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The amplification caused by gravitational bending of light

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Summary. Gravitational bending of light rays may not only lead to multiple imaging (gravitational lens effect), but also changes the apparent luminosity of a source. It is shown here that a general mass distribution always leads to an enhanced apparent luminosity relative to that one would observe in a lumpy universe if the lens were absent, for every relative position of source and observer. We then discuss that this does not violate flux conservation.

Key words: cosmology – gravitational lenses – quasars – galaxies – gravitation

1. Introduction

The gravitational bending of light rays, as predicted by the General Theory of Relativity, can lead to the gravitational lens effect, i.e. a source and an observer can be connected by more than one null geodesic. This causes the observer to see several images of the same source (see, e.g., Einstein, 1936; Refsdal, 1964a, b; Bourassa et al., 1973; Bourassa and Kantowski, 1975). In fact, the discovery of five multiple image quasars (Walsh et al., 1979; Weymann et al., 1980; Weedman et al., 1982; Lawrence et al., 1984; Djorgovski and Spinrad, 1983) shows, that the gravitational lens effect is realized in nature and has to be taken into account in the interpretation of cosmological observations.

This is mainly due to the fact that, aside from possible multiple imaging, gravitational bending of light rays gives rise to a change of apparent luminosity, compared with that one would see if no deflection occurs, so that, at least for some sources, we cannot derive the absolute luminosity of a source just from its apparent luminosity and redshift by means of a luminosity distance-redshift relation (Robertson, 1938). Especially, the $\ln N/\ln S$ relations, which are usually obtained by flux limited samples may be strongly influenced by this amplification effect caused by light bending, as this amplification gives rise to selection effects. Although the early proposal (Barnothy and Barnothy, 1965) that all quasars are gravitationally lensed Seyfert nuclei can be excluded (Tyson, 1981; Setti and Zamorani, 1983, and references therein) it is by no means clear to what extend the gravitational lens effect affects the results of cosmological observations. This question was investigated for several types of lenses (Canizares, 1982; Peacock, 1982; Setti and Zamorani, 1983; Turner, 1980); in particular, small-mass lens events are not detectable by multiple imaging as the image separation lies in the range of 10^{-5}–$10^{-6}\,(M/M_{\odot})^{1/2}$ arcsec; thus, they can only be proved by long time variability measurements (Canizares, 1982; Young, 1981) or by detection of the mutual coherence of their (unresolved) images (Schneider and Schmid-Burgk, 1984).

It is well-known that the Schwarzschild lens (the exterior of a spherically-symmetric mass distribution) always leads to amplification of at least one image (Refsdal, 1964a). This paper investigates the question whether there are lens geometries which lead to deamplification, where "amplification" means enhanced apparent luminosity relative to the unlensed source (see Sect. 5 below). We shall prove below, that *every transparent matter distribution causes an amplification factor* $\geqq 1$; hence, the Schwarzschild-lens is not an exceptional case. In Sect. 2 we formulate this theorem and discuss the assumptions under which it is valid. Section 3 presents a simple proof for the special case that the projected mass density of the lens is rotationally symmetric. In Sect. 4 we investigate the general case; Hereby, most of the assumption discussed in Sect. 2 will be dropped or at least very much weakened, so that the theorem should hold in nearly all situation of gravitational light bending. Finally (Sect. 5) we discuss this result in connection with cosmological apparent luminosity-redshift relations.

2. Formulation of the theorem

In this section, we derive the lens equations and discuss the validity of approximations used. We shall restrict ourselves to the case of geometrically thin lenses (in the sense that the transverse distance between the deflected and the undeflected ray is very much smaller than the length scale over which the gravitational potential varies significantly), to exclude multiple deflection of light rays. We also assume that the deflecting mass distribution is transparent (Bourassa and Kantowski, 1975), i.e. that every photon path traverses the lens; therefore, the Schwarzschild lens (Refsdal, 1964a, b) is excluded (but in that case we know that at least one image is amplified). Of course, gas in the lens may, apart from gravitating, absorb or scatter photons from a light beam, thus reducing the apparent luminosity of a source. The meaning of "amplification", therefore, is restricted to changes of apparent luminosity due solely to gravitational action of matter on a light bundle.

The third assumption we will make is the validity of linearized Einstein theory; hence, we require the deflection angle to the small (in the known cases of gravitational lensing the deflection angle is typically a few arcsecs). For a point mass, the deflection angle is given by $4G/c^2 M/r$, where M is the mass and r is the impact parameter of the light ray, provided it is much larger than the Schwarzschild radius $r_s = 2GM/c^2$. In linearized theory the total deflection angle is a sum of Schwarzschild angles if the extended mass distribution consists of point masses:

$$\boldsymbol{\alpha}=\sum_i \boldsymbol{\alpha}_i=\sum_i \frac{4G}{c^2}\frac{M_i}{|\boldsymbol{r}_i|}\frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|} \tag{1}$$

(Bourassa et al., 1973), where $\boldsymbol{r}_i$ is the impact vector for the point mass of mass M_i, and the sign of $\boldsymbol{\alpha}_i$ is chosen to be positive for deflection in the direction to the point mass M_i. In an exact treatment, the $\boldsymbol{r}_i$ should be the impact vectors of the distorted ray, but for small deflection angles and thin lenses the deflected ray will be not very different from the undisturbed one within the lens. The last two assumptions will be very much weakened later on (Sect. 4).

Consider a ray connecting a point source, which is far away from the deflecting mass distribution, and the "center" of this lens (which may be taken as the center of mass). Perpendicular to this ray we construct a plane through the mass distribution, which in the following will be called lens plane. Any ray from the point source, which crosses this plane near the lens, will in astrophysically relevant situations have a very small inclination angle with the normal of the plane, so they all hit the plane nearly perpendicularly. Therefore the total deflecting angle $\boldsymbol{\alpha}$ is given in sufficient approximation by

$$\boldsymbol{\alpha}(\boldsymbol{r})=\frac{4G}{c^2}\int\frac{\Sigma(\boldsymbol{r}')}{|\boldsymbol{r}'-\boldsymbol{r}|^2}(\boldsymbol{r}-\boldsymbol{r}')d^2r' \tag{2}$$

where Σ is the mass density projected into the lens plane [g cm^{-2}] (Bourassa and Kantowski, 1975) and the integral has to be taken over the whole plane. $\boldsymbol{\alpha}(\boldsymbol{r})$ is the deflection which a ray undergoes when crossing the lens plane at $\boldsymbol{r}$.

In Fig. 1 a typical lens geometry is illustrated. An observer at $\boldsymbol{x}$, whose distance from the lens is D_d, can see a point source S, which is at a distance D_{ds} above the lens plane, if there is a light ray emitted at S with an angle $\boldsymbol{\beta}$ with respect to the optical axis, SO, which hits the lens plane at $\boldsymbol{r}$ and is deflected by an angle $\boldsymbol{\alpha}(\boldsymbol{r})$, provided that

$$\boldsymbol{r}=D_{ds}\boldsymbol{\beta}\,, \tag{3a}$$

$$\boldsymbol{x}=(D_d+D_{ds})\boldsymbol{\beta}-D_d\boldsymbol{\alpha}(\boldsymbol{r})\,, \tag{3b}$$

where we have used $\tan(\beta)\approx\beta$ for the small angles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Eliminating $\boldsymbol{\beta}$, we get the lens equation

$$\boldsymbol{x}=\boldsymbol{r}\frac{D_d+D_{ds}}{D_{ds}}-D_d\boldsymbol{\alpha}(\boldsymbol{r}) \tag{4}$$

which describes the mapping of the lens plane into the observer plane, which is taken parallel to, and at a distance D_d from the lens plane.

Note that in general, where the light deflection takes place in an expanding universe, the distance between the source and the observer is not simply $D_s=D_d+D_{ds}$ (for a discussion of distance-redshift relations in lens theory see e.g. Kayser and Refsdal, 1983); however, this does not affect the validity of the following discussion.

We now write the lens Eq. (4) in the simplest form by defining

$$\tilde{\boldsymbol{x}}=D_{ds}/(D_d+D_{ds})\boldsymbol{x}$$

and

$$\tilde{\Sigma}(\boldsymbol{r})=4G/c^2D_dD_{ds}/(D_d+D_{ds})\Sigma(\boldsymbol{r});$$

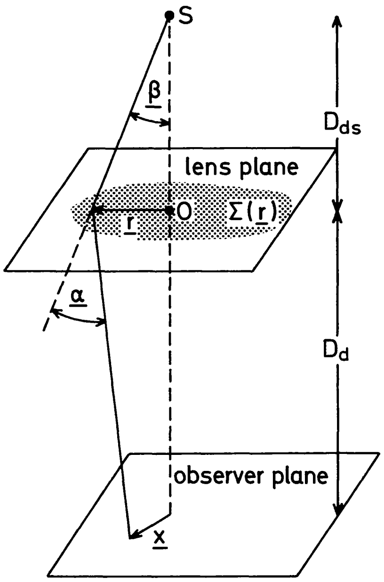

Fig. 1. The geometry of gravitational bending of light

we get

$$\tilde{\boldsymbol{x}}=\boldsymbol{r}-\tilde{\boldsymbol{\alpha}}(\boldsymbol{r}) \tag{5}$$

$$\tilde{\boldsymbol{\alpha}}(\boldsymbol{r})=\int\tilde{\Sigma}(\boldsymbol{r}')\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^2}d^2\boldsymbol{r}'\,. \tag{6}$$

In the following we will drop the *tilde*.

It is straightforward, for a given mass distribution Σ, to calculate the image $\boldsymbol{x}$ of $\boldsymbol{r}$. However, the inverse question, which points $\boldsymbol{r}$ are mapped to a given point $\boldsymbol{x}$, is very complicated and has been solved exactly only for the simplest mass distributions (Refsdal, 1964a; Bourassa and Kantowski, 1975; Chang and Refsdal, 1979; Chang, 1981). Burke (1981) has proved that for transparent lenses there is an odd number of $\boldsymbol{r}$'s for every $\boldsymbol{x}$; of course, this does not apply if one considers star disturbances in lensing galaxies (Chang and Refsdal, 1979, 1984), for they are not transparent.

The amplification factor I of a light bundle is given by the inverse Jacobian of the mapping (5), when the approximation of geometrical optics applies; this is always the case except near to the "critical lines" (Ohanian, 1983), where the Jacobian vanishes; hence,

$$I=\left[\det\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{r}}\right]^{-1}, \tag{7}$$

where negative values of I belong to images with negative parity, where, for example, a right-handed pair of vectors is mapped into a left-handed pair. Of course, for certain $\boldsymbol{r}$, $|I|$ may become less than one, but we *shall prove that for every observer position* $\boldsymbol{x}$ *there is an inverse image* $\boldsymbol{r}$ *for which* $|I|$ *is greater than or equal to 1.*

In Sect. 3 we prove this theorem for the special case of a rotationally symmetric mass distribution $\Sigma(\boldsymbol{r})=\Sigma(|\boldsymbol{r}|)$ where one could easily see that this theorem holds. In Sect. 4 we will treat the general case, which is a bit more technical.

3. The rotationally symmetric case

Because of its simplicity, the symmetric case will enable us to understand why the theorem is expected to hold; furthermore, it will be instructive to compare results which are derived in Sect. 4 with the symmetric lens.

Let $\Sigma(\boldsymbol{r})=\Sigma(|\boldsymbol{r}|)$ be the mass density which enters (6); introduce polar coordinates R, ϕ in the lens plane, then for the point $\boldsymbol{r}=\begin{pmatrix}\varrho\\0\end{pmatrix}$ the deflection angle is given by

$$\boldsymbol{\alpha}\begin{pmatrix}\varrho\\0\end{pmatrix}=\int_0^\infty dR\,R\,\Sigma(R)\int_0^{2\pi}d\phi \cdot\frac{1}{(R\cos\phi-\varrho)^2+(R\sin\phi)^2}\begin{pmatrix}\varrho-R\cos\phi\\-R\sin\phi\end{pmatrix}. \quad (8)$$

One easily sees that the second component of this equation vanishes as is expected from the symmetry; the first component can be written as ($\mu\equiv\varrho/R$)

$$\alpha_x\begin{pmatrix}\varrho\\0\end{pmatrix}=\int_0^\infty dR\,\Sigma(R)\int_0^{2\pi}d\phi\frac{\mu-\cos\phi}{1-2\mu\cos\phi+\mu^2}. \quad (9)$$

This second integral vanishes for $\mu<1 (R>\varrho)$, and for $\mu\geqq1$ it has the value $2\pi/\mu$, thus

$$\alpha_x\begin{pmatrix}\varrho\\0\end{pmatrix}=\frac{1}{\varrho}\int_0^\varrho 2\pi R\,dR\,\Sigma(R)\equiv\frac{m(\varrho)}{\varrho}, \quad (10)$$

where $m(\varrho)$ is the total mass enclosed by the circle of radius ϱ; the mass outside this circle does not contribute to the light deflection. The generalization of (10),

$$\boldsymbol{\alpha}(\boldsymbol{r})=\frac{m(|\boldsymbol{r}|)}{r^2}\boldsymbol{r} \quad (11)$$

leads to the Jacobian matrix $A\equiv\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{r}}$,

$$A=\begin{pmatrix}1-\frac{m(r)}{r^2}+\frac{2mr_x^2}{r^4}-\frac{m'r_x^2}{r^3} & \frac{2mr_xr_y}{r^4}-\frac{m'r_xr_y}{r^3}\\ \frac{2mr_xr_y}{r^4}-\frac{m'r_xr_y}{r^3} & 1-\frac{m}{r^2}+\frac{2mr_y^2}{r^4}-\frac{m'r_y^2}{r^3}\end{pmatrix}, \quad (12)$$

where $r=|\boldsymbol{r}|$ and $m'(r)=dm/dr$. The determinant of A is

$$\det A=\left(1-\frac{m}{r^2}\right)\left(1-\frac{d}{dr}\frac{m}{r}\right), \quad (13)$$

and its trace

$$\operatorname{tr}A=2-\frac{m'}{r}. \quad (14)$$

We now want to show that for every positive $x=|\boldsymbol{x}|$ there exist a positive r which satisfies the equation

$$x(r)=r-\frac{m}{r} \quad (15)$$

and for which $\det A$ is in the range $[0, 1]$. As $x(0)=0(m(r)$ rises as $\pi r^2\Sigma(0)$ for small r) and for large argument, r, $x(r)\to r$, and $r\to x$ is continuous and C^1 in $0\leqq r<\infty$, we know that we can always find, for every $x>0$ a value of $r\geqq0$ which satisfies (15) and where $dx/dr>0$; hence at this point, $1-m/r^2>0$ and $1-d/dr(m/r)>0$, so that $\det A$ is positive. Rewriting (13) as

$$\det A=\left(1-\left(\frac{m}{r^2}\right)^2\right)-m'/r(1-m/r^2), \quad (16)$$

we see that at the point considered $\det A\leqq1$, because $m'\geqq0$; thus we get the required result

$0<\det A\leqq1 \quad (<1$, if $m\neq0)$

and hence

$$I=(\det A)^{-1}\geqq1 \quad (17)$$

for at least one r which satisfies (15) (for every observer position x off the axis). We therefore conclude that for rotationally symmetric mass distributions the gravitational lens effect always leads to amplification of at least one image.

From (13) one can read off that the critical lines, which are circles, are given by the condition, that $a_1\equiv1-m/r^2=0$ or $a_2\equiv1-d/dr(m/r)=0$. In the first case, $a_1=0$, the critical line is mapped into the point $x=0$; one can also notice that $\operatorname{tr}A=a_1+a_2$, so there are two different kinds of regions where $\det A>0$, namely those where $a_1, a_2<0$, $\operatorname{tr}A<0$, and those where $a_1, a_2>0$, $\operatorname{tr}A>0$. As we shall see in Sect. 4, a similar distinction can be found for the general lens.

As a simple example, consider the case of a homogeneous mass disk of radius R, so that

$$m(r)=\begin{cases}\pi r^2\Sigma(0) & r<R\\ \pi R^2\Sigma(0) & r>R;\end{cases} \quad (18)$$

then, for $r<R$, $a_1\lesseqgtr0$ if $\pi\Sigma(0)\gtreqless1$, and $a_2\lesseqgtr0$ if $\pi\Sigma(0)\gtreqless1$; so in any case, we get for $r<R$ $\det A>0$ and $\operatorname{tr}A\lesseqgtr0$ depending on $\pi\Sigma(0)\gtreqless1$. All light rays which pass through the lens plane at $r<R$ go through a focus; depending on $\Sigma(0)$, this focus lies between the lens and observer (for $\pi\Sigma(0)>1$] or behind the observer ($\pi\Sigma(0)<1$). If $\pi\Sigma(0)>2$, we get $\det A>1$, i.e. deamplification. This Ricci focussing is due to matter in the light beam.

In the case that $\pi\Sigma(0)>1$, then for $r>R$, we have $a_2>0$, so that $r=R$ is one critical line; a_1 gets 0 for $r=(\pi\Sigma(0))^{1/2}R$ which is the second critical line, which is mapped, according to (15), into the point $x=0$. We thus have a ring between those critical lines where we get images of negative parity. Therefore, if $|\boldsymbol{x}|<R(\pi\Sigma(0)-1)$ the observer at $\boldsymbol{x}$ notices three images of the point source S. For $\pi\Sigma(0)<1$, no multiple imaging occurs.

Although this example is quite trivial, it shows all the important features we will meet in the next section. It should be noticed that there is always an image $\boldsymbol{r}$, for which $\det A>0$ and $\operatorname{tr}A>0$; for this image, $I>1$. The reasons for the occurrence of such an image lies in the fact that, provided $\pi\Sigma(0)>1$, the mass inside the circle $r=R$ is large enough to bend rays, which hit the lens plane just outside the region $r\leqq R$, at least to the center of the observer plane, $x=0$. For these light rays the mass distribution (18) acts like a simple Schwarzschild lens (Refsdal, 1964a), and therefore it always leads to amplification.

We now turn to the general transparent lens, where the arguments used are somewhat more elaborate, but the line of reasoning is the same as in this section.

4. The general case

We first note, that the deflection angle $\boldsymbol{\alpha}(\boldsymbol{r})$ (6) can be written as a gradient of the scalar function $\psi(\boldsymbol{r})$, $\boldsymbol{\alpha}(\boldsymbol{r})=\operatorname{grad}\psi(\boldsymbol{r})$, where

$$\psi(\boldsymbol{r})=\int \Sigma(\boldsymbol{r}')\ln|\boldsymbol{r}-\boldsymbol{r}'|d^2r', \tag{19}$$

which is the Coulomb potential in two dimensions. From this one concludes that the lens equation, $\boldsymbol{x}=\boldsymbol{r}-\boldsymbol{\alpha}(\boldsymbol{r})$ can be expressed in terms of another scalar function as

$$\boldsymbol{x}=\operatorname{grad}\phi, \tag{20a}$$

where

$$\phi(\boldsymbol{r})=\frac{|\boldsymbol{r}|^2}{2}-\psi(\boldsymbol{r}). \tag{20b}$$

In terms of ϕ, the Jacobian matrix is

$$A\equiv\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{r}}=\begin{pmatrix}\phi_{xx} & \phi_{xy}\\ \phi_{yx} & \phi_{yy}\end{pmatrix}, \tag{21}$$

and we use the notation $\phi_{ij}=\frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}$; the interchangeability of partial derivatives ensures the symmetry of A.

The determinant of A is given by

$$\begin{aligned}\det A&=\phi_{xx}\phi_{yy}-(\phi_{xy})^2\\ &=1-\Delta\psi+[\psi_{xx}\psi_{yy}-(\psi_{xy})^2],\end{aligned}$$

where $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$ denotes the two-dimensional Laplacian.

If one defines $\mu=\psi_{xy}$, $\lambda=\frac{1}{2}(\psi_{xx}-\psi_{yy})$, one sees that the determinant can be reexpressed as

$$\det A=(1-\tfrac{1}{2}\Delta\psi)^2-\lambda^2-\mu^2, \tag{22}$$

and the trace is given by

$$\operatorname{tr}A=\Delta\phi=2-\Delta\psi. \tag{23}$$

To calculate $\Delta\psi$ we first remark that $\boldsymbol{\alpha}(\boldsymbol{r})$ is a continuously differentiable function provided Σ is C°, has compact support and is C^1 on the support (these are no serious restrictions, for even the projection of a homogeneous sphere fulfils these conditions); thus, the second derivatives of ψ are well-defined; especially, $\Delta\psi$ is given by

$$\Delta\psi(\boldsymbol{r})=\int \Sigma(\boldsymbol{r}')\Delta\ln|\boldsymbol{r}-\boldsymbol{r}'|d^2r'. \tag{24}$$

The Laplacian in polar coordinates is

$$\Delta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+\frac{1}{r^2}\frac{d^2}{d\phi^2};$$

from this one concludes that $\Delta\ln|\boldsymbol{r}|$ vanishes for $\boldsymbol{r}\neq 0$; thus it is proportional to the Dirac δ-function. To obtain the constant of proportionality, we integrate $\Delta\ln|\boldsymbol{r}|$ over a circle of radius R:

$$\begin{aligned}\int_A \Delta\ln|\boldsymbol{r}|d^2r&=\int_A \operatorname{div}\operatorname{grad}\ln|\boldsymbol{r}|d^2r\\ &=\oint_A\left(\frac{d}{dx}\ln|\boldsymbol{r}|dy-\frac{d}{dy}\ln|\boldsymbol{r}|dx\right).\end{aligned}$$

At a point $R(\cos\phi,\sin\phi)$, $dx=-R\sin\phi d\phi$, $dy=R\cos\phi d\phi$, and $\operatorname{grad}\ln|\boldsymbol{r}|=1/R(\cos\phi,\sin\phi)$; therefore the last integral is independent of R and has the value 2π. Hence,

$$\Delta\ln|\boldsymbol{r}|=2\pi\delta^2(\boldsymbol{r}),$$

and from (24) we have

$$\Delta\psi(\boldsymbol{r})=2\pi\Sigma(\boldsymbol{r}),$$

which gives

$$\det A=(1-\pi\Sigma)^2-\lambda^2-\mu^2, \tag{25}$$

$$\operatorname{tr}A=2(1-\pi\Sigma). \tag{26}$$

The first term in (25) is a local quantity, describing the effect of material in the light beam; it is called Ricci-focussing, while $\lambda^2+\mu^2$ is a non-local quantity, called shear. (26) was first derived by Young (1981).

As Σ is a non-negative function, $\operatorname{tr}A\leqq 2$. If at a point $\boldsymbol{r}$, $\det A(\boldsymbol{r})\geqq 0$ and $\operatorname{tr}A(\boldsymbol{r})\geqq 0$, we have $(1-\pi\Sigma)\varepsilon[0,1]$ and $\det A\varepsilon[0,1]$, which implies that the amplification factor $I=(\det A)^{-1}\geqq 1$. We now show that for every observer position $\boldsymbol{x}$ there exists always an $\boldsymbol{r}$ which satisfies the lens equation (5) and where $\det A\geqq 0$ and $\operatorname{tr}A\geqq 0$.

Let the observer be at $\boldsymbol{x}=\boldsymbol{a}$; he will see an image of the source at the point $\boldsymbol{r}$ if $\boldsymbol{r}-\boldsymbol{\alpha}(\boldsymbol{r})=\boldsymbol{a}$, or where the vector field $\boldsymbol{y}(\boldsymbol{r})=\boldsymbol{r}-\boldsymbol{\alpha}(\boldsymbol{r})-\boldsymbol{a}$ vanishes. We express this new vector field as a gradient over a scalar function $\theta(\boldsymbol{r})$,

$$\boldsymbol{y}(\boldsymbol{r})=\operatorname{grad}\theta(\boldsymbol{r}), \tag{27a}$$

where

$$\theta(\boldsymbol{r})=\phi(\boldsymbol{r})-\boldsymbol{a}\cdot\boldsymbol{r}=\frac{|\boldsymbol{r}|^2}{2}-\psi(\boldsymbol{r})-\boldsymbol{a}\cdot\boldsymbol{r}. \tag{27b}$$

For large $|\boldsymbol{r}|$, $\theta(\boldsymbol{r})$ increases as $|\boldsymbol{r}|^2/2$, thus it must have a minimum. (Note that the conditions imposed on Σ stated below (23) imply that the total lens mass is finite). Let $\boldsymbol{r}_0$ be the location of a minimum, then θ must be minimal along every curve through $\boldsymbol{r}_0$. Define the functions

$$c_\phi(t)=\theta(\boldsymbol{r}_0+t\boldsymbol{e}_\phi),$$

where $\boldsymbol{e}_\phi=(\cos\phi,\sin\phi)$, then the condition that θ has a minimum at $\boldsymbol{r}_0$ translates into

$$\dot{c}_\phi(0)=0 \quad\text{and}\quad \ddot{c}_\phi(0)\geqq 0$$

for every ϕ. The first condition implies $\operatorname{grad}\theta(\boldsymbol{r}_0)=\boldsymbol{y}(\boldsymbol{r}_0)=0$, thus the point $\boldsymbol{r}_0$ satisfies the lens equation [besides, this shows that any point in the observer plane is hit by the mapping (5)].

The second condition implies

$$\theta_{xx}\cos^2\phi+2\theta_{xy}\cos\phi\sin\phi+\theta_{yy}\sin^2\phi\geqq 0$$

or

$$\boldsymbol{e}_\phi A\boldsymbol{e}_\phi\geqq 0; \tag{28}$$

note that according to the definition (27b), $\theta_{ij}=\phi_{ij}$. Choosing $\phi=0$ and $\pi/2$ in (28), one gets $\theta_{xx}\geqq 0$ and $\theta_{yy}\geqq 0$, respectively; hence,

$$\operatorname{tr}A\geqq 0. \tag{29}$$

The determinant of A is generally defined as

$$\det A=\boldsymbol{e}_1A\boldsymbol{e}_1\cdot\boldsymbol{e}_2A\boldsymbol{e}_2-\boldsymbol{e}_1A\boldsymbol{e}_2\cdot\boldsymbol{e}_2A\boldsymbol{e}_1, \tag{30}$$

where e_1 and e_2 form a orthonormal basis in R^2; especially we can write them as $e_1=(\cos\phi, \sin\phi)$, $e_2=(-\sin\phi, \cos\phi)$. As A is symmetric, one can choose ϕ such that the second term in (30) vanishes; this happens if the equation

$$\tan^2\phi+\left(\frac{\theta_{xx}-\theta_{yy}}{\theta_{xy}}\right)\tan\phi-1=0$$

is satisfied (if $\theta_{xy}=0$, choose $\phi=0$). From (28) and (30) we finally have

$$\det A\geqq 0\,, \tag{31}$$

which together with (29) implies $I\geqq 1$, so an observer at $\boldsymbol{a}$ sees a non-deamplified image at $\boldsymbol{r}_0$. This proves the general fact that every geometrically thin, transparent lens leads to at least one image of a source which amplification factor is greater than or equal to one.

What happens if the lens is not transparent everywhere? From the proof it is evident that the only condition that matters is the existence of a point $\boldsymbol{r}_0$ where θ has a minimum. If the lens is transparent in a surrounding of $\boldsymbol{r}_0$, an observer at $\boldsymbol{a}$ will see an image with $I\geqq 1$. From (25) and (26), at $\boldsymbol{r}_0$, $\pi\Sigma\leqq 1$, so that regions of non-transparency with $\pi\Sigma>1$ will not influence the result of the theorem. To give physical numbers, a surface mass density of ϱg/cm^2 leads to $\pi\Sigma=16.7\,\varrho\eta$/h, where η is $D_dD_{ds}/(D_d+D_{ds})$ in units of Hubble length, c/H_0, and $H_0=h\cdot 50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Furtermore, if the shear $\mu^2+\lambda^2$ is large enough in a region of opacity, then no point with $\det A\geqq 0$, $\mathrm{tr}\,A\geqq 0$ will occur in that region. For example, consider a star in a distant galaxy; the region where $\det A<0$ or $\pi\Sigma>1$ has a radius of roughly a few $10^{16}\,(M/M_\odot)^{1/2}$ cm for reasonable redshifts of the star and the source; this of course is much larger than the region of non-transparency of this star. We therefore expect that the theorem also holds for real galaxies and clusters which contain stars.

The assumption that the gravitational fields are weak can also be weakened. If, for example, the lens contains a black hole of mass M, the lens equation no longer holds for values of $\boldsymbol{r}$ near to the hole. But the region where $\det A<0$ has a radius of $R=(\eta c/H_0 4GM/c^2)^{1/2}$, so that $R/r_s=2.5\,10^{11}\,(\eta/h)^{1/2}\,(M/M_\odot)^{-1/2}$; thus, even for very massive black holes the strong field region is very much smaller than the region of negative parity. Therefore, the weak field (or linearized) approximation should hold in all points of interest.

5. Discussion

Starting with the simplest form of the lens equation we have shown that any mass distribution, lying between the source and the observer, leads to amplification, i.e. the apparent luminosity of at least one image is larger than it would be without light bending – as long as the regions of opacity within the mass distribution are small enough in the sense discussed above. There has been some confusion in the literature (e.g., Turner, 1980; Avni, 1981; Peacock, 1982; Canizares, 1982) about the point that light bending must not violate the law of flux conservation; this point deserves some clarifying comments (Weinberg, 1976; Peacock, 1983).

To discuss this we first remark that gravitational lenses of astrophysical interest are imbedded in the universe, that is, the lensing event takes place within a curved space-time. Given the geometry of the space-time manifold, the apparent luminosity of a given source depends on its position relative to an observer as well as on the properties of null geodesics on this manifold.

In the standard Friedman-Robertson-Walker (FRW) model the universe is described as being filled with homogeneous, isotropic ideal fluid. For this model, the null geodesics are known and the apparent luminosity can be calculated for a source of given absolute luminosity and redshift (Robertson, 1938). In particular, the propagation of light takes place in this perfect fluid; thus there is matter in the light beam which causes Ricci focussing.

While the perfect fluid description of the universe may be justified with regards to the large-scale geometry of the universe, it is certainly not correct to account for the details of light propagation; especially, a light ray from a distant galaxy which reaches us has not traversed a perfect fluid of the average mass density of the universe (Zel'dovich, 1964; Refsdal, 1970). To account for this, Dyer and Roeder (1972) have considered the light propagation in a lumpy universe, i.e. a universe where the large-scale geometry is of the FRW type but where the matter is concentrated in lumps, such as galaxies. A light bundle which happens to miss any lump will not experience Ricci focussing; so for a given source the apparent luminosity will be less than that in the standard FRW-model. Later (Dyer and Roeder, 1973) they extended their discussion to the case that there is some smeared-out matter, besides the lumps.

Weinberg (1976) gave the following argument: consider a source and a sphere, centered on it, with fixed radius (measured, say, with redshift), in a universe of given large-scale geometry. If the matter within that sphere is transparent, the integrated flux over the sphere of the light emanating from the source must be independent of the matter distribution within the sphere; hence, if the matter is concentrated in lumps, the *average* apparent luminosity must be the same as it were in a pure FRW-model. He thus concluded, that *on the average the gravitational lens effect must lead to an amplification relative to the luminosity-redshift relation obtained for the lumpy universe*.

Thus our result does not violate the law of flux conservation, as the amplification we consider is the *amplification relative to the lumpy universe model*, or more general, amplification relative to the case where the lens mass were absent (and not smoothed out!)

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