THE ASTROPHYSICAL JOURNAL, 267:488–510, 1983 April 15

THE STATISTICS OF GRAVITATIONAL LENSES: APPARENT CHANGES IN THE LUMINOSITY FUNCTION OF DISTANT SOURCES DUE TO PASSAGE OF LIGHT THROUGH A SINGLE GALAXY

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Received 1982 August 23; accepted 1982 September 21

ABSTRACT

We ask how the apparent distribution of fluxes (*N-F* relation) of point sources seen behind an intervening galaxy will change due to gravitational lensing of the galaxy as a whole or due to mini-lenses within it. The analysis is exact in the limit that the optical depth to lensing is small. We find that there should be a significant increase in the apparent density of quasars seen near galaxies but that a sample of more than $10^{4.5}$ galaxies will have to be studied before a statistically significant result is found. The resulting amplification of the *N-F* relation depends sensitively on the slope and curvature of the initial *N-F* relation. Because of this and requirements of flux conservation, there is expected to be a decrement of very faint quasars ($m > 26.5$) seen near galaxies. This, coupled with the scarcity of bright quasars, implies that searches should optimally be made in the vicinity of $m = 23$. The apparent amplification found by Canizares with a smaller sample ($N \approx 10^{3.3}$) of galaxies using relatively bright quasars ($m < 16$) is, if real and not a statistical anomaly, due to physical effects other than gravitational lensing.

Subject headings: cosmology — gravitation — quasars

I. INTRODUCTION

The recent discoveries of the double QSO 0957+561A,B (Walsh, Carswell, and Weymann 1979; Weymann *et al.* 1979) and of the triple QSO 1115+080A,B,C (Weymann *et al.* 1980; Young *et al.* 1981*a*) have reminded us of the potential that galaxies have for lensing distant objects.

Lately, theoretical attention (Gott and Gunn 1974; Press and Gunn 1973; Young *et al.* 1980) has concentrated on the ability of gravitational lenses to split images of background objects, a property which has been the basis for their identification. But in the past (Barnothy and Barnothy 1968), the ability of lenses to intensify images of lensed objects had drawn theorists' attention because of both its potential for the exploration of remote and otherwise unaccessible regions of space, and of the possibility of explaining disconcerting physical evidence. Gott (1981), Turner (1980), Canizares (1981), and Avni (1981) have reexamined the above mentioned property. Gott showed that gravitational lensing due to individual stars might be responsible, under certain circumstances, for the long time scale (~ 1 yr) luminosity variations of QSOs, thus providing a means to detect otherwise invisible "dark" matter in galactic halos. Turner (1980), modified slightly by Avni (1981), has considered the statistical effect of cosmologically distributed objects on classic statistical indicators of QSO evolution. Recently Canizares (1981) examined the possibility that lensing by nearby galaxies might be responsible for the enhancement of the density of bright QSOs in the proximity of bright galaxies, an effect studied in a different context by Tyson (1981).

The motivation for this paper and the following one stems from the desire to improve on both the mathematical and physical treatment of statistical lensing, whether due to a single nearby galaxy (Canizares 1981) or to a cosmological distribution of objects (Turner 1980), and to include properly both the effect of the smooth galactic potential and the spikes in the potential due to individual stars. The present paper (Paper I) deals with an individual galaxy. In Paper II we extend the analysis to convolve over all lenses along a given path, in a universe with a relativistic cosmology, to determine the importance of lensing for the statistical properties of QSOs. We have been able to formulate the problem in a manner similar to that used in radiative transfer. The single scalar differential/integral equation derived (eq. [8]) can include in a natural way all of the physical effects required to treat the propagation of rays from point images including cosmological effects, large and small amplification scattering events, etc. The method automatically conserves the number and total energy output of the source objects and allows some illuminating exact solutions to be found in certain cases. The general approach is valid in the limit of small optical depth but will become quite inaccurate if τ (optical depth) > 1 due to the tensorial nature of scattering events and the complex multiple paths possible in the high optical depth limit.

The plan of this paper is the following: In § II we develop the mathematical treatment of statistical lensing due to a composite object; we introduce a simple physical model for a galaxy and discuss some mathematical consequences of the model. We derive a differential/integral equation for the change in the number-magnitude relation of background objects seen through a screen of gravitational lenses which is valid for moderate "optical depth" of the screen. In § III we discuss the domain of validity of this approach. In § IV we apply this theoretical apparatus to the case of lensing of QSOs by a nearby single galaxy, discuss the physical input necessary for our calculations, and compare our results with those of Canizares and with the available observational results. In § V we summarize our conclusions.

II. MATHEMATICAL TREATMENT FOR A SINGLE GALAXY

a) Physical Construction

As mentioned above, our problem is that of describing, in a statistical fashion, the gravitational lensing effect of a galaxy. Previous numerical studies of lensing of finite size sources (Press and Gunn 1973; Young *et al.* 1980*a*, *b*; Young 1981) have concentrated on the distortion and formation of multiple images. We will study only the problem of magnification, considering the case in which the size of the image is negligible; this will allow us to deal with the problem analytically, treating as well the statistical effect of individual stars in the intervening galaxy which was neglected by the above authors. These, as Gott (1981) showed, though contributing negligibly to the bending of the light rays, may produce magnifying effects comparable, and even superior, to that of the galaxy as a whole.

We model a galaxy of stars in the following way: Suppose one replaces a given star with the sum of two density distributions. The first is obtained by distributing the whole stellar mass over the average interstellar volume containing one star (call it MD1). The other distribution (MD2) has no net mass and is made of two parts, one *negative* mass distribution over the same volume and the other a delta function of equal mass, the star itself. Thanks to the superposition principle for potentials (in Newtonian theory), the effect of these two mass distributions on the light ray paths will be the same as that of the original mass distribution.

Now, it is easy to see that the sum of all MD1's will give the usual smoothed galaxy, whose total effect has been computed by several authors. MD2's will give a different contribution. In fact, over impact parameters larger than the interstellar distance, $d_{is} = n_*^{-1/3}$, the total effect of this distribution will be null, because its total mass is zero, But for impact parameters comparable with a critical impact parameter α (defined in eq. [3]), the effect of this distribution is the same as that of the star alone, provided that $\alpha \ll d_{is}$. In fact, the fraction of the star mass, (with a negative sign!) included between the star surface and α is given by $(\alpha/d_{is})^3$, which, using standard galactic values for d_{is} and the values found for the double QSO, is found to be smaller than 10^{-10}. Notice, furthermore, that since these scatterings must conserve energy, some impact parameters must result in a deamplification which is not surprising since there is no net mass associated with MD2. We can understand this qualitatively in the following way. If one considers a normal lens, one finds that the reason why it focuses an image is that the farther away a light ray is from the symmetry axis, the more it is bent toward it. In the case of an isothermal galaxy, the bending toward the lens for photons passing at different impact parameters is changed; the one passing closer is bent more because of the $1/r^2$ dependence of the force for given mass, but this effect is balanced by the fact that the second light ray, which passes farther away, feels a stronger gravitational pull due to the larger mass which is contained within its impact parameter. The two effects exactly cancel out (Gott and Gunn 1974) so that there is still magnification. In the case of a point mass object, the balancing effect for larger impact parameters due to the encompassing of more matter no longer exists, so that now the bending angle decreases with increasing impact parameter; nonetheless, focusing in the perpendicular direction still occurs and a total net magnification ensues. In the case of MD2, rays with larger impact parameters are bent less not only because they are at greater distance, but also because they "see" less mass! This is only a qualitative argument, but computations, carried out in Appendix C, show this to be exactly the case. The attractive feature of this construction is that, since the influence of the MD2 distributions beyond a radius $b = d_{is}$ is null, they have a finite total cross section σ_{TOT}. This can then be used to define a total optical depth τ by means of

$$d\tau = n_* \sigma_{TOT} ds. \tag{1}$$

Since we also know the differential cross section for amplification, we can also obtain the probability distribution as a function of total (= final) amplification A, provided we have a way to "superpose" the amplifications due to two or more lumps. In Appendix A we use the formalism of Bourassa and Kantowsky (1975) to show that, in the case of two scattering masses, as long as at least one of the scatterings is weak (in the sense $b \gg \alpha$), then the total average amplification A_{TOT} is approximately $A_{TOT} = A_1 A_2$, where A_1 and A_2 are the amplifications which each scatterer would produce in the absence of the other, and we can convolve the individual probability distributions to obtain the distribution of A_{TOT}.

b) *Treatment of a Single Scattering*

We now turn to the problem of determining the probability distribution for a single scattering. Following Press and Gunn (1973), the magnification of the two images formed by a single star is given by

$$A_{\pm}=\left|\frac{1}{1-\left(\alpha^2/b_{\pm}^2\right)^2}\right|, \tag{2}$$

where

$$\alpha^2 \equiv \frac{4GM_*(1+z_l)}{c^2}\,\frac{(L-\lambda)\lambda}{L} \tag{3}$$

and

$$b_{\pm}\equiv \tfrac{1}{2}\left[-l \pm (l^2+4\alpha^2)^{1/2}\right], \tag{4}$$

and l is the distance of the deflected source from the line of sight to the deflector (see Fig. 1). Here L and λ are the affine distances to the source and lens, respectively, and (M_*, z_l) the mass and redshift of the lens. Then define

$$f \equiv l/\alpha, \qquad A \equiv A_+ + A_-, \tag{5}$$

and obtain the total amplification, A, of the two images (which are assumed to be so close together as to be unresolvable). In other words, after passing by the lens at impact parameter l, the flux seen by an observer at the origin is a factor A larger than would have been the case in the absence of a lens. From equations (2)–(5) we find the total amplification for a point mass:

$$A=\frac{f^2+2}{f(f^2+4)^{1/2}}. \tag{6a}$$

The quantity f, which measures the actual impact parameter in units of the critical value, parameterizes the scattering event. For $f \ll 1, A \approx f^{-1}(1+3f^2/8)$ and the amplification is large. For $f \gg 1, A \approx 1+2/f^4$, the total amplification rapidly approaches negligible values, with the total effect of weak scattering being clearly convergent. For $f=1$, the two amplifications are $(A_+, A_-)=(1.17, 0.17)$ or $\Delta M = 0.32$ mag. It is thus convenient to characterize the cross sections for $f<1$ as σ_{STRONG}. The condition $l=\alpha$ or $f=1$ also gives Gott's (1981) critical impact parameter. Then, $\sigma_{\text{STRONG}}=\pi\alpha^2$ is the area of strong scatterings, the one which gives total amplification $A \geq 1.34$. In what follows, when treating strong scattering, we have not used equation (6a), but the approximation

$$A=1/f \quad (f<1), \tag{6b}$$

which has the advantage over equation (6a) of being analytically handier, a feature which will be greatly appreciated in the following. Figure 2 shows the correct $A(f)$, its approximation, and the equivalent curve for an isothermal sphere, rather than a point mass. It shows our approximation to be excellent over the range $f \leq 1$. For impact parameters

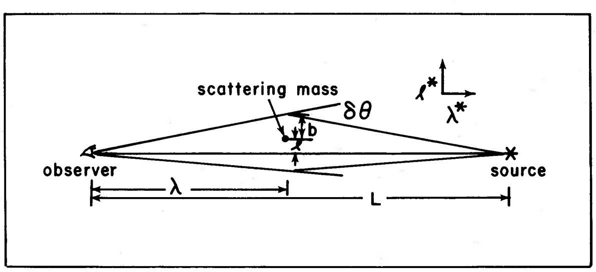

FIG. 1.—Geometry and nomenclature of scattering events (from Press and Gunn 1973)

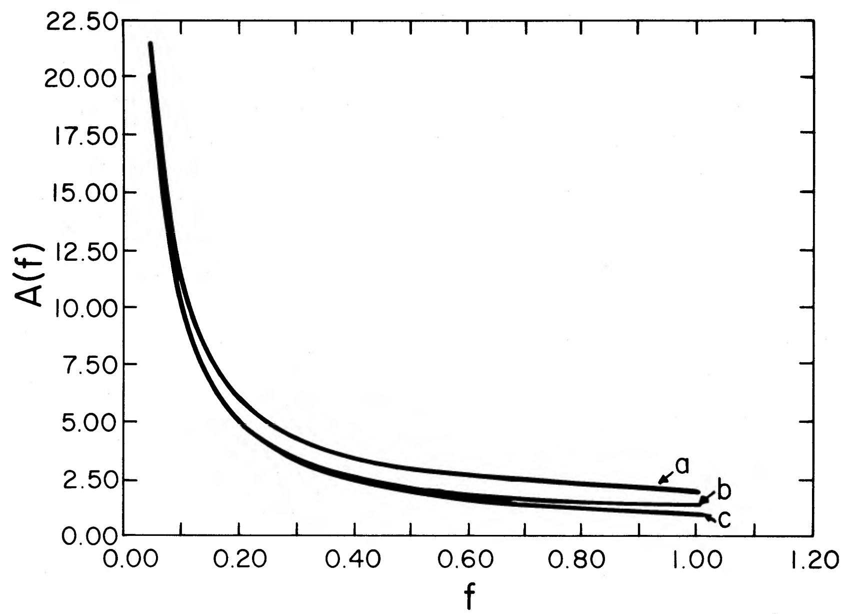

FIG. 2.—Total amplification A as a function of impact parameter f (in normalized units, eq. [5]) for (a) an isothermal smooth galaxy; (b) a real star, eq. (6a); (c) the approximation (6b).

larger than the critical one ($f=1$) we take $A=-A_0$, with A_0 defined to produce energy conservation. Another advantage equation (6b) has over (6a) is described at the end of this section. For a light ray passing through one cell in a galaxy, there are then two possible fates: it either hits an area where it is "strongly" scattered (probability $d\tau$), or it hits another area where it is demagnified (probability $1-d\tau$). No third possibility exists because we have constructed the MD2 scattering regions to completely fill the galaxy. In analogy with equation (1) we can say that the differential probability of suffering a strong scattering in distance ds is simple $d\tau_{\text{STRONG}} = \sigma_{\text{STRONG}} n_* \, ds$. Henceforth we use this definition dropping, for convenience, the subscript STRONG.

Before proceeding further with a detailed mathematical discussion, let us demonstrate a key qualitative result of this paper. For strong scatterings, the probability of a given range of impact parameters f less than the critical value ($=1$) is clearly $dP = \text{const} \times 2\pi f \, df$. This with (6b) gives a probability distribution for amplification A:

$$dP = \text{const} \times dA/A^3 \quad \text{(strong scattering)}. \tag{6c}$$

If a lensing event for either a point mass or a singular isothermal sphere is strong enough to increase the brightness of the image by more than $\Delta M = 1/3$, *then the statistical distribution of lensed images goes as* $dP = \text{const} \times (L/L_i)^{-3} \, dL/L_i$, *where L is the final apparent luminosity and* L_i *the initial apparent luminosity*. This result for a single event will be found to hold, statistically, for the results of many scatterers as well. As the beam passes through the whole galaxy, we convolve together the probabilities in the standard fashion. For an image which has been strongly scattered, the differential probability of being magnified from magnitude m' to m *per strong scattering* is

$$P(\Delta M) \, d\Delta M = 2K \, 10^{0.8\Delta M} d\Delta M = 2k(L_i/L)^2 \, d\Delta M, \tag{7a}$$

where

$$k \equiv 0.4 \ln_e(10) \quad \text{and} \quad \Delta M = m - m', \tag{7b}$$

for A given by (6b). For an image which has been deamplified, in accordance with our previous assumption, we find that the differential probability of being "weakly" scattered to amplitude m is $d\Delta M \delta(\Delta M - \Delta_1)$, where the constant Δ_1 is to be determined from the requirement of energy conservation. We can now derive the evolution of a luminosity function through the above described tube of cells (MD2) each being a zero total mass lump.

c) *Integral Equation of Transfer*

i) *Derivation and Conservation Laws*

We now proceed to show how multiple scatterings can affect the apparent luminosity function $\mathfrak{L}(m, \tau = 0)$ of objects seen through the galaxy. An alternative and equivalent treatment is given in Appendix A. The change in the number-magnitude relation in passing through a layer of optical depth $d\tau$ has two terms, one due to the amplifications characterized by the function $P(\Delta M)$ (eq. [7]) and one due to deamplification which is taken at an average value Δ_1:

$$\mathfrak{L}(m, \tau + d\tau) = d\tau \int_{+m}^{+\infty} P(m - m')\mathfrak{L}(m', \tau)\, dm' + (1 - d\tau)\int \delta(m - m' - \Delta_1)\mathfrak{L}(m', \tau)\, dm',$$

or, on expanding, and retaining first-order terms,

$$\frac{\partial \mathfrak{L}(m, \tau)}{\partial \tau} = \int P(m - m')\mathfrak{L}(m', \tau)\, dm' - \mathfrak{L}(m, \tau) - \Delta_1 \frac{\partial \mathfrak{L}(m, \tau)}{\partial m}. \tag{8}$$

This is our fundamental equation, and is valid for any conservative $P(\Delta M)$.

Notice that equation (8) satisfies the number conservation requirement:

$$\frac{\partial}{\partial \tau} N = \frac{\partial}{\partial \tau} \int_{-\infty}^{+\infty} \mathfrak{L}(m, \tau)\, dm = \int \int P(m - m')\mathfrak{L}(m')\, dm'\, dm - \int_{-\infty}^{+\infty} \mathfrak{L}(m)\, dm - \Delta_1 \int_{-\infty}^{+\infty} \frac{\partial \mathfrak{L}}{\partial m} dm.$$

The last term is obviously zero. In the first, change integration variables to obtain $\int\int P(m - m')\mathfrak{L}(m')\, dm'\, dm = \int P(m - m')\, d(m - m') \int \mathfrak{L}(m)\, dm = \int \mathfrak{L}(m)\, dm \equiv N$, which, of course, yields $\partial N/\partial \tau = 0$. Next consider energy conservation. Notice first that, since $m = -2.5 \log L + c$,

$$E = \int L \mathfrak{L}(m, \tau)\, dm = c \int e^{-km} \mathfrak{L}(m, \tau)\, dm.$$

Then $\partial E/\partial \tau = 0$ implies that

$$\int \int e^{-km} P(m - m')\mathfrak{L}(m')\, dm'\, dm - \int e^{-km} \mathfrak{L}(m, \tau)\, dm - \Delta_1 \int e^{-km} \frac{\partial \mathfrak{L}}{\partial m} dm = 0.$$

Notice that

$$c \int e^{-k(m - m')} P(m - m')\, d(m - m') = \int \frac{L}{L'} P(m - m')\, d(m - m') \equiv \bar{A},$$

where $\bar{A}$ is the average strong amplification. Then, after simple algebra, one gets

$$\partial E/\partial \tau = (\bar{A} - 1 - \Delta_1 k) E = 0, \tag{9a}$$

or

$$\Delta_1 = (\bar{A} - 1)/k, \tag{9b}$$

which defines Δ_1 in terms of the requirement of energy conservation. Δ_1, evaluated in Appendix B with the approximation (6b), is 1.086. If one passes through optical depth $\Delta\tau = 1$ (which of course requires passage through a very large number of cells) and does not suffer a strong scattering event, then on average the image suffers about 1 mag of demagnification. Of course, most paths have $\Delta\tau \ll 1$.

We have chosen for simplicity to divide the scattering process into two parts. A "hit" on a scatterer gives strong positive amplification. A "miss" gives a small deamplification. With this characterization, $P(m)$ exists only for negative values of its argument. We could alternatively have omitted the $-\Delta_1 \partial \mathfrak{L}/\partial m$ term and included negative scattering in $P(m)$. Finally, we could have considered all of the weak scatterings (the positive part due, for example, to $f > 1$ in [6a]) by treating them as a diffusive term $\partial^2 \mathfrak{L}/\partial m^2$ which would have been derived from the integral equation with application of the steepest-descent approximation to the integral. We are in this paper treating the simplest problem that can be handled within the formulation. The exact solution presented below can easily be generalized to more complex cases as well.

ii) *Formal Solution of Equation of Transfer*

We now proceed to the derivation of the explicit analytical solution of equation (8). The choice of m, rather than L, as an independent variable is motivated by the fact that the first term on the right is thus seen to be a normal convolution product. This suggests that one make a Fourier transformation with respect to m. Defining

$$\tilde{P}(\lambda) \equiv \int_{-\infty}^{0} P(m) e^{2\pi i \lambda m} \, dm, \tag{10}$$

$$\tilde{\mathfrak{L}}(\lambda) \equiv \int_{-\infty}^{+\infty} \mathfrak{L}(m, \tau) e^{2\pi i \lambda m} \, dm, \tag{11}$$

one gets

$$\frac{\partial \tilde{\mathfrak{L}}(\lambda, \tau)}{\partial \tau} = \left(\tilde{P}(\lambda) - 1 + 2\pi i \Delta_1\right) \tilde{\mathfrak{L}}(\lambda, \tau),$$

which has the solution

$$\tilde{\mathfrak{L}}(\lambda, \tau) = \tilde{\mathfrak{L}}_0(\lambda) \exp\left[\left(\tilde{P}(\lambda) - 1 + 2\pi i \Delta_1 \lambda\right)\tau\right].$$

This yields, on inverting the Fourier transform,

$$\mathfrak{L}(m, \tau) = \int_{-\infty}^{+\infty} \tilde{\mathfrak{L}}_0(\lambda) \exp\left[\left(\tilde{P}(\lambda) - 1 + 2\pi i \Delta_1 \lambda\right)\tau\right] e^{-2\pi i m \lambda} \, d\lambda,$$

or

$$\mathfrak{L}(m, \tau) = \int_{-\infty}^{+\infty} \mathfrak{L}_0(m') G(m - m', \tau) \, dm', \tag{12}$$

where

$$G(m, \tau) = \int_{-\infty}^{+\infty} \exp\left[\left(\tilde{P}(\lambda) - 1 + 2\pi i \Delta_1 \lambda\right)\tau\right] e^{-2\pi i \lambda m} \, d\lambda, \tag{13}$$

which shows that $\tilde{\mathfrak{L}}_0(\lambda)$ is just the Fourier transform of the initial condition [$\mathfrak{L}(m, \tau)$ at $\tau = 0$]. Also, both equations (8) and (12) reveal their similarity with the diffusion equation; this, of course, implies that any attempt at numerical reconstruction of the original luminosity function from the observed one is bound to be at most only partially successful.

For any scattering function $P(m)$ we compute the Fourier transform via $\tilde{P}(\lambda)$ using equation (10), then the redistribution function $G(m, \tau)$ via equation (13), and convolve with the initial distribution with equation (12) to obtain the final state. Equation (13) can be rewritten as

$$G(m, \tau) = e^{-\tau}\left\{\int_{-\infty}^{+\infty} \left(e^{\tau \tilde{P}} - 1\right) \exp\left[-2\pi i \lambda (m - \Delta_1 \tau)\right] d\lambda + \int_{-\infty}^{+\infty} \exp\left[-2\pi i \lambda (m - \Delta_1 \tau)\right] d\lambda\right\},$$

or

$$G(m, \tau) = e^{-\tau}\left[F(m - \Delta_1 \tau, \tau) + \delta(m - \Delta_1 \tau)\right], \tag{14}$$

with

$$F(m, \tau) \equiv \int_{-\infty}^{+\infty} \left(e^{\tau \tilde{P}(\lambda)} - 1\right) e^{-2\pi i \lambda m} \, d\lambda. \tag{15}$$

Equation (15) shows F to be a smooth and finite ($< +\infty$) function, provided $\tilde{P}(\lambda) \to 0$ for $\lambda \to +\infty$. In order to

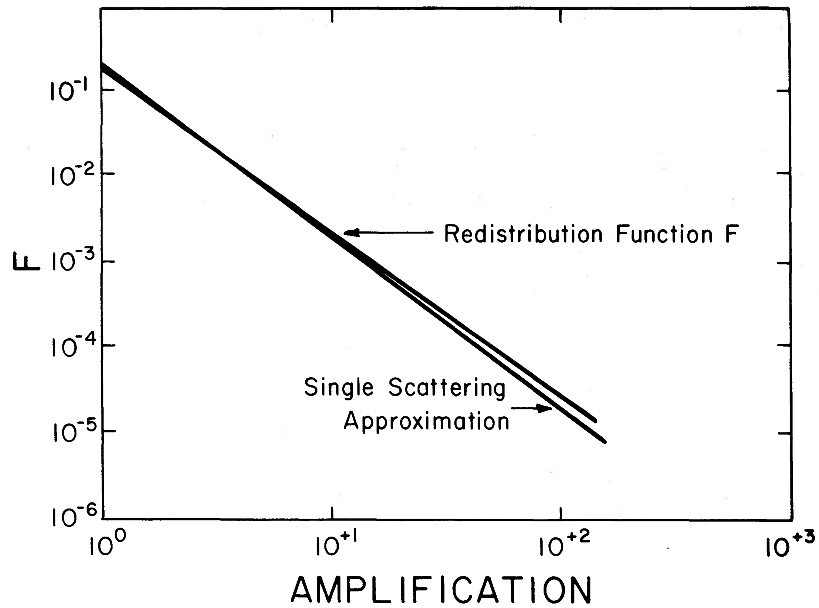

FIG. 3.—The redistribution function $F(m, \tau)$, as a function of amplification A, for $\tau = 0.1$, compared with its single strong scattering approximation.

understand what G is, consider a luminosity function representing a unique QSO luminosity, that is, $\mathfrak{L}_0(m) = \delta(m - m_0)$. Then equation (12) shows that $\mathfrak{L}(m, \tau) = G(m - m_0, \tau)$, or G is the luminosity probability distribution after one QSO's image has gone through a finite τ layer of stars. Equation (14) shows that it is the superposition of a decaying delta function which moves left as τ increases and is made up of all images which have only been demagnified, plus a long tail of scattered images whose norm increases with increasing τ.

Computation of F requires knowledge of $\tilde{P}(\lambda)$, or, by means of equations (10) and (7), of A as a function of f. Appendix B shows that with the approximation (6b) F can be written as

$$F(m, \tau) = \left(\frac{2k\tau}{|m|}\right)^{1/2} e^{2km} I_1\left((8k\tau|m|)^{1/2}\right), \tag{16}$$

for $m < 0$ and $F(m, \tau) = 0$ for $m > 0$ where I_1 is the usual modified Bessel function of first order. Under the same approximation, $\Delta_1 = k^{-1} = 1.086$. In the following, we will always use equation (16) for F. To first order in τ, Canizares's equation (5) can be seen to reproduce our equation (16) for the case of a critical isothermal sphere. However, both equations neglect the effects of the negative masses and the overall magnifying effect of the galaxy, which effects are included in our final redistribution function (cf. eq. [23]). Equation (16) is shown in Figure 3 for $\tau = 0.1$. The dashed line, for single scattering, shows the $\mathfrak{L}(L) \propto L^{-3}$ result noted earlier. Young (1981) treated numerically the case in which the QSO image size is not negligible, and where superposition of stars makes the one-scattering hypothesis invalid. His case $\tau_* = 1/2$, $\gamma = 0$ (notice that his τ_* equals our τ), which is shown in his Figure 1, is the closest one to the domain of validity of (16). It can be seen that both distributions have a maximum, at about $A = 1.6$, at approximately the same position. Also, his high-amplification tail is consistent with our A^{-3} tail.

d) The Smooth Galactic Potential Well

Before we treat the total probability distribution function for lensing by a galaxy, we must consider the effect of the smooth potential well. Since the amount of magnification depends on the amount of matter within the impact parameter (Press and Gunn 1973; Gott and Gunn 1974), it is necessary to adopt a model for the total mass distribution within the galaxy. For sake of simplicity, we have adopted a critical isothermal density profile:

$$\rho = \frac{\sigma_v^2}{2\pi G} \frac{1}{r^2}. \tag{17}$$

This profile, which has the advantage of being an analytical expression, has been chosen explicitly because the resulting mass distribution,

$$M(r) = \left(2\sigma_v^2/G\right)r, \tag{18}$$

reproduces the H I rotation velocity observations in many galaxies. From Gott and Gunn (1974), the total magnification for this mass distribution, for an object located at angular distance b from the line of sight to the galaxy, is

$$A_{\pm} \equiv I/I_0 = 1 \pm b_{\rm cr}/b, \tag{19a}$$

$$b_{\rm cr} \equiv \frac{4\pi\sigma_v^2}{c^2}(1+z_l)\frac{L-\lambda}{L}\lambda, \tag{19b}$$

and the two signs are due to the fact that (at most) two images may be formed in this potential, provided $b < b_{\rm crit}$. For $b > b_{\rm crit}$, only the image whose amplification is $A_+ = 1 + b_{\rm crit}/b$ exists. Having assumed a mass distribution, we can also compute the optical depth τ for scattering by stars along a given path. Gott (1981) found

$$\tau = b_{\rm cr}/2b \quad \text{or} \quad A = 1 + 2\tau \qquad (b > b_{\rm cr}), \tag{20}$$

using as a cross section what we call the "strong" scattering cross section.

We can now put together all the pieces. We consider a specific ray through the galaxy, a specific initial luminosity and, as explained above, we treat separately the magnification due to the smooth potential well and that due to the single stars. The effect of the first is just to change the QSO's magnitude from M to

$$M' = M - 2.5 \log A, \tag{21}$$

where A is given by (19). This simply shifts the luminosity function (delta function) by an amount $\Delta m = 2.5 \log A$. The second effect is to spread the probability distribution of an assumed delta function input over an interval

$$M' + \Delta_1\tau < M' < -\infty, \tag{22}$$

where the form of the distribution is

$$\begin{aligned} P(M'', M', \tau) &= 0 && \text{for } M'' > M' + \Delta_1\tau, \\ &= e^{-\tau}\delta(M'' - M' - \Delta_1\tau) + e^{-\tau}F(M'' - M' - \Delta_1\tau, \tau) && \text{for } M'' < M' + \Delta_1\tau, \end{aligned} \tag{23}$$

where $P(M'', M', \tau)\, dM''$ is the probability that an object of magnitude M' is amplified to the interval $(M'', M'' + dM'')$ (see eqs. [12], [13], [14]), where F is given by equation (15), and τ by (20). Equation (23) is our final result.

The question of which sign we should use in (19a) does not arise, since the application we have in mind is that of the apparent enhancement of bright QSOs near bright galaxies; in this case, in fact, one only deals with b's which are considerably larger than $b_{\rm crit}$. Equation (23) is easily seen to be nonzero between $M' + \Delta_1\tau$ and M', a feature which depends on our approximation (6b); in fact, if we had adopted equation (6a) the amplification due to our zero total mass (MD2) objects, for $f \leq 1$, then the minimum strong amplification would have been $A_{\rm min} = 1.34$, since for $f \geq 1$ we took A to be $A = -A_0$. In this case, the redistribution function would have been given by a delta function, due to the images which have never undergone a strong scattering, and by a long tail, due to all images that have been strongly scattered at least once; this tail is made of objects which are *at least* 1.34 times brighter than those which have never undergone strong scattering. This hole in the total redistribution function would have been an artifact of our having approximated $A(f)$ with a discontinuous function. For this reason, the approximation in equation (6b) is a better choice, for small impact parameters, than the exact equation (6a).

III. DOMAIN OF VALIDITY OF INTEGRAL EQUATION FOR THE CHANGE IN THE LUMINOSITY FUNCTION

Equation (8), which we have derived to describe the evolution of the number-magnitude relation of QSOs as the light rays pass through a set of zero-total-mass objects, is somewhat more general then previously stated. For instance, use of an appropriate probability distribution function $P(\Delta m)$ for whole galaxies will allow us, in Paper II, to derive the cosmological evolution of the luminosity function of QSOs. It has the advantage that it provides a single scalar equation for which we have derived a formal solution which can be expressed in close form (eqs. [12], [13]). The results reproduce successfully the results of Monte Carlo simulations (Young 1981) for sufficiently small image sizes. But, in the derivation of equation (8) we have tacitly assumed that the evolution of the number-magnitude relation from τ to $\tau + d\tau$ is essentially independent of its past history. The method of derivation was modeled on the theory of radiative transfer, and the independence of changes in the range $\tau \to \tau + d\tau$ on the prior path was called by Chandrasekhar the principle of invariance. It is not strictly valid for a treatment of gravitational lensing. In order to see what this assumption implies for the individual scattering, let us consider an initial distribution function which corresponds to a unique luminosity; i.e., $\mathfrak{L}(m,0) = \delta(m - m_0)$. Then, if $\tau_2 > \tau_1$, our solution of equation (8) (see eqs. [12], [13]) implies

$$\mathfrak{L}(m', \tau_2) = \int \delta(m - m_0) G(m' - m, \tau_2)\, dm',$$

while our previous assumption requires that

$$\begin{aligned}
\mathfrak{L}(m', \tau_2) &= \int \mathfrak{L}(m, \tau_1) G(m' - m, \tau_2 - \tau_1)\, dm \\
&= \iint \delta(m'' - m_0) \cdot G(m - m'', \tau_1) G(m' - m, \tau_2 - \tau_1)\, dm''\, dm \\
&= \int \delta(m'' - m_0) \cdot G(m' - m'', \tau_2)\, dm'',
\end{aligned} \tag{24}$$

where

$$G(m' - m'', \tau_2) = \int G(m' - m, \tau_2 - \tau_1) G(m - m'', \tau_1)\, dm.$$

This is equivalent to saying that the total probability of being magnified by an amount A must equal the product of being magnified by an amount A/A' times the probability of an amplification A' integrated over A'. In terms of the individual scatterings (both strong and weak), this means that, if we examine a situation where our image passes next to two scattering centers, its total amplification must be the product of two amplifications which each of the two scatterers would cause in the absence of the other. Let us call this the superposition principle. Due to the tensorial nature of the amplification phenomenon (see, for instance, Bourassa and Kantowski 1975), this is well known to be, in general, false (Press and Gunn 1973; Young 1981). In particular, it is this tensorial character that is responsible for the deformation of images going through regions containing many scattering centers. Is this problem serious? If we rewrite equation (24) by means of equation (14), we can actually see that this *strong* form of the superposition principle is not required, in our case. In fact,

$$\begin{aligned}
G(m' - m'', \tau_2) &= \int \exp\left[-(\tau_2 - \tau_1)\right]\{F[m' - m - \Delta_1(\tau_2 - \tau_1), \tau_2 - \tau_1] + \delta[m' - m - \Delta_1(\tau_2 - \tau_1)]\} \\
&\quad \times e^{-\tau_1}[F(m - m'' - \Delta_1\tau_1, \tau_1) + \delta(m - m'' - \Delta_1\tau_1)]\, dm \\
&= e^{-\tau_2}[\delta(m' - m'' - \Delta_1\tau_2) + F(m' - m'' - \Delta_1\tau_2, \tau_2 - \tau_1) + F(m' - m'' - \Delta_1\tau_2, \tau_1)] \\
&\quad + \int F[m' - m - \Delta_1(\tau_2 - \tau_1), \tau_2 - \tau_1] F(m - m'' - \Delta_1\tau_1, \tau_1)\, dm.
\end{aligned}$$

This illustrates the following fact: If we restrict ourselves to the case $\tau \leq 1$, where two double strong scatterings (probability $\tau^2 e^{-\tau}/2$) are relatively rare (corresponding to the fourth term above), then we only need a restricted version of the superposition principle, one where at least one of the two scatters is weak. It is shown in Appendix A that a version of this restricted superposition principle holds, in this sense: first, A_{TOT} is the average over the angle θ between the two scatterers (the unperturbed source image lies at the origin of the coordinate system on the plane of the sky on which the angle is taken). Second, not only $A \approx A_1 A_2$ but also

$$\lim_{R \to +\infty} \frac{A - A_1 A_2}{A_1 A_2} = 0$$

where R is the distance, on the plane of the sky of the furthest away of two scatterers from the unperturbed image. Thus $A = A_1 A_2$ is generally a good approximation.

In the same Appendix we show, furthermore, that the ratio between the probability that n strong scatters result in a total amplification A to the same probability for $n+1$ strong scatters is approximately

$$\frac{P_n}{P_{n+1}} = \frac{n(n+1)}{2\tau \ln A};$$

the condition that single strong scatters dominate is then $P_1/P_2 \gg 1$, or

$$\ln A \ll 1/\tau.$$

This proves the following: on the average (see Fig. 5) any light ray will be strong-scattered by our lumps once and only once, to very high accuracy, at least for all regions for which $\tau \lesssim 1$, which include all but the innermost $1''$–$2''$ of a galaxy (see later). All the rest of the time the light ray will be weakly scattered by our zero-total-mass lumps, and in this case their effect on the total amplification is always multiplicative. Thus, the total effect of our tube of MD2 is the total product of all individual amplifications, only one of which is strong. This completes our description of the total combined effect of MD2. The problem now remains of studying the validity of the superposition principle for the tube of material and the smooth galaxy which is obtained by adding all of the MD1 distributions.

In order to do this, we recall that the mass distributions that we adopted for our galaxy is the critical isothermal one. In this case $\tau = b_{\text{crit}}/2b$, where b_{crit} as given by equation (19b) denotes the radius within which the unperturbed image of the QSO must lie in order to form two images. Thus, our condition $\ln A \lesssim 1/\tau$ (which becomes $\tau \lesssim 0.5$ if we want to be able to retain accuracy for $A \leq 10$) shows that our previous argument limits us anyway to $b \gtrsim b_{\text{crit}}$, a condition which by means of a simple generalization of the results obtained in Appendix A is seen to imply a weak scattering due to the smooth potential well of the galaxy. Since we know already that we can always superpose multiplicatively the effects of a weak scattering with anything else, we conclude that, on one side, the validity of our approximation confines us to the region $b > b_{\text{crit}}$, on the other, since for these values of τ the single strong scattering approximation holds, the total amplification will be given by the product of all the individual magnifications, including that of the smooth gravitational well of the galaxy. This is in agreement with the tacit hypothesis we made in deriving equation (8), and shows that it is correct, provided $\tau \lesssim 1$.

IV. AN APPLICATION: LENSING BY NEARBY GALAXIES

a) *Mathematical Development*

The theory developed, which is applicable for $b \geq b_{\text{crit}}$, is ideal for testing the suggestion made by Canizares (1981), that the apparent enhancement of bright QSOs near bright galaxies is due to gravitational lensing of quasar images by individual stars in the outer halos of these galaxies since $b \gg b_{\text{crit}}$ in these regions. Canizares (1981) computed the probability of scattering by exactly one star. He neglected the effect of the smooth potential well and also the effect of "negative scatterings." By doing so, he was able to derive an expression for the overabundance of quasars close to a bright galaxy which appears to agree fairly well with the available observational evidence (see Canizares 1981 for references). We now analyze this problem in the framework of our theory. Our quantitative predictions are similar to those of Canizares in terms of $\delta N/N$, but in the next section we show that a very large sample is required for a significant detection. The advantages of our theory are that it incorporates in a natural fashion the effect of distant and nearby objects, and that it is consistent with low-τ previous calculations. The requirement of total energy conservation by lensing does not play any significant role here, since we are considering light rays passing relatively close to the center of a galaxy.

Since we are interested in a nearby galaxy, lensing a background object, we may make the useful approximation

$$\lambda \ll L, \quad \beta_{\text{cr}} = b_{\text{cr}}/\lambda \approx 4\pi(1+z_l)(\sigma_v/c)^2,$$

or (25a)

$$\beta_{\text{cr}} \approx 1.7(1+z_l)(\sigma_v/\sigma_*)^2 \text{ arcsec},$$

where σ_* is the one-component velocity dispersion of an L_* galaxy, here taken to be 240 km s^{-1}. For most studies concerning the overabundances of bright QSOs near bright galaxies, the survey is made by counting all quasars brighter than a definite apparent magnitude within a circle of radius R centered on the galaxies in question. For all surveys to date, $R \geq 15'$. Consider the average value of τ within this circle. From equation (20),

$$\bar{\tau} = \left(\frac{\beta_{cr}}{2\beta}\right) = \left(\int_0^R \frac{\beta_{cr}}{2\beta} 2\pi\beta\, d\beta\right)/\pi R^2 = \frac{\beta_{cr}}{R} = 0.003. \tag{25b}$$

This means $\beta \gg \beta_{cr}$, and the analysis we have developed for small optical depth is valid.

Let us discuss at this point the validity of assuming the galaxy to be a critical isothermal sphere. We expect that there will be a radius beyond which the heavy halo will start to dominate the mass distribution. For instance, Caldwell and Ostriker (1981) find that the total mass interior to the Sun in our galaxy is roughly equally divided between the normal matter and dark halo. The core radius for the halo is found to be $r_c = 15$ kpc; beyond this radius, $\rho \propto r^{-2}$ and the galaxy resembles a critical isothermal sphere. Let us assume that it holds, at least to within half an order of magnitude, for every galaxy. Essentially it is equivalent to saying that rotation curves become flat no farther than 15 kpc from the center of all galaxies. Then the condition that the $\beta_{\rm core} \equiv r_c/\lambda$ is $\beta_c \lesssim \beta_{cr}$ is that $\lambda \geq 600$ Mpc $(r_c/10$ kpc), which corresponds to $z = 0.2$. In addition, $\beta_{\rm core} \approx 1'.5$ for a typical galaxy in RC2. Since $\lambda_{\rm core}$ is small compared to $(R \geq 15')$, we will thus assume that galaxies are well represented by critical isothermal spheres.

Let us now compute the apparent luminosity function after lensing, $L_a(m)\, dm$, defining the one before lensing as $L_b(m)\, dm$. From equation (12) (with G replaced by P) one obtains

$$L_a(m) = \int P(m, m', \tau) L_b(m')\, dm'. \tag{26}$$

Now, using equations (20), (21), and (23) to define P, one may rewrite equation (26) as

$$L_a(m) = e^{-\tau}\left[L_b(m-\Delta) + \int F(m - m' - \Delta, \tau) L_b(m')\, dm'\right], \tag{27a}$$

where

$$\Delta \equiv \Delta_1 \tau - 2.5 \log A. \tag{27b}$$

Using (19) and (20), we find that to first order in τ

$$\Delta = k^{-1}[\tau - \ln(1 + 2\tau)] \approx -\tau/k, \tag{27c}$$

which yields

$$m - (m' + \Delta) \approx -2.5 \log\left(\frac{L}{L' e^{\tau}}\right). \tag{28}$$

Use of (28) gives, for the first term on the right-hand side of (27), to first order in τ,

$$e^{-\tau} L_b(m - \Delta) \approx L_b(m) - [L_b(m) - k^{-1} L_b'(m)]\tau + O(\tau^2), \tag{29}$$

where a prime denotes differentiation with respect to m. This shows the interesting fact that, to first order, the magnification effect produced by the whole galaxy is half balanced by the demagnification due to individual star scatterings. Development of F in powers of τ yields

$$F(m - (m' + \Delta), \tau) \equiv 2k\tau L'^2/L^2 + O(\tau^2), \tag{30}$$

as is evident from equation (7). Equations (29) and (30) yield, by substitution in (27),

$$L_a(m) - L_b(m) = \left[\frac{2k}{L^2}\int_m^\infty L'^2 L_b(m')\, dm' - L_b(m) + k^{-1} L_b'\right]\tau + O(\tau^2). \tag{31}$$

Our result, equation (31), gives the change in the luminosity function due to a nearby galaxy. It is clear that gravitational lensing does not necessarily produce an enhancement of objects at every magnitude. Rather, there exists an initial-luminosity-function–dependent limiting magnitude at which the sign of the enhancement changes. Equation (31) also shows that the number of quasars which are strongly magnified is balanced by the reduction of the luminosity function as it passes through a tube of optical depth τ and is not scattered, due to the requirement for number conservation. Notice, furthermore, that the apparent magnitude at which the change in sign takes place does not depend on τ or radius.

b) Application: Choice of Luminosity Function L_b and the Problem of Endpoints

Equation (31) explicitly reveals a difficulty, which is a consequence of the asymptotic behavior of F and, as such, does not depend on the limiting form (for $\tau \to 0$), but is a general property of gravitational lensing.

Consider a possible choice for L_b: suppose it were a power law, or, in our units, $L_b(m) = \text{const} \cdot 10^{am}$ for some a. Next consider the integral in (31):

$$I = \int_m^{+\infty} L'^2 L_b(m')\, dm' = \int_m^{+\infty} 10^{-0.8m'} 10^{am'}\, dm'. \tag{32}$$

Equation (32), in order to be convergent, requires $a \leq 0.8$, but this is in direct contrast to some observations: for instance, Green and Schmidt (1978) have cited $a = 0.93$ (for $15.5 < m < 18$). (For a resume of work done to date, see, for instance, Koo and Kron 1981).

There are two solutions to this problem: The first is to introduce a cutoff in $L_b(m)$. This seems very unphysical, because then the final result would depend critically on this cutoff; furthermore, it appears to be in contrast to available evidence. The other way out is to introduce a number-magnitude relation which flattens out at low luminosities. Koo and Kron (1981) have investigated the differential count versus magnitude distribution of quasars at faint ($J \geq 23$) magnitudes. They have also collected all the results obtained by other workers (see Koo and Kron for references, especially their Fig. 7). They show that the number-magnitude relation flattens out at about $J \approx 21$ mag, a result consistent with both radio and X-ray observations. In order to use the resulting number-magnitude relation in equation (31) we have fitted it with an analytical expression

$$\log A(m) = am^2 + bm + c \qquad (a = -0.048, b = +2.418). \tag{33}$$

The quantity c has been evaluated by imposing the condition that the number-magnitude relation be normalized within some range of magnitudes. The results are of course independent of this range. Notice that, besides reproducing the flattenings at low luminosities, our expression (33) also yields

$$\frac{d \log A(m)}{dm} = 0.9 \text{ at } m = 15.5,$$

in good agreement with Green and Schmidt's (1978) result. *An important conclusion of this work is that the degree, even sign of enhancement, in counts depends on the curvature of the initial number-magnitude relation.* Substitution of equation (33) into (31) allows us to draw some definite conclusions. The results are shown in Figure 4 for differential number-magnitude relations: the solid curve is L_a, the dotted one is L_b. The main feature of this plot is, of course, the different slope which they have at high luminosities. L_a has an l^{-2} tail [or an l^{-3} tail for $L(l)\, dl$], while L_b has the original one, which has been chosen to be $l^{-2.3}$, or $d \log A(m)/dm = 1$. The effect of lensing is considerable: at $m = 16$, it is about half an order of magnitude, even though it should be remembered that Figure 4 has been obtained for a value of $\beta = 3\beta_{\text{cr}}$ or $\tau = 1/6$, and that the final result, when integrated over θ, will be scaled down by a factor $3\beta_{\text{cr}}/R$, as shown in equation (31). Figure 4 also shows that the predicted crossing of the two luminosity functions happens at about $m = 25.5$. For *integral* number-magnitude relations this point will be moved toward even fainter magnitudes. Consideration of Figure 4 (and similar ones, for other values of β) supports our belief that our first-order analysis in τ is justified.

Let us cast our results into a form useful for comparison with observations and previous work. In order to do this, we integrate equation (31) in m from $-\infty$ to m, and in θ from 0, to θ to obtain, for the fractional enhancement $\delta N/N$ of all QSOs brighter than magnitude m within a radius of a galaxy,

$$\frac{\delta N}{N} = \frac{\int_0^\theta 2\pi\theta'\, d\theta' \int_{-\infty}^m [L_a(m) - L_b(m)]\, dm}{\pi\theta^2 \int_{-\infty}^m L_b(m)\, dm} \equiv f(m) \times \frac{2\beta_{\text{cr}}}{\theta}. \tag{34}$$

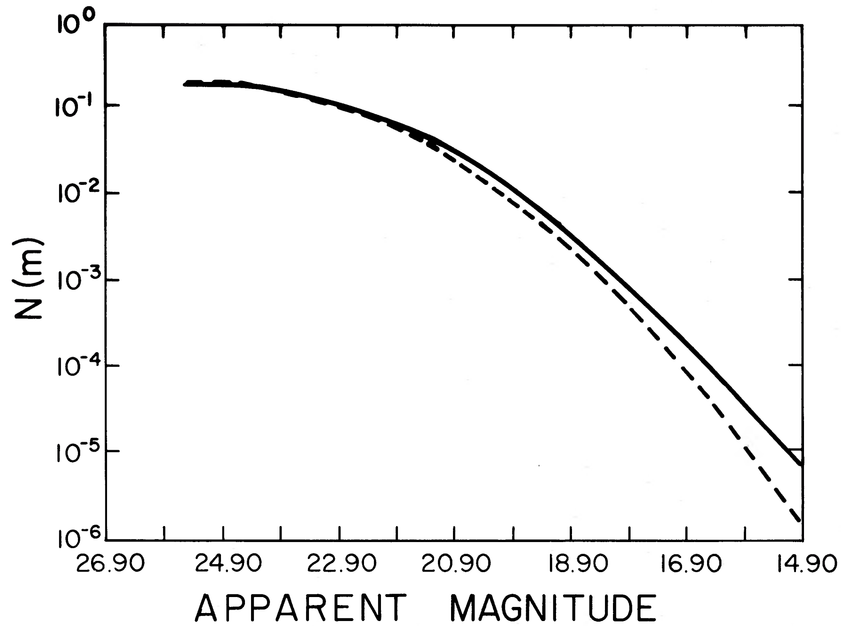

FIG. 4.—Differential number count relation (arbitrary normalizations) before (*dashed*) and after (*solid curve*) going through a $\tau = 1/6$ tube of matter.

TABLE 1

COMPUTED EFFECTS OF LENSING BY A SINGLE GALAXY

m (1)	$(\delta N/N)_{\rm int}$ (2)	$(\delta N/N)_{\rm diff}$ (3)	α (4)
15......	17.30	13.05	3.40
16......	10.39	8.19	2.24
17......	6.93	5.62	1.66
18......	4.97	4.07	1.33
19......	3.72	3.04	1.13
20......	2.85	2.28	1.00
21......	2.20	1.69	0.92
22......	1.69	1.19	0.85
23......	1.27	0.75	0.80

In Table 1, $f(m)$ is shown in column (2) using as initial number magnitude relation for quasars equation (33) adapted from Koo and Kron (1981). Using equation (25), this gives, for total enhancement,

$$\frac{\delta N}{N} = 0.02 f(m)(1+z_1)\left(\frac{\sigma_v}{\sigma_*}\right)^2 \theta^{-1}, \tag{35}$$

where θ is measured in arcminutes. Equation (35) gives an observable excess consistent with Canizares's computation (notice that our f, at equal m, is smaller than Canizares's), at least up to values of $m \lesssim 22$. After that, the depopulation of the number-magnitude relation required by number conservation can be seen from the data displayed in Table 1; the change of sign occurs beyond $m \approx 25$.

How significant is the effect described by equation (35), when compared with random statistical fluctuation in the surface density of quasars near bright galaxies? If $\Sigma(m)$ is the surface density per arcmin^2 of all quasars brighter than m, then their total number within an angular distance θ from the center of the galaxy is $N(m)$ is $\pi\theta^2\Sigma(m)$, and the statistical uncertainty is $\pm\sqrt{N(m)} = \pm[\pi\Sigma(m)^{1/2}\theta$. Then, from equation (35), we get

$$\frac{\delta N}{\sigma} = 0.02\pi^{1/2}(1+z_1)\left(\frac{\sigma_v}{\sigma_*}\right)^2\{f(m)[\Sigma(m)]^{1/2}\} \equiv 3.5\times10^{-2}(1+z_1)\left(\frac{\sigma_v}{240}\right)^2 g(m). \tag{36}$$

TABLE 2

THE FUNCTION $g(m)$ IN EQUATION (37)

m	$g(m)/\Sigma_0^{1/2}$
18	0.07
19	0.11
20	0.18
21	0.26
22	0.33
23	0.38
24	0.37

Equation (36) shows that the signal-to-noise ratio is independent of the angular distance θ, provided our line of sight still crosses the halo of the galaxy. Also, using for Σm: equation (33), we obtain

$$g(m) \equiv f(m)[\Sigma(m)]^{1/2} = \left(\frac{\Sigma_0}{\pi}\right)^{1/2} f(m) \int_{-\infty}^{\Xi} e^{-\xi^2} d\xi, \tag{37}$$

where $\Xi \equiv (m + b/2a)(|a| \ln 10)^{1/2}$ and where Σ_0 is the total number of quasars per arcmin^2. It is obvious that $g(m)$ must have a maximum. In fact, for $m \to -\infty$, $g(m) \to 0$, and $g(m) = 0$ for $m \approx 26.5$, since, for $m = 26.5$, $f(m)$ changes sign. Table 2 shows the values of $g(m)/\Sigma_0^{1/2}$ for $18 \leq m \leq 26$. From it we can see that, for a limiting magnitude $m \approx 23$, our signal-to-noise ratio will be maximum. The normalization factor can be extrapolated from Koo and Kron's (1981) finding that the surface density of QSOs between $J = 22.5$ and $J = 21.5$ is 140 ± 22 QSOs per square degree, using the previously adopted functional form for $\Sigma(m)$. We obtain $\Sigma_0 = 0.5$ arcmin^{-2}, which implies a *maximum* signal-to-noise ratio, for an L_* galaxy, $\delta N/\sigma = 5.6 \times 10^{-3}$. Obviously, this makes it impossible to detect the effect of gravitational lenses in any single galaxy. Analysis of a large statistical sample is required. In fact, even in this most favorable situation, since

$$\frac{\delta N}{\sigma} = N_{\text{gal}}^{1/2}\left(\frac{\delta N}{\sigma}\right)_{\text{gal}} = 5.6 \times 10^{-3} N_{\text{gal}}^{1/2},$$

where N_{gal} is the total number of galaxies considered, in order to have $\delta N/\sigma \geq 1$, we need $N_{\text{gal}} \geq 10^{4.5}$. It is thus obvious that the effect which Canizares proposes to have detected cannot be due to gravitational lensing, since, in all of the analysis quoted by him, $N_{\text{gal}} \sim 10^3$.

Although no conclusions can be reached on the basis of presently available data, we agree with Canizares that galaxy lensing may be significant and is, in principle, detectable. We address next the question: What is the relative importance in lensing of individual stars and the galaxy potential? In order to obtain the number-magnitude relation after lensing by the simple smoothed potential well of the galaxy, set $\tau = 0$ in equation (27) then develop in powers of β_{cr}/β using equation (19) to obtain

$$L_a(m) = L_b(m) + \frac{5}{\ln 10} \frac{\beta_{\text{cr}}}{2\beta} \frac{dL_b}{dm}. \tag{38}$$

Then define the parameter

$$\alpha \equiv \frac{(\delta N)^{\text{stars + galaxy}}}{(\delta N)_{\text{galaxy only}}},$$

where δN is defined as in equation (34) to represent the relative strength; α is presented in column (4) of Table 1, which shows the large preponderance of single stars in the creation of the luminous tail. This is again qualitatively consistent with the conclusions of Canizares (1981) and of Gott (1981).

The behavior of the parameter α is not easy to understand. In fact, there are two processes occurring: in regions of steep number-magnitude relation (NS) even a small magnification due to the galaxy potential well will correspond to a large change of the NS; on the other side, since regions of steep NS are those of very high luminosities, even a small τ will produce a high-luminosity tail (by acting on the bulk of the luminosity function, which is some magnitudes weaker), tending to compensate for the effect of the potential well. It then appears that the ratio α should be of roughly order of magnitude 1, which it is (see Table 1), even though its exact determination, which is of absolutely fundamental importance for the problem of discriminating between heavy halos made of neutrinos and of stars, shows it to be half a magnitude away from this value, at luminosities which are easily accessible.

As emphasized by Gott, there are circumstances in which the granularity of the galactic potential will not produce minilensing. If the region of the quasar which produces the optical emission has angular size $\beta_Q = r_{Q/L}$, then the approximations upon which geometrical optics are based will break down for observed wavelengths $w > \min[\beta_Q^2 \lambda, 4GM_l(1+z_l)(L-\lambda)/L_c^2]$. This will never occur for lenses of astronomical size but always for subatomic "lenses" such as neutrinos. More serious is the requirement that the angular size of the quasar be small compared to that of the lens: specifically there will be no minilensing if $\beta_Q > \beta_{cr}(I/I_0)^{1/2}$ which translates roughly to a requirement *for* minilensing of $M_l > 10^{-4}\ M_\odot(r_Q/10^{15}\ \text{cm})^2$. Table 2 also shows the signal-to-noise ratio for a single L_* galaxy as a function of limiting magnitude if we neglect the contribution of individual stars.

V. CONCLUSIONS

We find that a significant increase in the apparent density of quasars should, in principle, be observable in the vicinity of bright galaxies. Because of the small surface density of quasars the effect cannot be securely determined except with relatively large samples ($N_{gal} > 10^{4.5}$) of galaxies. The dependence of the increase on the apparent magnitude of the quasar, the absolute magnitude of the galaxy, and the projected separation of the galaxy and quasar can be predicted so that it should be possible to determine whether positive effects (assuming they are securely detected) are attributable to lensing. Since mini-lenses within the lensing galaxies contribute a large part of the postulated effect, a measurement should allow one to say something definite about the size of the typical mass unit in galactic halos. The predicted effect depends quite sensitively on the shape of the average "unlensed" number-magnitude relation for quasars, so any test of the lensing effect will require a careful simultaneous determination of this important observational relation.

Our analysis differs from previous work in several respects. The first point is that negative scatterings are included; we have shown that mass distributions made of a (positive) delta function plus a smooth (negative) density distribution do require them: we have shown it both by means of the principle of energy conservation and by a more detailed analysis. These negative scatterings, which are analogous to those discussed by Avni (1981), are due to the specific density distributions we considered. In our problem, there is a total net amplification since we are assuming our light rays to pass close to a galaxy; the main effect of these negative scatterings is to (half) balance the magnifying effect of the smooth potential well due to the whole galaxy, as shown by equation (31).

Another important result is the L^{-2} tail (see eq. [31]) in the number-magnitude relation which is caused by the presence of stars in the outer part of the galaxy: we have shown, in fact, that this tail dominates the effect due to the smooth potential well at high luminosities. As we mentioned above, it is obvious that the slope (-3), $[L(l)\,dl = \phi_0 l^{-3}\,dl]$, is a critical slope: that is, for steeper luminosity functions, the individual stars tend to deform this slope toward the value -3, while shallower number-magnitude relation are insensitive to their effect. In this sense, it is then a lucky chance that the slope of the unlensed number-magnitude relation is found to be 3.3 (Green and Schmidt 1978), allowing us to detect this phenomenon.

We have also considered the relative importance of stars and the potential well in changing the initial number-magnitude relation. This ratio has been found large still, but considerably smaller than that found by Canizares.

Furthermore, we have shown that it is absolutely necessary to use a correct initial number count which flattens out at faint magnitudes, since the observed effects depend critically on its curvature.

Since at very faint magnitudes we predict a *depletion* of the quasar surface density and at very bright magnitudes the sample size is very small, it is clear that there is an optimal apparent magnitude at which the signal-to-noise ratio for the expected enhancement is maximal. On the basis of this preliminary study we suggest that searches be made in the vicinity of m (B or J magnitude) ≈ 23.

Using, as an approximate calibration for the quasar number-flux relation, a numerical fit to results of Kron and Koo (1981) we find that Canizares's study of quasars to a limiting magnitude of 19 near 2000 galaxies should not have detected a statistically significant enhancement of quasars. A sample of galaxies at least 10 times more numerous and a quasar search to 3–5 mag fainter would be required before the predicted effect could be found.

We are grateful for the advice of many of our colleagues, especially J. Gunn, J. Goodman, and S. Chitre, and to C. Canizares for sending us preprints in advance of publication. The research was supported in part by National Science Foundation grant AST 80-22785.

APPENDIX A

In this Appendix we derive a theorem on the superposition of amplification due to two scatterers. Our goal is to show that two objects will cause a gravitation amplification on the light of a background source, such that, when averaged over the angle on the plane of the sky, it is equal to the product of the amplifications which each object would have caused in the absence of the other. In order to do this, we will employ the formalism of Bourassa and Kantowski (1975). In their formalism they introduce a complex function I, defined as

$$I(x_0, y_0) = \frac{c}{2G}\left[\int_{-\infty}^{+\infty} \frac{\partial\phi(x_0, y_0, ct)}{\partial x} dt - i\int_{-\infty}^{+\infty} \frac{\partial\phi(x_0, y_0, ct)}{\partial y} dt\right], \tag{A1}$$

which is related to the bending angle α by

$$\alpha = -(4G/c^2) I^*, \tag{A2}$$

where the asterisk denotes complex conjugation. The variables in which both α and I are expressed are the location, on the plane of the sky, of the image of the source, in physical units. The source itself is located at z, where z is given by

$$z = z_0 - (4GD/c^2) I^*, \tag{A3}$$

where $D = [(L-\lambda)\lambda]/L$, with L = observer-source distance, $L-\lambda$ = scatterer-source distance, λ = scatterer-observer distance. The total amplification A is given by

$$A = \frac{1}{1-|F|^2}, \tag{A4}$$

where

$$F = \frac{4GD}{c^2}\frac{dI}{dz_0}. \tag{A5}$$

We will confine our attention to the case where only two point sources are present. Then

$$\phi(r) = -G\left(\frac{M_1}{|r-r_1|} + \frac{M_2}{|r-r_2|}\right),$$

where r_1, r_2 are the locations of the point sources. Then

$$I = \frac{M_1}{z-z_1} + \frac{M_2}{z-z_2},$$

and

$$F = -\frac{4\pi GD}{c^2}\left[\frac{M_1}{(z-z_1)^2} + \frac{M_2}{(z-z_2)^2}\right].$$

Let us confine ourselves to the case where the two masses are equal and lie at the same distance R, on the plane of the sky, from the undeflected source. Furthermore, let the first object lie on the x-axis. Then we have

$$\frac{c^4}{16\pi^2G^2D^2M^2}|F|^2 = \left\{\frac{(x-R)^2 - y^2}{\left[(x-R)^2+y^2\right]^2} + \frac{(x-R\cos\theta)^2-(y-R\sin\theta)^2}{\left[x^2+y^2+R^2-2xR\cos\theta-2yR\sin\theta\right]^2}\right\}^2$$
$$+\left\{\frac{2y(x-R)}{\left[(x-R)^2+y^2\right]^2} + \frac{2(x-R\cos\theta)(y-R\sin\theta)}{\left[x^2+y^2+R^2-2xR\cos\theta-2yR\sin\theta\right]^2}\right\}^2, \tag{A6}$$

where the terms due to the first object are the first ones in each set of parentheses, and vice versa for the second object. The coordinates appearing in (A6) are those of the deflected image. We need to compare situations where it is the source to be held fixed. We will always assume, in the following, that the unscattered image lies at the origin of our reference system on the plane of the sky.

The approximation we are looking for does not need to hold for every scattering, as we shall see below. All that we need is a form for the superposition of one scattering whose net result is only a weak amplification, with any other amplification. Let us tackle first the problem in which both scatterings end up in weak amplifications; that is, let $R \to \infty$. The equation which ties the position of the source (the origin!) and that of the image is

$$z = \frac{4GD}{c^2}\left[\frac{M}{(z-z_1)^*} + \frac{M}{(z-z_2)^*}\right]. \tag{A7}$$

In the limit $R \to \infty$, it is easy to see that the two images forming close to the two scatterers, which they approach as $R \to +\infty$, become fainter and fainter. The only other existing solution is close to the origin, where the undeflected source lies; its brightness tends to that of the source. In this case, $|z| \ll R$, and, to the first nonzero order we obtain

$$z \approx -\lambda^2\left(\frac{1}{z_1} + \frac{1}{z_2}\right)^*, \quad \text{where } \lambda^2 = \frac{4GM}{c^2}D,$$

which shows that, in this limit, the displacements of the image with respect to the source add vectorially. Using, as assumed above, $z_1 = (R,0)$, $z_2 = (R\cos\theta, R\sin\theta)$,

$$z \approx -(\lambda^2/R)(1+e^{i\theta}). \tag{A8}$$

This gives us a condition for our approximation to be valid. In fact, we must have $|z| \ll R$, or $R < \lambda = (4GDM/c^2)^{1/2}$, which is exactly the opposite of our seemingly arbitrary condition for strong scattering. Thus we see that gravitation and geometry provide us with a natural scale length in our problem.

We are now ready to insert (A8) into (A6). Before doing that, though, we notice that, since $|z|$ is $O(1/r)$, we can expand each one of the two terms in the two parentheses in Taylor series in z, retaining only the first order term. With some rearranging,

$$\frac{c^4}{16\pi^2G^2D^2M^2}\left[|F|^2 - \left(|F_1|^2 + |F_2|^2\right)\right] \approx c, \tag{A9}$$

where $|F_1|^2$ and $|F_2|^2$ are the terms due to either one of the two scatterers, in the absence of the other, each one computed at the point where the image would be in the absence of the other; and c represents the cross terms. We now average c over θ, the angular separation between the two deflectors. We have

$$c/\lambda^4 = \sum_{j=1}^{7} c_j,$$

where the sum extends over the only nonzero terms, up to order R^{-6}. We have

$$c_1 = \frac{1}{2\pi}\int_0^{2\pi} c_1(\theta)\,d\theta = \overline{-\frac{4\lambda^2}{R^6}\cos\theta} = 0, \qquad c_2 = -\frac{2\lambda^2}{R^6}\,\overline{\cos\theta(2\cos 2\theta - 1)(1+\cos\theta)} = 0,$$

$$c_3 = -\frac{2\lambda^2}{R^6}\,\overline{\sin^2\theta(4\cos^2\theta - 3)} = \frac{4\lambda^2}{R^6}, \qquad c_4 = -\frac{4\lambda^2}{R^6}\,\overline{(1+\cos\theta)(2\cos^2\theta - 1)} = 0,$$

$$c_5 = -\frac{4\lambda^2}{R^6}\,\overline{\cos\theta(4\cos^2\theta - 3)(2\cos^2\theta - 1)} = 0, \qquad c_6 = -\frac{8\lambda^2}{R^6}\,\overline{\sin^2\theta\cos\theta(4\cos^2\theta - 1)} = 0,$$

$$c_7 = \frac{8\lambda^2}{R^6}\,\overline{\sin^2\theta\cos\theta} = 0,$$

from which $c = 4(\lambda/R)^6$.

Notice, now, that in (A9), while $c \propto R^{-6}$, F_1^2 and F_2^2 are $O(R^{-4})$. Then we have, using (A4),

$$A = \frac{1}{1-|F|^2} \approx 1 + |F|^2 \approx 1 + |F_1|^2 + |F_2|^2 + 4(\lambda/R)^6 \geq 1 + |F_1|^2 + |F_2|^2 \approx \frac{1}{\left(1-|F_1|^2\right)} \frac{1}{1-|F_2|^2} = A_1 A_2 .$$

We thus have our simple theorem: the amplification A of a source, due to two equal-mass scatterers at a common distance R from the source, is such that $A \geq A_1 A_2$, where A_1 and A_2 are the amplification of the one source when the other is missing altogether. Furthermore, since $(A-1)$ is of order R^{-4}, and $A - A_1 A_2$ is of order R^{-6}, one has that $A = A_1 A_2$ is an excellent approximation, for $R > \lambda = (4GDM/c^2)^{1/2}$.

We tackle now the case where only one of the amplifications is weak; that is, we keep one of the scatterers at fixed, nonlarge distance R, from the source, and let the other scatterer go to infinity ($R_2 \to +\infty$). We are going to prove a theorem totally analogous to the one stated above.

We proceed as before. The equation which determines the location of the images

$$z = -\lambda^2 \left(\frac{1}{z - z_1} + \frac{1}{z - z_2} \right)^* ,$$

can now be replaced, taking into account the fact that $|z| < R_2$, by

$$z \approx -\lambda^2/(z - z_1)^* , \tag{A10}$$

or, the location of the image is determined only by the close deflector. Substitution of (A10) into (A6), where now the first terms in the parentheses have $R = R_1$ and the second ones have $R = R_2$, shows that the second terms can be approximated by the first order Taylor series, while we maintain the terms depending on the first scatter in their exact form. Having done this, and having rearranged the terms so that also $|F_2|^2$ depends on the positions of the scattered image in the absence of the first deflector, we obtain

$$|F|^2 = |F_1|^2 + |F_2|^2 + C ,$$

where, again, c represents the cross terms, and expresses the nonlinearity of the amplification process. Averaging c over θ, we get

$$c = \frac{7}{2} (\lambda/R_2)^6 ,$$

and the total amplification A,

$$A = \frac{1}{1-|F|^2} \approx \frac{1}{1 - \left(|F_1|^2 + |F_2|^2\right) + \frac{7}{2}(\lambda/R_2)^6} ,$$

where $|F_1|^2$ is $O(R^0)$, $|F_2|^2$ is $O(R^4)$. Then

$$A \approx \frac{1}{1 - \left(|F_1|^2 + |F_2|^2\right)} \geq \frac{1}{1-|F_1|^2} \frac{1}{1-|F_2|^2} = A_1 A_2 ;$$

and our previous theorem holds again. It is also easy to see that neither the assumption of equal masses nor that of point sources are important in the proof of these theorems, which then hold in a larger class of cases than has been shown here.

In the main article, what has just been shown is used to superpose the effects of the (weak) deamplification of our negative mass objects with what is defined as a "strong scattering." Is this enough? Indeed it is, since, as we are going to show, at a fixed optical depth and fixed total amplification, most of contribution comes from one strong scattering, and the contribution of two or more is negligible.

In order to show this, let us first remark that, in a galaxy, τ is small. In fact, we have

$$\tau = \lambda^2 \cdot l \cdot n_0 = 0.01 \left(\frac{D}{10^3 \text{ Mpc}} \right) \left(\frac{M}{M_\odot} \right) \left(\frac{l}{30 \text{ kpc}} \right) \left(\frac{N}{10^{11} \text{ stars}} \right) \left(\frac{30 \text{ kpc}}{r} \right)^3 ,$$

where l is the path length through a galaxy with N stars of mass M, and a radius r. Thus τ is a small quantity.

Now, we know that the probability that n scatters take place, while moving through an optical depth τ, is $\tau^n e^{-\tau}/n!$. We want to know the probability that, given that there are n scatters, their final result is an amplification A. We suppose all scatters' amplifications to be multiplicative (that is, $A = A_1 A_2$) and we approximate the total amplification due to a scatter with

$$A_1 = \frac{f^2+2}{f(f^2+4)^{1/2}} \approx \frac{1}{f} \text{ (see eq. [6a], [6b]).}$$

This approximation is discussed in the following Appendix. Then

$$A = A_1 A_2 \ldots A_n = \frac{1}{f_1 f_2 \cdots f_n}.$$

The probability that n scatters give a total amplification A can be obtained by integrating the product

$$\prod_{i=1}^{n} 2f_i\, df_i$$

of the normalized differential cross sections over the hypersurface $A = 1/(f_1 f_2 \ldots f_n) = \text{const}$. To do this, we change variables. We put $f_n = 1/(Af_1 f_2 \ldots f_n)$ and obtain

$$P(A)\, dA = dA \int df_1 \int df_2 \ldots \int df_{n-1} \prod_{i=1}^{N} (2f_i) J\left(\frac{f_1, f_2 \ldots f_n}{f_1, f_2 \ldots A}\right),$$

which yields

$$P(A)\, dA = \frac{dA\, n}{A^3} 2 \int_{1/A}^{1} \frac{df_1}{f_1} \int_{1/Af_1}^{1} \frac{df_2}{f_2} \cdots \int_{1/Af_1 f_2 \ldots f_{n-2}}^{1} \frac{df_{n-1}}{f_n - 1}$$

where the integration limits are determined by the condition $f_n \le 1$ (which implies $f_{n-1} \ge 1/[Af_1 f_2 \ldots f_{n-2}]$) which descends from our definition of strong scattering. Introducing the variables $\mu_i = \ln f_i$, we obtain

$$P(A)\, dA = \frac{dA}{A^3} 2^n \int_{\alpha}^{0} d\mu_1 \int_{\alpha - \mu_2}^{0} d\mu_2 \ldots \int_{\alpha - \xi}^{0},$$

where $\xi \equiv \Sigma_{i=1}^{n-2} \mu_1$, and $\alpha \equiv -\ln A$, which can be evaluated by induction and yields

$$P_n(A) = \frac{2^n}{A^3} \frac{(\ln A)^{n-1}}{(n-1)!}.$$

Then, the probability that n scatters take place, and result in an amplification A, is

$$P_n(A) = \frac{\tau^n e^{-\tau}}{n!} \frac{2^n}{(n-1)!} \frac{(\ln A)^{n-1}}{A^3}.$$

The condition for simple scatterings to dominate is thus seen to be $P_1/P_2 \ge 1$, or

$$\tau(\ln A) < 1.$$

For τ's of the same order of magnitude as the one computed above, we see that double scatterings dominate only for $A > e^{100} \approx 10^{43}$; this proves our point that two or more strong scatterings are negligible, and thus that the superposition of two of them is irrelevant in our problem. We can also compute analytically both the average number of scatters as a function of A for a given τ, and the associated rms. We have in fact that the total probability of being amplified by an amount A is

$$P_{\rm TOT}(A) = \sum_{n=1}^{+\infty} \frac{\tau^n e^{-\tau}}{n!} \frac{2^n}{(n-1)!} \frac{(\ln A)^{n-1}}{A^3} = \frac{2\tau e^{-\tau}}{A^3} \frac{I_1\left[(8\tau \ln A)^{1/2}\right]}{(2\tau \ln A)^{1/2}} .$$

If one obtains $P_{\rm TOT}(m)$ from the above, this is easily seen to be equal to (B13). The average number of scatters is thus seen to be

$$N = \left[\Sigma n \frac{\tau^n e^{-\tau}}{n!} \frac{2^n}{(n-1)!} \frac{(\ln A)^{n-1}}{A^3}\right] \bigg/ \left\{ \frac{2\tau e^{-\tau}}{A^3} \frac{I_1\left[(8\tau \ln A)^{1/2}\right]}{(2\tau \ln A)^{1/2}} \right\} = (2\tau \ln A)^{1/2} \frac{I_0\left[(8\tau \ln A)^{1/2}\right]}{I_1\left[(8\tau \ln A)\right]^{1/2}} ,$$

and we obtain also

$$\overline{N^2} = \overline{N} + 2\tau \ln A ,$$

and then $\sigma = [N^2 - (N)^2]^{1/2}$. For the case $\tau = 0.01$, N, $N + \sigma$, and $N - \sigma$ are illustrated in Figure 5.

APPENDIX B

We derive here equations (16) and (16b). Let us recall that

$$F(m,\tau) \equiv \int_{-\infty}^{+\infty} \left(e^{\tau P(\lambda)} - 1\right) e^{-2\pi i \lambda m}\, d\lambda , \tag{B1}$$

where

$$P(\lambda) \equiv \int_{-\infty}^{M_{\rm min}} e^{2\pi i \lambda m} P(m)\, dm , \tag{B2}$$

where $P(m)$ is the probability that strong scattering results in an amplification A such that $m = -2.5 \log A$. Since strong scatterings for MD2 objects have been defined as those for which the impact parameter f in rationalized units

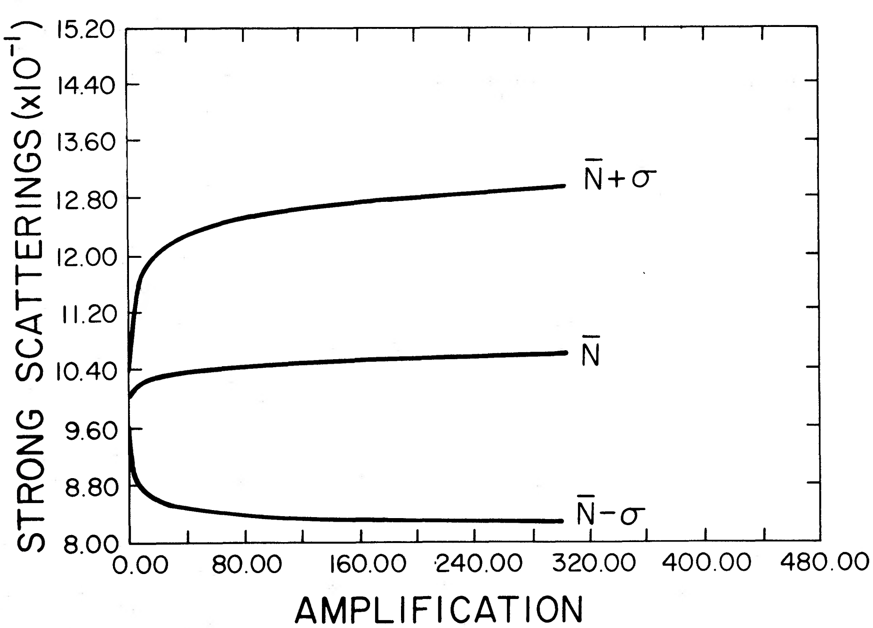

FIG. 5.—Average number of strong scatterings ($\overline{N}$) as a function of total amplification and the ± 1 σ deviations for the case $\tau = 0.01$

($f = l/\alpha$; see eq. [4]) is less than 1, the integral in equation (A2) should not be carried out from $-\infty$ to zero, but rather to some limiting amplification. $P(m)\,dm$ can be obtained from

$$P(f)\,df = P(m)\,dm, \tag{B3}$$

where $df P(f) = 2\pi f\,df/\pi f_{\rm lim}^2$, and, since by definition $f_{\rm lim} = 1$ for strong interactions, $P(f) = 2f$. The exact functional dependence of $A \equiv 10^{-m/2.5}$ on f is

$$A = \frac{1}{f}\frac{f^2+2}{(f^2+4)^{1/2}}, \tag{B4}$$

which yields, once substituted in (B3), an expression too difficult to be integrated analytically in (B2). Thus we introduce the approximation

$$A \approx 1/f, \tag{B5}$$

which actually underestimates the amount of amplification due to these lumps, even though only by 10%. In fact, the average amplification due to strong scattering resulting from (B4) is

$$\bar{A} = \int_0^1 A(f)P(f)\,df = \sqrt{5},$$

while the approximation yields

$$\bar{A}_{\rm app} = \int_0^1 \frac{1}{f}P(f)\,df = 2,$$

which proves our point. $\bar{A}_{\rm app}$ having been computed, equation (9b) yields

$$\Delta_1 = \frac{\bar{A}-1}{k} = \frac{1}{k} = \frac{2.5}{\ln 10} = 1.086. \tag{B6}$$

Substitution of equation (B5) into (B3) yields

$$P(m) = 2ke^{2km}. \tag{B7}$$

Consider now the superior limit in the integral appearing in equation (A2). It is defined as $m_{\rm lim} = 2.5\log A(f=1)$ which, given the approximation (A5), is $m_{\rm lim} = 0$. Then we have

$$P(\lambda) = \int_{-\infty}^{0} 2ke^{2\pi i\lambda m}e^{2km}\,dm = \frac{1}{1+i\pi\lambda/k} \tag{B8}$$

and

$$F(m,\tau) = \int_{-\infty}^{+\infty}\left[\exp\left(\frac{\tau}{1+i\pi\lambda/k}\right)-1\right]\exp(-2\pi i\lambda m)\,d\lambda$$

or

$$F = \sum_{n=1}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{n!}\frac{\tau^{n}e^{-2\pi i\lambda m}}{(1+i\pi\lambda/k)^n}\,d\lambda. \tag{B9}$$

Integration by means of complex variables techniques shows that $f(m) = 0$ for $m > 0$, which is exactly what we expected: that is, strong scattering alone cannot produce demagnification. Furthermore, given that

$$\int_{-\infty}^{+\infty}\frac{e^{-2\pi i\lambda m}}{(1+i\pi\lambda/k)^n}\,d\lambda = \frac{1}{(n-1)!}\frac{(2k|m|)^n}{|m|}e^{2km},$$

we have

$$F(m,\tau) = e^{2km}(2k\tau)\sum \frac{1}{n!(n-1)!}(2k\tau|m|)^{n-1}. \tag{B10}$$

For the evaluation of (B10), let us rewrite F as

$$F = -e^{2km}\frac{d}{dm}\sum_{n=0}^{+\infty}\frac{1}{(n!)^2}(2k\tau|m|)^n. \tag{B11}$$

Recalling the series expansion of Bessel's I functions,

$$I_0(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2},$$

one obtains

$$\sum_{n=0}^{+\infty}\frac{1}{(n!)^2}(2k\tau|m|)^n = I_0\left[(8k\tau|m|)^{1/2}\right] \tag{B12}$$

Furthermore, let us use $dI_0(x)/dx = I_1(x)$ and (B12) in (B11) to obtain

$$F(m,\tau) = \left(\frac{2k\tau}{|m|}\right)^{1/2} e^{2km} I_1\left[(8k\tau|m|)^{1/2}\right] \tag{B13}$$

for $m < 0$, $F = 0$ for $m > 0$. (B13) is equation (16).

APPENDIX C

We show here the existence of negative scatterings (i.e., scatterings which produce a total demagnification of the source, $|A| < 1$) for the zero total mass objects referred to as lumps in the text. Their mass distribution is

$$M(r) = M_0\left[1-(r/l)^3\right],$$

where M_0 is the mass of the star and l is the interstellar mean distance. For such a mass distribution, whose center is located at the origin, $\nabla\phi = +[GM(r)/r^2]\hat{r}$, and, again in Bourassa and Kantowsky's (1975) formalism, we have

$$I = M_0\frac{\left(1 - z_0 z_0^*/l^2\right)^{3/2}}{z_0}, \qquad z_0 = x_0 + iy_0,$$

$$F = \frac{4GD}{c^2}\frac{1}{2}\left(\frac{\partial I}{\partial x_0} - i\frac{\partial I}{\partial y_0}\right) = -\frac{4GDM}{c^2}\frac{\left(1 - z_0 z_0^*/l^2\right)^{1/2}}{z_0^2}\left(1 + \frac{1}{2}\frac{z_0 z_0^*}{l^2}\right),$$

$$G = 1 - \frac{4GD}{c^2}\frac{1}{2}\left(\frac{\partial I}{\partial x_0} + i\frac{\partial I}{\partial y_0}\right) = 1 + \frac{3}{2}\frac{4GDM}{c^2 l^2}\left(1 - \frac{z_0 z_0^*}{l^2}\right)^{1/2}.$$

In this formalism, defining $\alpha \equiv 4GDM/c^2l^2$ and $x = |z_0|/l$, we get

$$A = \frac{1}{G^2 - |F|^2} = \left[1 + \frac{9\alpha^2}{4}(1-x^2) + 3\alpha(1-x^2)^{1/2} - \alpha^2\frac{1-x^2}{x^4}\left(1 + \frac{x^2}{2}\right)^2\right]^{-1}, \tag{C1}$$

and the impact parameter x is related to the position r of the unperturbed source by

$$r = x + \alpha(1 - x^2)^{3/2}/x, \tag{C2}$$

where all of these formulae hold only for $x \leq 1$, which can be seen from (C2) to imply $r < 1$, since, for larger values of r, the light rays pass outside a lump of zero total mass. From (C2) it is easily seen that, for $r \to 1$, $x \to 1$. In this case, the term $1 + 3(1 - x^2)^{1/2}$ dominates the denominator of (C1), thus implying that, for sufficiently large impact parameters, $A < 1$. Remember that, in these units, $R_{\mathrm{STRONG}} \approx 10^{-3}$.

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