ constant always.

When $t = t_0$, equations (2.7) and (2.21) show that $\rho = a(\alpha^2 + \beta^2)t_0^{-2} > 0$, but p is not necessarily positive. The desirable condition $p > 0$ when $t = t_0$ imposes a further restriction which, from equations (2.8) and (2.21), is

$$\alpha^2(t_0) > 6a - 5/2 . \tag{2.23}$$

We now attempt to satisfy these conditions for a physically acceptable solution by choosing a specific functional form for α and β, namely

$$\alpha = (1 + ht^{-b})(1 + 2ht^{-b})^{-1/2} , \quad \beta = ht^{-b}(1 + 2ht^{-b})^{-1/2} , \tag{2.24}$$

where h and b are positive constant parameters. This choice ensures that $\dot{\alpha} < 0$ and conditions 1 and 2 are satisfied. Condition 3, i.e., equation (2.22), will be satisfied if

$$4m - 2 + b(2l - 1) = 0 ; \tag{2.25}$$

and, by picking out powers of ht^{-b}, we find that the conditions (2.15) and (2.16) are satisfied if

$$a - b \geq 0 , \quad 2a - b \geq 0 , \tag{2.26}$$

$$a - m - b \geq 0 , \quad 3a - 3m - 2b \geq 0 , \quad 2a - 2m - (l + 1)b \geq 0 . \tag{2.27}$$

The first of the inequalities (2.27) is sufficient to ensure the second and also both of the inequalities (2.26). From equation (2.25) the third of the inequalities (2.27) yields

$$b \leq \tfrac{2}{3}(2a - 1) , \tag{2.28}$$

which, since $b > 0$, implies that $a > \frac{1}{2}$, so that the only zero-curvature FRW models satisfying the conditions imposed here are those for which the standard perfect fluid model has equation of state $0 \leq p/\rho < \frac{1}{3}$, i.e., $\frac{2}{3} \geq a > \frac{1}{2}$, in keeping with the condition (2.20).

There remains the condition (2.23) for $p > 0$ when $t = t_0$, i.e.,

$$(1 + ht_0^{-b})^2(1 + 2ht_0^{-b})^{-1} > 6a - 5/2 .$$

Rewriting this in the form

$$h^2 t_0^{-2b} + (7 - 12a)ht_0^{-b} + \tfrac{1}{2}(7 - 12a) > 0 ,$$

we see that this is always satisfied, irrespective of h and A_0, if

$a \leq 7/12$ (i.e., $\gamma \geq 1/7$). However, if $a > 7/12$, the condition is satisfied if

$$ht_0{}^{-b} > 6a - \tfrac{7}{2} + (36a^2 - 36a + \tfrac{35}{4})^{1/2} . \qquad (2.29)$$

By combining this condition with equation (2.21) we obtain a relation between h and A_0 which, if satisfied, ensures that $p > 0$ at the initial time t_0. In fact this relation can be adjusted to yield any desired value in the interval $0 \leq p/\rho < \frac{1}{3}$ for the ratio p/ρ at $t = t_0$.

The magnetic field vector, B_μ, is of the form

$$B_\mu = A_0 t^{-2a}(\beta, 0, 0, -\alpha t^a) . \qquad (2.30)$$

However, our set of preferred observers with comoving velocity $v^\mu = (1, 0, 0, 0)$ orthogonal to the spacelike hypersurfaces see a magnetic field of the form

$$\bar{B}_\mu = -A_0 t^{-a}(0, 0, 0, 1) . \qquad (2.31)$$

We have thus shown that the zero-curvature FRW model with metric (2.1) and $\frac{1}{2} < a \leq \frac{2}{3}$ is an exact solution of the VMHD field equations (1.1) with an axially directed magnetic field, an axially directed velocity vector given by equations (2.3) and (2.24), density and pressure by equations (2.7) and (2.8), shear conductivity by equation (2.10), temperature by equation (2.13), and thermal conductivity by equation (2.15). The model starts at time $t = t_0$ given by equation (2.21) and is a physically acceptable solution satisfying all energy conditions, thermodynamic conditions, and positivity requirements provided that the conditions (2.25), (2.27), (2.28), and (2.29) are satisfied. We shall show that these conditions can be satisfied by choosing a specific example of an FRW model.

Consider the Einstein–de Sitter model for which $a = \frac{2}{3}$ and $\gamma = 0$. Equation (2.28) becomes $b \leq \frac{2}{9}$ and we must satisfy equation (2.25) and the first of the inequalities (2.27) which now reads $m + b \leq \frac{2}{3}$. These conditions can be satisfied, for example, by

$$b = \tfrac{1}{6} , \quad m = \tfrac{1}{3} , \quad l = \tfrac{5}{2} . \qquad (2.32)$$

With this choice for the values of the numerical parameters the various physical quantities are given by

$$\rho = \tfrac{4}{3}(1 + ht^{-1/6})^2(1 + 2ht^{-1/6})^{-1}t^{-2} - \tfrac{1}{2}A_0^2 t^{-8/3} , \qquad (2.33)$$

$$p = \tfrac{4}{9}h^2 t^{-7/3}(1 + 2ht^{-1/6})^{-1} - \tfrac{1}{6}A_0^2 t^{-8/3} , \qquad (2.34)$$

$$\eta = [4 + 3A_0^2 h^{-2} t^{-1/3}(1 + 2ht^{-1/6})](1 + 2ht^{-1/6})^{1/2} t^{-1} , \qquad (2.35)$$

$$\kappa = \tfrac{8}{3}[T_0 t(1 + \tfrac{7}{3}ht^{-1/6})(1 + ht^{-1/6})^{-1}(1 + 2ht^{-1/6})^{-1/2} + \tfrac{1}{6}T_1 t^{2/3}(2 + 8ht^{-1/6} + h^2 t^{-1/3})(1 + ht^{-1/6})^{1/2} \times (1 + 2ht^{-1/6})^{-7/4}]^{-1} , \qquad (2.36)$$

from which we see that all quantities approach infinity as $t \to 0$ and approach zero as $t \to \infty$.

To ensure that $p > 0$ at $t = t_0$, equation (2.29) leads to

$$ht_0{}^{-1/6} > \tfrac{1}{2}(1 + \sqrt{3}) ,$$

which, from equation (2.21), implies that

$$h^2 > 1.616\ A_0 . \qquad (2.37)$$

However, in general, p is increasing when $t = t_0$, reaches a maximum value, and then decreases forever. We can ensure that t_0 coincides with the time of maximum p (or later) by choosing

$$h^2 \geq 2.187\ A_0 , \qquad (2.38)$$

in which case p is always decreasing from the initial time t_0. Of course, when the magnetic field is absent, we have $p > 0$ and $\dot{p} < 0$ for all $t \geq 0$ irrespective of the value of h. Note that when p has its maximum value, i.e., when $h^2 = 2.187\ A_0$, the ratio $p/\rho = 0.031$, so that the presence of the magnetic field causes a considerable reduction in the initial value of p/ρ from the value $\frac{1}{3}$ at $t = 0$ when $A_0 = 0$.

Hence, there does exist one FRW model (in fact, there are many) which is an exact VMHD solution satisfying the required energy and thermodynamic conditions for a physically acceptable solution. Note that the standard perfect fluid solution is a special case of the solution presented here corresponding to $h = A_0 = 0$.

Finally, we note that for a general $\alpha = \alpha(t)$ the velocity congruence u^μ has volume expansion and shear given by

$$\Theta = \dot{\alpha} + 3a\alpha t^{-1} , \qquad (2.39)$$

$$\sigma^2 = \tfrac{1}{3}\dot{\alpha}^2 \qquad (2.40)$$

and the expansion components Θ_z and $\Theta_\perp$ along and perpendicular to the z-direction, respectively, as measured by an observer moving with the VMHD fluid, are

$$\Theta_z = \dot{\alpha} + a\alpha t^{-1} , \quad \Theta_\perp = a\alpha t^{-1} , \qquad (2.41)$$

implying a directionally dependent Hubble constant. The corresponding shear components are

$$\sigma_z = \tfrac{2}{3}\dot{\alpha} , \quad \sigma_\perp = -\tfrac{1}{3}\dot{\alpha} . \qquad (2.42)$$

The representative length l, defined by $l_{,\mu} u^\mu l^{-1} = \frac{1}{3}\Theta$, is given by

$$l = t^a \alpha^{1/3} ; \qquad (2.43)$$

and the declaration parameter, q, is given by

$$q = (2\alpha^2 t^{-2} - 10\alpha\dot{\alpha} t^{-1} - 3\alpha\ddot{\alpha} - \dot{\alpha}^2)(2\alpha t^{-1} + \dot{\alpha})^{-2} . \qquad (2.44)$$

Some insight into the physical properties of our specific model (represented by eqs. [2.24] and [2.32]) can be obtained by investigating the behavior of these kinematical quantities. For example, if we fix h and A_0 by assuming that

$$h^2 = 3A_0 , \qquad (2.45)$$

so that the condition (2.38) is satisfied, we find that the ratio $\sigma\Theta^{-1}$ is given by

$$\sigma\Theta^{-1} \begin{cases} \to 0.0251 & \text{as} \quad t \to 0 , \\ = 0.0163 & \text{when} \quad t = t_0 , \\ \to 0 & \text{as} \quad t \to \infty ; \end{cases} \qquad (2.46)$$

and the ratio of the directional expansion rates is given by

$$\Theta_\perp \Theta_z{}^{-1} \begin{cases} \to 1.143 & \text{as} \quad t \to 0 , \\ = 1.068 & \text{when} \quad t = t_0 , \\ \to 1 & \text{as} \quad t \to \infty . \end{cases} \qquad (2.47)$$

The deceleration parameter is given by

$$q \begin{cases} \to 0.696 & \text{as} \quad t \to 0 , \\ = 0.595 & \text{when} \quad t = t_0 , \\ \to 0.500 & \text{as} \quad t \to \infty ; \end{cases} \qquad (2.48)$$

and the density parameter, $\frac{1}{6}\rho H^{-2}$, where $H = \frac{1}{3}\Theta$ is the averaged Hubble constant, decreases monotonically from 0.544 at $t = 0$ and approaches 0.5 as $t \to \infty$.

III. THE RADIAL SOLUTION

For simplicity, in this section we shall confine our attention to the case of a viscous fluid alone; i.e., we shall not include an electromagnetic field, and we shall consider only the Einstein–de Sitter model for which the form of the metric in spherical coordinates is

$$ds^2 = -dt^2 + t^{4/3}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) . \quad (3.1)$$

At the cost of some algebraic complexity, we can generalize the work described in this section to other $k = 0$ FRW models and also to include a magnetic field, as in the previous section.

In this case we choose a "tilting" velocity with spacelike component in the radial direction, i.e.,

$$u_\mu = (-\alpha, \beta t^{2/3}, 0, 0) , \quad (3.2)$$

where $\alpha^2 - \beta^2 = 1$, and α and β are assumed to be functions of t and r only. We choose q_μ to be of the form

$$q_\mu = Q(\beta, -\alpha t^{2/3}, 0, 0) ; \quad (3.3)$$

and, again assuming that the bulk viscosity is zero, we find that the viscous fluid field equations have the following nontrivial components:

$$\begin{aligned} \tfrac{4}{3}t^{-2} &= \rho\alpha^2 + p\beta^2 - \tfrac{4}{3}\eta X \beta^2 - 2Q\alpha\beta , \\ 0 &= \rho\beta^2 + p\alpha^2 - \tfrac{4}{3}\eta X \alpha^2 - 2Q\alpha\beta , \\ 0 &= (\rho + p) - \tfrac{4}{3}\eta X - Q(\alpha^2 + \beta^2)(\alpha\beta)^{-1} , \\ 0 &= p + \tfrac{2}{3}\eta X , \end{aligned} \quad (3.4)$$

where

$$X = \dot{\alpha} + (\beta' - \beta r^{-1})t^{-2/3} , \quad (3.5)$$

and the dot and the prime denote partial differentiation with respect to t and r, respectively. Equations (3.4) lead to

$$\rho = \tfrac{4}{3}\alpha^2 t^{-2} , \quad p = \tfrac{4}{9}\beta^2 t^{-2} , \quad Q = \tfrac{4}{3}\alpha\beta t^{-2} , \quad \eta X = -\tfrac{2}{3}\beta^2 t^{-2} , \quad (3.6)$$

so that we must have $X \leq 0$ in order that $\eta \geq 0$.

The baryon conservation equation (1.5) becomes

$$(\dot{n}\alpha + n\dot{\alpha} + 2n\alpha t^{-1}) + t^{-2/3}(n'\beta + n\beta' + 2n\beta r^{-1}) = 0 . \quad (3.7)$$

We shall seek a solution of this equation by assuming that each of the bracketed quantities is zero, i.e.,

$$\dot{n}\alpha + n\dot{\alpha} + 2n\alpha t^{-1} = 0 , \quad (3.8)$$

$$n'\beta + n\beta' + 2n\beta r^{-1} = 0 , \quad (3.9)$$

The solution of these two equations is

$$\alpha = [1 - H^2(t)K^2(r)]^{-1/2} , \quad \beta = H(t)K(r)[1 - H^2(t)K^2(r)]^{-1/2} , \quad (3.10)$$

$$n = n_0 K^{-1}(r)[1 - H^2(t)K^2(r)]^{1/2} r^{-2} t^{-2} , \quad (3.11)$$

where n_0 is a constant and $H(t)$ and $K(r)$ are arbitrary functions of t and r, respectively. We shall specify these functions by choosing

$$H(t) = \text{constant} \times t^c , \quad K(r) = \text{constant} \times r^b ,$$

where b and c are constant parameters, so that equations (3.10) and (3.11) become

$$\alpha = (1 - h^2 t^{2c} r^{2b})^{-1/2} , \quad \beta = h t^c r^b (1 - h^2 t^{2c} r^{2b})^{-1/2} , \quad (3.12)$$

$$n = n_0 t^{-2} r^{-b-2} \alpha^{-1} , \quad (3.13)$$

where h is an arbitrary constant parameter which may be positive or negative but not zero. Note that, unlike the solution of § II, this solution is valid only for certain values of the coordinates, namely the region for which $1 - h^2 t^{2a} r^{2b} \geq 0$.

By noting that the expressions (3.12) imply

$$\dot{\alpha} = c\alpha\beta^2 t^{-1} , \quad \dot{\beta} = c\alpha^2 \beta t^{-1} , \quad \alpha' = b\alpha\beta^2 r^{-1} , \quad \beta' = b\alpha^2 \beta r^{-1} ,$$

we find that the Gibbs relation (1.6) becomes

$$d(S/n) = \tfrac{4}{9} n_0^{-1} T^{-1} \alpha t^{-1} r^{b+1} \{ r\beta^2 (10c\alpha^2 + 2 - c) dt + t[10b\alpha^4 + (8 - 7b)\alpha^2 - 2] dr \} , \quad (3.14)$$

and the integrability condition for this expression is

$$\{T^{-1} r^{b+2} \alpha\beta^2 [10c\alpha^2 + 2 - c]\}' t^{-1} = \{T^{-1}\alpha[10b\alpha^4 + (8 - 7b)\alpha^2 - 2]\}^{\cdot} r^{b+1} . \quad (3.15)$$

In order to solve this equation for T we shall assume that T is of the form

$$T = T_0 t^m r^n \alpha^s , \quad (3.16)$$

where m, n, and s are constants. Substituting this expression into equation (3.15) and simplifying, we obtain a polynomial equation in β which must be satisfied at all points in the domain of validity of the solution. This leads to the following equations:

$$5cn - 5bm - (2bc - b + 4c)s = -bc + 3b - 2c , \quad (3.17)$$

$$(2 + 9c)n - (8 + 13b)m - 2c(b + 2)s = -2bc + 6b - 4c + 4 , \quad (3.18)$$

$$(b + 2)m = 0 . \quad (3.19)$$

By introducing the simplifying notation $x = ht^c r^b$, so that the domain of validity of the solution is $x^2 \leq 1$, we find that equations (3.3), (3.6), and (3.16) lead to the following expression for equation (1.8):

$$\kappa\{t^{-1/3} r x^2 [(cs - m - \tfrac{2}{3})x^2 + (c + m + \tfrac{2}{3})] + (b + bs - n)x^3 + nx\} = \tfrac{4}{3} x^2 T_0^{-1} t^{-m-4/3} r^{1-n} \alpha^{1-s} , \quad (3.20)$$

and the condition $\kappa \geq 0$ becomes

$$t^{-1/3} r x^2 [(cs - m - \tfrac{2}{3})x^2 + (c + m + \tfrac{2}{3})] + (b + bs - n)x^3 + nx \geq 0 . \quad (3.21)$$

From equations (3.5) and (3.6) the condition $\eta \geq 0$, i.e., $X \leq 0$, takes the form

$$x^3 + ct^{-1/3} r x^2 + (b - 1)x \leq 0 . \quad (3.22)$$

In order to find solutions satisfying all the required physical conditions we need to find values of b, c, m, n, and s which satisfy equations (3.17), (3.18), and (3.19) and which are such that the conditions (3.21) and (3.22) are satisfied throughout the entire domain of validity $x^2 \leq 1$. There are many such solutions, of which the following is a small selection.

Solution 1.—We choose

$$b = -1\,, \quad c = \tfrac{1}{3}\,, \tag{3.23}$$

and equations (3.17)–(3.19) yield

$$m = 0\,, \quad n = -\tfrac{1}{6}\,, \quad s = \tfrac{11}{6}\,. \tag{3.24}$$

In this case $t^{-1/3}r = hx^{-1}$, and equation (3.22) becomes

$$x(x^2 + \tfrac{1}{3}h - 2) \leq 0\,,$$

which is satisfied everywhere in $x^2 \leq 1$ if and only if $x > 0$ (i.e., $\beta > 0$, $h > 0$), and $0 < h \leq 3$. The condition (3.21) is found to be satisfied everywhere in $x^2 \leq 1$ if and only if $h \geq 3$. Hence all conditions are satisfied if and only if $h = 3$. Thus we have found the following solution which is valid in the spacetime region $9t^{2/3} \leq r^2$:

$$\alpha = (1 - 9t^{2/3}r^{-2})^{-1/2}\,,$$
$$\beta = 3t^{1/3}r^{-1}(1 - 9t^{2/3}r^{-2})^{-1/2}\,, \tag{3.25}$$

$$T = T_0 r^{-1/6}(1 - 9t^{2/3}r^{-2})^{-11/12}\,, \tag{3.26}$$

$$\kappa = \tfrac{24}{17}T_0{}^{-1}t^{-1}r^{1/6}(1 - 9t^{2/3}r^{-2})^{-7/12}\,, \tag{3.27}$$

$$\eta = 2t^{-1}(1 - 9t^{2/3}r^{-2})^{-1/2}\,, \tag{3.28}$$

$$n = n_0 t^{-2}r^{-1}(1 - 9t^{2/3}r^{-2})^{1/2}\,, \tag{3.29}$$

with ρ, p, and Q given by equations (3.6) and (3.25). The model has no vorticity, but the expansion and shear are given by

$$\Theta = 2t^{-1}(1 + 3t^{2/3}r^{-2})(1 - 9t^{2/3}r^{-2})^{-1/2}\,, \tag{3.30}$$

$$\sigma^2 = 3t^{-2/3}r^{-4}(1 - 9t^{2/3}r^{-2})^{-1}\,. \tag{3.31}$$

Note that Θ is always positive and all physical quantities are infinite on the bounding hypersurface $9t^{2/3} = r^2$ except for n, which is zero. Since $\beta > 0$, the spacelike component of the 4-velocity is in the outward radial direction.

Solution 2.—For this solution we choose

$$b = 1\,, \quad c = -\tfrac{1}{3}\,, \tag{3.32}$$

so that, from equations (3.17)–(3.19),

$$m = 0\,, \quad n = 12\,, \quad s = 8\,. \tag{3.33}$$

Now $t^{-1/3}r = h^{-1}x$, so the condition (3.22) leads to

$$x^3(1 - \tfrac{1}{3}h^{-1}) \leq 0\,,$$

and, since h and x have the same sign, we have that either

$$h < 0 \quad \text{or} \quad 0 < h \leq \tfrac{1}{3}\,.$$

However, the condition (3.21) is satisfied for all $x^2 \leq 1$ if and only if

$$h \geq \tfrac{1}{3}\,.$$

These two conditions on h are satisfied if and only if $h = \frac{1}{3}$, and the solution is then valid in the spacetime region $9t^{2/3} \geq r^2$, which is the complementary region to that of solution 1. The two solutions share a common singular bounding surface, and together they cover the entire spacetime region of the Einstein–de Sitter universe. Note that this value of h leads to $X = 0$, so that the shear is zero everywhere and the shear viscosity coefficient, η, is infinite everywhere. This is analogous to the situation in "perfect" magnetohydrodynamics (Lichnerowicz 1967) in which the electric field is zero and the current nonzero, so that the conductivity is infinite. The solution in detail is

$$\alpha = (1 - \tfrac{1}{9}t^{-2/3}r^2)^{-1/2}\,, \quad \beta = \tfrac{1}{3}t^{-1/3}r(1 - \tfrac{1}{9}t^{-2/3}r^2)^{-1/2}\,, \tag{3.34}$$

$$T = T_0 r^{12}(1 - \tfrac{1}{9}t^{-2/3}r^2)^{-4}\,, \tag{3.35}$$

$$\kappa = \tfrac{2}{9}T_0{}^{-1}t^{-5/3}r^{-10}(1 - \tfrac{1}{9}t^{-2/3}r^2)^{5/2}(6 + \tfrac{5}{9}t^{-2/3}r^2)^{-1}\,, \tag{3.36}$$

$$\eta \text{ infinite}\,, \tag{3.37}$$

$$n = n_0 t^{-2}r^{-3}(1 - \tfrac{1}{9}t^{-2/3}r^2)^{1/2}\,, \tag{3.38}$$

together with equations (3.6). This model has no vorticity or shear, but the expansion is given by

$$\Theta = 3t^{-1}(1 - \tfrac{1}{9}t^{-2/3}r^2)^{-1/2}\,, \tag{3.39}$$

and is always positive. Note that κ is zero on the singular boundary hypersurface (which is now an initial hypersurface) and not infinite, as in the previous solution.

Solutions 1 and 2, which were based on the assumption that the factor $t^{-1/3}r$ in the conditions (3.21) and (3.22) is proportional to a power of x, are "critical solutions" in the sense that they satisfy the conditions only for a particular value of h. However, solutions exist for which h is arbitrary, an example of which is:

Solution 3.—By choosing

$$b = -2\,, \quad c = -\tfrac{2}{9}\,, \tag{3.40}$$

we find that m is not defined by equation (3.19) and we have only the equations (3.17) and (3.18) to define m, n, and s. These can be satisfied by

$$m = -\tfrac{4}{9}\,, \quad n = \tfrac{16}{5}\,, \quad s = -1\,. \tag{3.41}$$

which result in the conditions (3.21) and (3.22) being satisfied for all $x^2 \leq 1$ provided that $h > 0$, so that again we have $\beta > 0$. The complete solution is

$$\alpha = (1 - h^2t^{-4/9}r^{-4})^{-1/2}\,,$$
$$\beta = ht^{-2/9}r^{-2}(1 - h^2t^{-4/9}r^{-4})^{-1/2}\,, \tag{3.42}$$

$$T = T_0 t^{-4/9}r^{16/5}(1 - h^2t^{-4/9}r^{-4})^{1/2}\,, \tag{3.43}$$

$$\kappa = \tfrac{5}{12}hT_0{}^{-1}t^{-2}r^{-21/5}(1 - h^2t^{-4/9}r^{-4})^{-2}\,, \tag{3.44}$$

$$\eta = \tfrac{2}{3}ht^{-14/9}r^{-1}(1 - h^2t^{-4/9}r^{-4})^{1/2} \times (3 + \tfrac{2}{9}ht^{-5/9}r^{-1} - h^2t^{-4/9}r^{-4})^{-1}\,, \tag{3.45}$$

$$n = n_0 t^{-2}(1 - h^2t^{-4/9}r^{-4})^{1/2}\,, \tag{3.46}$$

with ρ, p, and Q given by equations (3.6) and (3.42). The solution is valid in the region $t^{2/9}r^2 \geq h > 0$, where h is arbitrary. The expansion and shear of the velocity congruence u^μ are given by

$$\Theta = 2t^{-1}(1 - \tfrac{10}{9}h^2t^{-4/9}r^{-4} - h^3t^{-1/3}r^{-7}) \times (1 - h^2t^{-4/9}r^{-4})^{-3/2}\,, \tag{3.47}$$

$$\sigma^2 = \tfrac{1}{243}h^2t^{-26/9}r^{-8}(9h^2t^{1/9}r^{-3} - 2h - 27t^{5/9}r)^2\,, \tag{3.48}$$

and the vorticity is zero. Note that Θ is negative on the initial bounding hypersurface $h^2t^{-4/9}r^{-4} = 1$ and becomes positive as $t \rightarrow \infty$.

Finally, we note that a radial solution exists in which α and β are constants, namely:

Solution 4.—In this case

$$b = c = 0 , \tag{3.49}$$

and equations (3.17)–(3.19) yield

$$m = s = 0 , \quad n = 2 . \tag{3.50}$$

(In fact, s is arbitrary, but may be taken to be zero since α is a constant.) The conditions $\kappa \geq 0$ and $\eta \geq 0$ become, respectively,

$$\tfrac{2}{3}t^{-1/3} r \beta^2 \alpha^{-4} + 2\beta\alpha^{-3} \geq 0 ,$$

$$-\beta\alpha^{-2} \leq 0 ,$$

and these conditions are satisfied everywhere by any $\beta > 0$. The complete solution, which is valid at all points of the spacetime, is

$$\alpha = \text{constant} , \quad \beta = \text{constant} , \tag{3.51}$$

$$T = T_0 r^2 , \tag{3.52}$$

$$\kappa = 2T_0^{-1}\alpha^3 \beta t^{-4/3} r^{-2} (\beta t^{-1/3} + 3\alpha r^{-1})^{-1} , \tag{3.53}$$

$$\eta = \tfrac{2}{3}\beta t^{-4/3} r , \tag{3.54}$$

$$n = n_0 t^{-2} r^{-2} , \tag{3.55}$$

and equations (3.6). The expansion and shear of the velocity congruence u^μ are given by

$$\Theta = 2\alpha t^{-1} + 2\beta r^{-1} t^{-2/3} , \tag{3.56}$$

which is always positive, and

$$\sigma^2 = \tfrac{1}{3}\beta^2 t^{-4/3} r^{-2} . \tag{3.57}$$

IV. CONCLUSION

The solutions presented here are the first known solutions of the VMHD equations which also satisfy the thermodynamic relations (1.5) to (1.9). They are remarkable in that the spacetime metrics corresponding to these solutions of a set of complex field equations are, in fact, the metrics of the simplest known perfect fluid solutions, namely the zero-curvature FRW models. As in previous articles (Coley and Tupper 1983*a*, *b*; Tupper 1983*b*), we see that the FRW models and, in particular, the Einstein–de Sitter model, each correspond to an infinite set of interpretations as VMHD fluid models with the standard perfect fluid interpretation being just the special case in which the fluid velocity is comoving with the preferred observer.

We have taken a very conservative physical stance in seeking these solutions in that we have insisted that the temperature gradient law (1.9) should hold, that the viscous fluid and the electromagnetic field should, independently, satisfy the dominant energy condition, and that the pressure should be positive and decreasing. It may be possible to argue that some or all of the conditions could be relaxed; but, whatever conditions are substituted for these, there appears to be no reason for corresponding physically acceptable solutions not to exist.

The axial solution of § II is the simplest generalization of the standard perfect fluid FRW models in that all physical quantities are functions of time only. These models have an axial velocity, an axial magnetic field, and a shear viscosity which combine in such a way that the total stress-energy, and thus the spacetime geometry, is homogeneous and isotropic. The solutions given by the expression (2.24) for positive values of b and h form an infinite set of solutions each of which differs only slightly from the standard perfect fluid models at a time corresponding to the present value of t, but differs considerably from the standard model as one extrapolates back to $t \rightarrow 0$. Thus small amounts of axial magnetic field, viscosity, etc., in the universe at present would suggest a history for the model which is quite different from, and perhaps more physically realistic than, that of the standard models. On the other hand, the radial solutions, with their radial velocity and natural boundary, are less easy to interpret as cosmological models, but may be suitable for interpretation in some astrophysical context.

Our intention in this article is to show the existence of exact solutions of the field equations (1.1) which satisfy all the normally required conditions for physical acceptability, and for which the spacetime metric is that of the zero-curvature FRW model. We have, in fact, found many such solutions; the solutions presented here are chosen because they illustrate the essential properties of the solutions that we have found. Having demonstrated the existence of these solutions, we believe that priority should be given not to the generation of more solutions but to a discussion of the physical consequences of this work and its possible significance to current cosmology. We note that the observed cell structure of the universe (Jôeveer and Einasto 1978), which is predicted by the adiabatic theory of galaxy formation (Zel'dovich 1979), requires dissipative processes, such as viscosity, in the early universe. Furthermore, the adiabatic theory is based on the introduction of small-scale inhomogeneities in standard perfect fluid FRW models (Doroshkevich, Sunyaev, and Zel'dovich 1974) whereas it would be more natural to consider perturbations in a model with viscosity. Thus there do appear to be aspects of cosmology and astrophysics which which may benefit from an analysis of the models presented here. This analysis, together with a discussion of observations in these models, is in preparation and will be presented in due course.

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A. A. Coley: Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, B3H 3H8, Canada

B. O. J. Tupper: Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick E3B 5A3, Canada