THE DEFINITION OF THE ECLIPTIC

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Abstract. The ecliptic as a mean orbital plane of the Sun in Le Verrier's theory is a mean orbital plane determined from the secular parts of the longitude of the ascending node and the inclination of the Sun with respect to a reference plane. On the other hand, the ecliptic in Newcomb's theory is so chosen that the latitude with respect to his ecliptic does not have $\cos g$ nor $\sin g$ where g is the mean anomaly of the Sun. The two definitions are really different in spite of their apparent similarity. Standish (1981) defined the ecliptic from a kinematical point of view, and it is shown that the ecliptic defined by Standish (in the rotating sense) does coincide with the ecliptic defined by Newcomb.

1. Introduction

The ecliptic, which is one of the fundamental reference planes in the dynamics of the solar system and astrometry, is usually understood as a mean orbital plane of the Sun or more exactly the barycenter of the Earth and the Moon system respect to the barycenter of the solar system. The definition of the ecliptic as a mean orbital plane of the Sun seems to be unique but the various authors have given various definitions of the ecliptic. In this paper, we discuss such various definitions of the ecliptic and investigate the relationships among them. As a result, we recognize only the two different ones in principle.

The first one comes from the Le Verrier's theory and is a mean orbital plane determined from the secular parts of the longitude of the ascending node and the inclination of the Sun with respect to an inertial reference frame. His definition of the ecliptic is simple from a theoretical point of view. We call the ecliptic coordinates by Le Verrier simply Le Verrier's framework. If we use Le Verrier's framework, the observed declinations of the Sun near the soltices have a constant perturbation and the observed declinations of the Sun is not zero near the equinoxes. The second one comes from Newcomb who, avoiding the above situation, defined the ecliptic so that the latitude with respect his ecliptic does not have $\cos u$ or $\sin u$ (u is the argument of latitude). Newcomb did not explicitly mention the above definition, but we could only guess what Newcomb's ecliptic is from his Solar Tables. In fact his tables do not contain terms with argument u in the periodic perturbations. In Section 2 we discuss the relationships between Le Verrier's framework and Newcomb's framework and give the numerical differences between them. Newcomb defined ecliptic from a geometrical point of view. In Section 3 we give a kinematical interpretation of Newcomb's framework. Standish (1981) defined the mean ecliptic in the rotating

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framework. In Section 4 we show the ecliptic defined by Standish does coincide with the ecliptic defined by Newcomb.

We recommend that the definition by Newcomb is better from a rather intuitive point of view, similar to the definitions of the eccentricity and inclination of the Moon by Brown in his Lunar Theory. Furthermore, the observation analyses have been referred to his framework so far, and thus Newcomb's definition provides us with the continuity of the results.

2. Newcomb's Definition of the Ecliptic

The coordinates **r** of the Sun referred to an inertial reference frame, say the equator at an epoch, are expressed by

$$\mathbf{r} = R_3(-\Omega)R_1(-I)\begin{bmatrix} r\cos u \\ r\sin u \\ 0 \end{bmatrix}, \tag{1}$$

$R_1(\theta)$ and $R_3(\theta)$ are rotational matrices by the angle θ around the x-axis and the z-axis, respectively, and are given explicitly by Mueller (1969). Ω is an osculating longitude of the ascending node and I is an osculating inclination (obliquity) with respect to a fixed reference plane, i.e., the equator at a definite epoch, and u is the argument of latitude. Ω and I are expressed by

$$\Omega = \Omega_s + \delta\Omega_p \quad \text{and} \quad I = I_s + \delta I_p, \tag{2}$$

where the subscripts s and p stand for the secular part and periodic part of each element, respectively. Here the secular parts mean those which do not depend on longitudes of the Sun and disturbing planets among the perturbations. Now we introduce a moving reference plane, P_s, defined by Ω_s and I_s (see Figure 1): the $\tilde{x}$-axis is along the ascending node of this plane and the $\tilde{y}$-axis is in this plane. The coordinates of the Sun referred to this plane are given by

$$\tilde{\mathbf{r}} = \begin{bmatrix} r\cos\tilde{u}\cos\tilde{\beta} \\ r\sin\tilde{u}\cos\tilde{\beta} \\ r\sin\tilde{\beta} \end{bmatrix} = R_1(I_s)R_3(\Omega_s)\mathbf{r}. \tag{3}$$

We call $\tilde{\mathbf{r}}$-coordinates the ecliptic coordinates by Le Verrier or simply Le Verrier's framework. Substituting Equations (1) and (2) into (3) and keeping first-order terms with respect to $\delta\Omega_p$ and δI_p, we have

$$\tilde{u} = u + \delta\Omega_p \cos I_s, \tag{4}$$

$$\sin\tilde{\beta} \doteqdot \tilde{\beta} = -\delta\Omega_p \sin I_s \cos u + \delta I_p \sin u. \tag{5}$$

The periodic perturbations, $\delta\Omega_p$ and δI_p due to disturbing planets, have terms of argument $2u$:

$$\sin I_s \delta\Omega_p = A\sin 2u + B\cos 2u + (\text{periodic terms}), \tag{6}$$

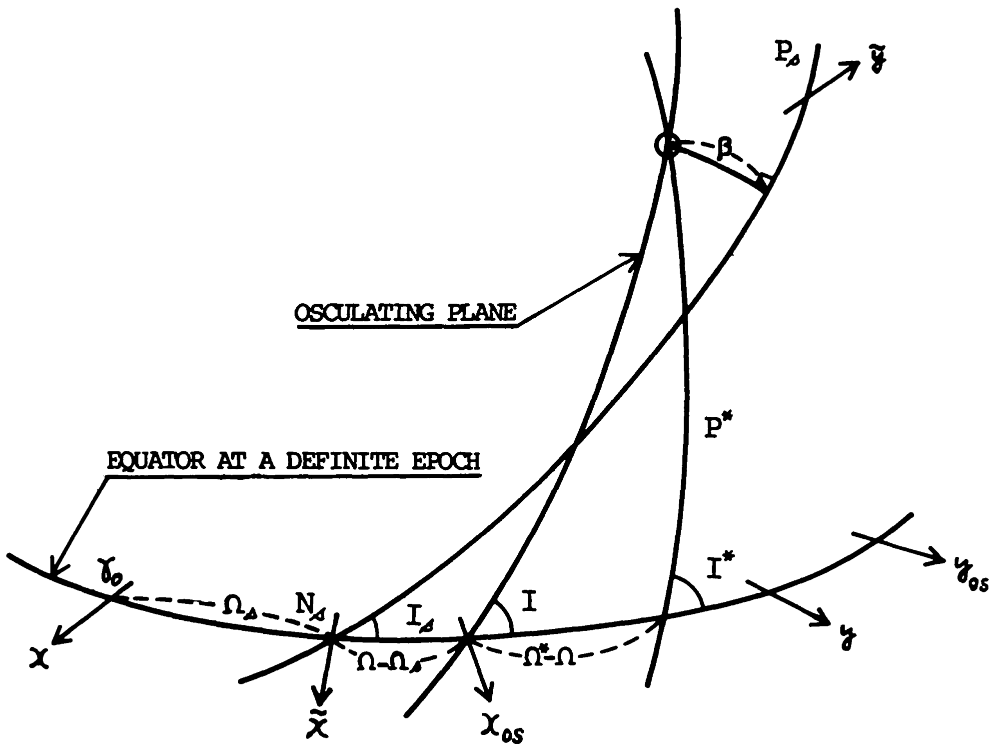

Fig. 1. Relations of various coordinate systems: P_s is the mean ecliptic defined by Le Verrier; P^* is the osculating plane referred to the moving Le Verrier coordinate system (P_s).

and

$$\delta I_p = C \sin 2u + D \cos 2u + \text{(periodic terms)}.$$

From the terms with argument $2u$, periodic terms with argument u appear in the expression of the latitude:

$$\tilde{\beta} = M \sin u + N \cos u + \text{(periodic terms)}, \tag{7}$$

where

$$M = -(A + D)/2 \quad \text{and} \quad N = (C - B)/2.$$

The remaining periodic terms include terms with argument $3u$, which are extremely small and are ignored in the following discussion. The obliquity of the ecliptic is well determined from observations of the declinations of the Sun around the soltices. If we use the ecliptic as a mean orbital plane defined by Ω_s and I_s, observed declinations near the soltices ($u \doteqdot +\pi/2$) include always M and the latitude is not zero when $u \doteqdot 0$. In order to avoid this difficulty arising from use of this ecliptic, we introduce another moving reference plane defined by

$$\Omega_N = \Omega_s + \delta'\Omega \quad \text{where} \quad \delta'\Omega = -N/\sin I_s, \tag{8}$$

and

$$I_N = I_s + \delta' I \quad \text{where} \quad \delta' I = M.$$

The latitude referred to this new reference plane is

$$\beta_N = \tilde{\beta} + \sin I_s \cos u \, \delta'\Omega - \sin u \, \delta' I$$
$$= (M - \delta' I) \sin u + (N + \sin I_s \delta'\Omega) \cos u + (\text{periodic terms})$$
$$= 0 \times \sin u + 0 \times \cos u + (\text{periodic terms}). \quad (9)$$

The ecliptic in Newcomb's Solar Tables is so defined that the ecliptic does not have the motion of short periodic terms with respect to a fixed reference plane and that the latitude referred to this ecliptic (or the latitude with respect to Newcomb's framework) does not have $\cos u$ or $\sin u$ terms. The moving plane defined by Ω_N and I_N satisfies Newcomb's requirements of the ecliptic.

The expression of the latitude, so as to satisfy the Newcomb's requirement, can be derived from the fundamental numbers of the amplitudes of $\sin I_s \delta\Omega_p$ and δI_p given by Le Verrier (1858). Using Le Verrier's numerical values we can calculate $\delta' I$ and $\delta'\Omega$:

$$\delta' I = 0''.004, \text{ and } \delta'\Omega = 0''.091, \quad (10)$$

which are referred to the equator of 1850.0. When we use VSOP80, which is newly developed by Bretagnon (1980), $\delta' I$ and $\delta'\Omega$ referred to the equator of 2000.0 are

$$\delta' I = 0''.00329, \text{ and } \delta'\Omega = 0''.09351. \quad (11)$$

Standish (1981) derived $\delta' I = 0''.00334$ and $\delta'\Omega = 0''.093\,66$ from the different point of view (see Section 4). These numerical values are slightly different from those in Equation (11); this difference originates from the fact that Standish's values are essentially based on the secular perturbation by Newcomb while the values of Equation (11) are based on the periodic perturbations by Bretagnon.

It is worthy to note that the latitude obtained by Le Verrier does include $\sin u$ and $\cos u$ terms. Therefore, the ecliptic in Le Verrier's Solar Tables (1858) is defined Ω_s and I_s, which is different respectively, from the ecliptic defined by Newcomb, by the quantities $\delta'\Omega$ and $\delta' I$.

3. A Kinematical Interpretation

We are constructing the relation between Newcomb's framework of the ecliptic coordinates and the osculating elements kinematically. First of all, we consider only the secularly changing case for the longitude of the ascending node and the obliquity, and assume that the Sun is revolving uniformly within this orbital plane, for brevity, taking aside the equation of center as well as the perturbations. A justification for this procedure is given at the end of this section.

Let u_N be the argument of latitude in this case; then the equatorial coordinates

r is, under the above assumption, given by

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_3(-\Omega_N)R_1(-I_N)\mathbf{r}_0, \tag{12}$$

where

$$\mathbf{r}_0 = \begin{bmatrix} \cos u_N \\ \sin u_N \\ 0 \end{bmatrix} \tag{13}$$

provided that only the motion on the sphere of the unit length is here considered for brevity. I_N and Ω_N are assumed to be expressed by

$$I_N = I_0 + I_1 t + I_2 t^2 + I_3 t^3 + \cdots$$

and

$$\Omega_N = \Omega_1 t + \Omega_2 t^2 + \Omega_3 t^3 + \cdots. \tag{14}$$

It is true that the *instantaneous* velocity vector of this hypothetical sun (hereafter we omit 'hypothetical') does not lie within the moving orbital plane defined by I_N and Ω_N. On the contrary, we may obtain an osculating plane of the Sun, which is not necessarily in coincidence with the moving orbital plane, by the following conditions:

$$\mathbf{r} = R_3(-\Omega')R_1(-I')\mathbf{r}_0(u'), \tag{15}$$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (n + \delta_N n)R_3(-\Omega')R_1(-I')\frac{\partial \mathbf{r}_0(u')}{\partial u'} \tag{16}$$

where

$$I' = I_N + \delta_N I, \tag{17}$$

$$\Omega' = \Omega_N + \delta_N \Omega,$$

$$u' = u_N + \delta_N u,$$

and

$$n = \frac{\mathrm{d}u_N}{\mathrm{d}t} (= \text{constant}).$$

After some manipulation, we have indeed the osculating elements, and the deviations (the quantities affixed by δ_N) from the mean elements given by Newcomb, as follows:

$$\delta_N I = \frac{\dot{I}_N}{n} \sin u_N \cos u_N - \frac{\dot{\Omega}_N}{n} \sin I_N \cos^2 u_N,$$

$$\delta_N \Omega = \frac{\dot{I}_N}{n \sin I_N} \sin^2 u_N - \frac{\dot{\Omega}_N}{n} \sin u_N \cos u_N, \tag{18}$$

$$\delta_N u = -\frac{\dot{I}_N \cos I_N}{n \sin I_N} \sin u_N + \frac{\dot{\Omega}_N}{n} \cos I_N \sin u_N \cos u_N,$$

and

$$\delta_N n = \dot{\Omega}_N \cos I_N,$$

where the dot above characters represents the derivation with respect to the time argument t.

It is true, therefore, that, if we take the plane defined by the osculating elements, Ω' and I', (neglecting δu_N from the argument of longitude in this new plane), we have again a uniform motin (of $n + \delta_N n$) within this plane. In other words, we can have the latitude referred to this plane always being zero, from the condition (15). Now, this case includes short periodic terms in the obliquity and the longitude of ascending node, as will be seen from Equation (18).

If we take only the slowly varying part taking aside the periodic parts of $\sin 2u_N$ or $\cos 2u_N$ in $\delta_N I$ and $\delta_N \Omega$, we have

$$\begin{aligned}
\delta'_N I &= -\dot{\Omega}_N \sin I_N / 2n, \\
\delta'_N \Omega &= \dot{I}_N / (2n \sin I_N), \\
\delta'_N u &= -\dot{I}_N \cos I_N / (2n \sin I_N), \\
\delta'_N n &= \delta_N n = \dot{\Omega}_N \cos I_N.
\end{aligned} \tag{19}$$

If we take the coordinates referred to this framework, we have

$$\begin{bmatrix} \cos \lambda' \cos \beta' \\ \sin \lambda' \cos \beta' \\ \sin \beta' \end{bmatrix} = R_1(I_N + \delta'_N I) R_3(\Omega_N + \delta'_N \Omega)\mathbf{r}$$

$$= \begin{bmatrix} \cos(u_N - \dot{I}_N \cos I_N/(2n \sin I_N)) \\ \sin(u_N - \dot{I}_N \cos I_N/(2n \sin I_N)) \\ (\dot{I}_N \cos u_N + \dot{\Omega}_N \sin I_N \sin I_N)/2n \end{bmatrix}, \tag{20}$$

from which we can easily have

$$\sin \beta' \doteqdot \beta' = (\dot{I}_N \cos u_N + \dot{\Omega}_N \sin I_N \sin u_N)/(2n), \tag{21}$$

retaining only the first order perturbation. Therefore, we must have

$$\begin{aligned}
\delta'_N I &= -\delta' I, \\
\delta'_N \Omega &= -\delta' \Omega,
\end{aligned} \tag{22}$$

comparing Equation (21) with Equations (7), (8), (9) and (19); this shows that the application of $\delta'_N I$ and $\delta'_N \Omega$ to the Newcomb's framework provides that of Le Verrier (see also Equation (36)).

In this section we have taken aside the equation of center and the perturbation in longitude; however, this is justified because such periodic terms do not affect directly the position of osculating plane. Moreover, the change of Newcomb's framework to that of Le Verrier can only affect the sin u and cos u terms in latitude but not other periodic perturbations. From this point of view, our procedure in this section, even though simple, holds the essential part of the problem, and can be justified.

4. Another Kinematical Interpretation by Standish

The velocity vector of the Sun in the moving reference plane defined by I_s and Ω_s is

$$\frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} r\left(\frac{\tilde{\mathbf{r}}}{r}\right) = \frac{\tilde{\mathbf{r}}}{r}\frac{dr}{\mathrm{d}t} + r\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tilde{\mathbf{r}}}{r}\right), \tag{23}$$

where r is the radial distance. The definition of $\tilde{r}$ should be referred to Section 2. The angular momentum vector $\tilde{G}$ with respect to the ecliptic coordinates by Le Verrier is given by

$$\tilde{\mathbf{G}} = \tilde{\mathbf{r}} \times \left(\frac{\mathrm{d}\tilde{\mathbf{r}}}{\mathrm{d}t}\right) = r\left(\tilde{\mathbf{r}} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tilde{\mathbf{r}}}{r}\right)\right). \tag{24}$$

The vector $\tilde{G}$ is not equal to the angular momentum vector in the inertial reference frame because of the motion of the reference plane. The components of $\tilde{G}$ in the Le Verrier framework

$$\begin{aligned} \tilde{G}_{\tilde{x}} &= r^2(b \sin u + \dot{\tilde{u}}\, \delta\Omega_p \sin I_s), \\ \tilde{G}_{\tilde{y}} &= -\, r^2(b \cos u + \dot{\tilde{u}}\, \delta I_p), \\ \tilde{G}_{\tilde{z}} &= r^2 \dot{\tilde{u}}, \end{aligned} \tag{25}$$

where

$$b = -\,\delta\dot{\Omega}_p \sin I_s \cos u + \delta\dot{I}_p \sin u,$$

which are easily calculated with use of Equations (3), (4), (5), and (24). Here we neglect the second-order terms with respect to $\delta\Omega_p$ and δI_p. Now we determine a plane, P^*, which is perpendicular to $\tilde{\mathbf{G}}$. In order to determine the longitude of the node Ω^* and the inclination I^* of the osculating plane P^*, we express components of $\tilde{\mathbf{G}}$ in a reference frame of which x_{os} axis is towards the node of the osculating plane and y_{os} axis is in the fixed reference plane (see Figure 1):

$$\tilde{\mathbf{G}}_{os} = R_3(\Omega - \Omega_s) R_1(-I_s) \tilde{\mathbf{G}}. \tag{26}$$

From (26), we have

$$\tilde{G}_{x_{os}} = r^2 b \sin u,$$
$$\tilde{G}_{y_{os}} = r^2(-b\cos I_s \cos u - \dot{u}\cos I_s \delta I_p - \dot{\tilde{u}}\sin I_s), \quad (27)$$
$$\tilde{G}_{z_{os}} = r^2(-b\sin I_s \cos u - \dot{u}\sin I_s \delta I_p - \dot{\tilde{u}}\cos I_s).$$

The components of $\tilde{\mathbf{G}}$ are expressed in $x_{os} - y_{os} - z_{os}$ system

$$\tilde{G}_{x_{os}} = \tilde{G}\sin I^* \sin(\Omega^* - \Omega),$$
$$\tilde{G}_{y_{os}} = -\tilde{G}\sin I^* \cos(\Omega^* - \Omega), \quad (28)$$
$$\tilde{G}_{z_{os}} = \tilde{G}\cos I^*,$$

where $\tilde{G}$ is the angular momentum: $\tilde{G} = r^2\dot{\tilde{u}}$. Now we define δI^* and $\delta\Omega^*$ by

$$I^* = I_s + \delta I^* \quad \text{and} \quad \Omega^* = \Omega_s + \delta\Omega^*. \quad (29)$$

Substituting (29) into the third equation of (28) and comparing its result with the third equation of (27), we easily derive

$$\delta I^* = \delta I_p + (b\cos u)/\dot{u}. \quad (30)$$

From the remaining Equations (27) and (28), we obtain

$$\delta\Omega^* = \delta\Omega_p + (b\sin u)/(\dot{u}\sin I_s). \quad (31)$$

From the definition of the osculating elements of Ω and I, we have (see Brown and Shook, 1933)

$$\frac{dI}{dt}\sin u - \frac{d\Omega}{dt}\sin I\cos u = 0, \quad (32)$$

which means that the velocity of a particle does not have the component perpendicular to the osculating plane. Substituting $I = I_s + \delta I_p$ and $\Omega = \Omega_s + \delta\Omega_p$ into Equation (32) and keeping the first-order terms of $\delta\Omega_p$ and δI_p, we have

$$b = -\delta\dot{\Omega}_p \sin I_s \cos u + \delta\dot{I}_p \sin u$$
$$= \dot{\Omega}_s \sin I_s \cos u - \dot{I}_s \sin u. \quad (33)$$

Then we obtain from Equations (30), (31), and (33)

$$\delta I^* = \delta I_p + (\dot{\Omega}_s \sin I_s)/(2\dot{u}) + (\dot{\Omega}_s \sin I_s \cos 2u - \dot{I}_s \sin 2u)/(2\dot{u}), \quad (34)$$

and

$$\delta\Omega^* = \delta\Omega^*_p - \dot{I}_s/(2\dot{u}\sin I_s) + (\dot{\Omega}_s \sin I_s \sin 2u + \dot{I}_s \cos 2u)/(2\dot{u}\sin I_s). \quad (35)$$

The secular parts of δI^* and $\delta\Omega^*$ and are not zeros, but we have

$$(\delta I^*)_s = (\dot{\Omega}_s \sin I_s)/(2n), \quad (36)$$

and

$$(\delta\Omega^*)_s = -\dot{I}_s/(2n \sin I_s),$$

where n is the sidereal mean motion of the Sun. Here we neglect the square of the eccentricity of the Sun. Now we are another mean orbital plane defined by

$$I_S = I_s + (\delta I^*)_s \quad \text{and} \quad \Omega_S = \Omega_s + (\delta\Omega^*)_s. \tag{37}$$

The mean orbital plane thus derived does not coincide with the mean orbital plane defined by Ω_s and I_s. We shall show relationships between $(\delta\Omega^*)_s$ and $(\delta I^*)_s$ of (36) and $\delta' I$ and $\delta'\Omega$ of (8). By substituting the expressions of periodic perturbations of $\sin I_S \delta\Omega_p$ and δI_p (Equations (6)) into (33) and comparing coefficients of $\sin u$ and $\cos u$, we obtain

$$\dot{I}_s = (C - B)\dot{u} \quad \text{and} \quad \dot{\Omega}_s \sin I_s = -(A + D)\dot{u}. \tag{38}$$

Combining Equations (38), (7), (8), and (36), we finally obtain

$$(\delta I^*)_s = \delta' I \quad \text{and} \quad (\delta\Omega^*)_s = \delta'\Omega \tag{39}$$

thus we have

$$I_S = I_N \quad \text{and} \quad \Omega_S = \Omega_N. \tag{40}$$

The relation of Equation (40) shows that a mean orbital plane defined by Standish kinematically is such that the latitude referred to this plane does not have $\sin u$ nor $\cos u$ terms. Therefore, the ecliptic defined by Standish (in the rotating sense) does coincide with the ecliptic defined by Newcomb.

The expression (36) can be expressed in terms of precessional quantities π and

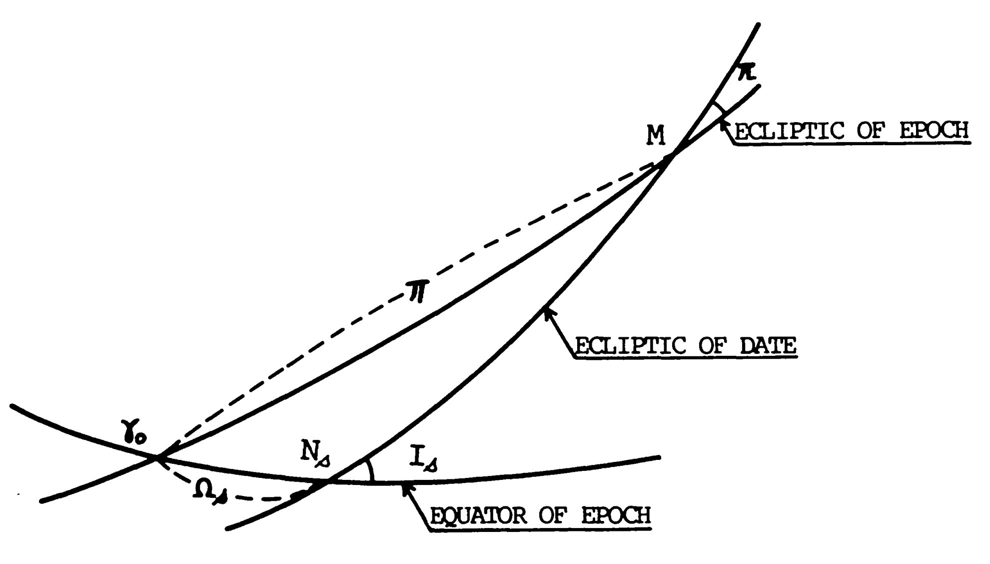

Fig. 2. Rotation of the mean ecliptic.

Π (Lieske *et al.*, 1977). From the spherical triangle $\gamma_0 N_s M$ (see Figure 2), we have

$$dI_s = -\cos N_s M \, d\pi - \sin \pi \sin N_s M \, d\Pi,$$

$$\sin I_s \, d\Omega_s = \sin N_s M \, d\pi + \sin \pi \cos N_s M \, d\Pi. \tag{41}$$

Since contribution from $d\Pi$ is of second-order, we obtain

$$\dot{I}_s = \dot{\pi} \cos \Pi \quad \text{and} \quad \dot{\Omega}_s = (\dot{\pi} \sin \Pi)/\sin I_s. \tag{42}$$

and then

$$(\delta I^*)_s = (\dot{\pi} \sin \Pi)/(2n) \quad \text{and} \quad (\delta\Omega^*)_s = -(\dot{\pi} \cos \Pi)/(2n \sin I_s), \tag{43}$$

which coincides with the result obtained by Standish (1981).

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