

REVERBERATION MAPPING OF THE EMISSION LINE REGIONS OF SEYFERT GALAXIES AND QUASARS

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ABSTRACT

Variations in the strengths of the central photoionizing source in a quasar or Seyfert galaxy will generate variations in the strengths and profiles of the emission lines. These “reverberations” in the emission lines will lag behind the continuum variations due to light travel time effects. A procedure is described for analyzing a time series of measurements of both the continuum and the lines. This procedure permits direct verification of the assumed causal connection of the lines to the continuum. We demonstrate that if the emission line region has a high degree of symmetry, then it is possible to invert the time-dependent line profiles and obtain the phase space distribution of the emission-line gas—i.e., its emissivity and the moments of its velocity distribution as functions of position. The cases of spherical and disk symmetry are considered in detail; the case of a straight jet, which may be relevant to correlated optical and radio variations, is discussed briefly. Explicit calculations of expected line variations have been carried out for several simple models. We suggest that with recently developed instrumentation it should now be possible to apply this technique to Seyfert galaxies. Long term, highly accurate observations will be required for the application to quasars.

Subject headings: galaxies: Seyfert — line profiles — quasars — radiative transfer

I. INTRODUCTION

The emission line spectra of quasars and Seyfert I galaxies are conventionally explained in terms of emission from gas irradiated by a central source of ionizing radiation (e.g., Bahcall and Kozlovsky 1969; Davidson 1972). This model is remarkably successful: the strength of virtually every broad emission line from the far-UV to the near-IR can be accounted for to within a factor 2 or so provided the gas is extremely thick in the Lyman continuum (e.g., Kwan and Krolik 1981). The hydrogen density n and the temperature T of the gas are estimated to be $n \sim 10^{9-10} \text{ cm}^{-3}$ and $T \sim 10^4 \text{ K}$. The distance of the gas from the central source of ionizing radiation must therefore be fairly well localized to about $L_{c46}^{1/2} \text{ pc}$, where L_{c46} is the ionizing luminosity in units of $10^{46} \text{ ergs s}^{-1}$.

The velocity widths of the broad emission lines are typically in the range $10^{3-4} \text{ km s}^{-1}$. These widths are generally ascribed to bulk motions of the emitting gas, since thermal and pressure broadening are negligible for the inferred density and temperature. The profiles of the emission lines have been analyzed in order to infer the velocity structure of the emitting gas, but unfortunately the procedure is not unique. Many quasars and Seyferts have approximately logarithmic profiles (i.e., flux $F_\nu \propto \log |\Delta\nu/(\nu - \nu_0)|$, where ν_0 is the line center and $\Delta\nu$ the half-width at zero intensity), which are consistent with radiative acceleration models (Blumenthal and Mathews 1975). However, such profiles can also arise in flow into or out of a gravitational potential well, in disks, or in ballistic flow models (Capriotti, Foltz, and Byard 1980).

The small size of the broad line regions means that it will be impossible to resolve them spatially at optical wavelengths in the foreseeable future. Time variations in the central ionizing source can act as a sensitive probe of the spatial and velocity structure, however. The response of the ambient gas to a flash of radiation—the “reverberation”—has been used to account for the observations of the apparent expansion of Nova Persei in 1901 (Couderc 1939), and it has been proposed as a mechanism to explain the light curves of Type I supernovae (Morrison and Sartori 1969) and X-ray bursts (Canizares 1976). At any instant of time the observer sees radiation emanating from an expanding paraboloid centered on the flash. The intensity of this radiation as a function of time is determined by the variation of the emissivity from one paraboloid to the next.

The first analysis of reverberations in the emission lines of quasars and Seyferts was carried out by Bahcall, Kozlovsky, and Salpeter (1972), who calculated the response to the total line intensity in a spherical distribution of gas. When the recombination time is short, the response of a spherical shell of gas is essentially instantaneous: for an ionizing flux which is constant except in a time interval $0 \leq t \leq \tau_{\text{on}}$, when it is larger by a constant factor A , the line intensity rises linearly with time for $0 \leq t \leq \tau_{\text{on}}$, is constant for $\tau_{\text{on}} < t < 2R/c$, and then decays to its original value in a time τ_{on} , where $R \geq c\tau_{\text{on}}/2$ is the radius of the shell. The amplitude of the increase of the line luminosity is proportional to $(A-1)L_c\tau_{\text{on}}(2R/c)^{-1}$, which is small if the shell is very large. Observation of variations in both the continuum and the line luminosities thus allows one to infer the size of the emission line region; in fact, we shall show that, granted certain assumptions, one can infer the emissivity as a function of radius.

Variations in the strengths and profiles of emission lines from Seyfert and radio galaxies have been reported by Cromwell and Weymann (1970), Anderson and Kraft (1971), Osterbrock, Koski, and Phillips (1976), Tohline and Osterbrock (1976), and Oke, Readhead, and Sargent (1980), among others, but so far no firm evidence for variations in quasar emission lines has been obtained. Yee and Oke (1981) have made the most detailed analysis: for 3C 390.3 they find evidence for gas which responds rapidly to variations in the continuum and for other gas which has a response time of order a year. In one instance, the blue wing of the line appeared to strengthen before the red wing, which is consistent with radial outflow.

Recent developments in instrumentation will allow the profiles of emission lines to be monitored with greater accuracy and spectral resolution than heretofore possible. Here we show how observations of the reverberations in emission line profiles enable one to map the spatial velocity structure of the emission line region of the object. In the short run these techniques will be useful primarily for Seyferts, since they vary on a time scale of months, rather than for quasars, which vary on a time scale of years.

In the next section we define the transfer function which relates the line output to the continuum input and show how it can be measured. Our objective is to use this transfer function to learn about the phase space distribution of emission line gas within the nucleus of the Seyfert galaxy or quasar. However, the transfer function is a function of only two variables, the velocity shift and the time. We can therefore only hope to derive the distribution function if this latter is a function of no more than two variables—i.e., we must assume a specific symmetry to the distribution. There are two methods for determining the gas distribution. The first is to invert the equation for the transfer function. This is the more elegant approach; it has the advantage that fewer assumptions need to be made about the source, but it requires data of high quality. The second method is model-fitting. Specific functional forms for the emissivity as a function of position and velocity are used to calculate the transfer function, and the parameters are adjusted until the agreement with the observed transfer function is optimized. In §§ III–V we discuss the first method—inverting the equation—for three geometrical assumptions present in existing models of galactic nuclei: a spherically symmetric distribution of emitting gas (§ III), a thin disk (§ IV), and a straight jet (§ V). The second method is illustrated in § VI by several simple models worked out in the three geometries. Readers who are not interested in the mathematics of the inversion procedure can skip §§ III–V and find the qualitative results of our work in the figures in § VI. Finally, in § VII we summarize our assumptions, present our conclusions, and discuss the feasibility of this procedure.

II. TRANSFER FUNCTION

a) Definition and Measurement

A basic consequence of photoionization models for the emission line region is that the line emissivity depends upon the incident continuum flux. If the gas is optically thick to the radiation which excites the lines, then the total line emission is linearly proportional to the incident flux. More generally, the emissivity of an individual line will have a functional dependence on the incident flux which can be determined from photoionization models; we shall adopt a linear function for simplicity, although our method works for any well behaved function. In general, the line emissivity depends on the amount of emitting gas as well. Variations in the amount of gas should occur on a time scale $\gtrsim t_{\text{dyn}}$, the dynamical time, which is $R/w \sim 100 R_{\text{pc}}/w_9 \text{ yr} \sim 100 L_{46}^{1/2}/w_9 \text{ yr}$, where w_9 is the flow velocity in units of 10^9 cm s^{-1} . We shall assume that variations in the amount of emission line gas have a negligible effect on variation in the line profiles, which should be reasonably valid if the data spans a time interval less than t_{dyn} .

Under these hypotheses, a quasar or Seyfert galaxy may be regarded as a linear system which outputs a line of luminosity $L_l(v, t)$ in response to an input continuum luminosity $L_c(t)$. Here t is the observer time and $L_l(v, t)$ is the spectral luminosity in a line per unit velocity interval as a function of the observed velocity Doppler shift v . Different lines with identical profiles can be added together to improve the signal-to-noise in $L_l(v, t)$. Because of light travel time effects, the observer sees emission from gas excited by the ionizing source at all previous times, so the system must

be characterized by a transfer function $\Psi(v, t)$ defined by

$$L_l(v, t) = \int_{-\infty}^{\infty} dt' L_c(t') \Psi(v, t - t'), \quad (2.1)$$

where

$$\begin{aligned} \Psi(v, t) &= 0 \quad (t < 0) \\ \Psi(v, t) &\geq 0 \quad (t \geq 0). \end{aligned} \quad (2.2)$$

The transfer function is just the line luminosity produced in response to a δ -function continuum luminosity at $t=0$; i.e., it is the Green's function for the system. It can be obtained in terms of line and continuum luminosities by Fourier transforming in time. Let

$$\tilde{L}_l(v, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} L_l(v, t) \quad (2.3)$$

be the Fourier transform of $L_l(v, t)$, etc. Then, applying the convolution theorem, we find

$$\Psi(v, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{\tilde{L}_l(v, \omega)}{\tilde{L}_c(\omega)}. \quad (2.4)$$

In order to apply these results, a further assumption must be made: the observed continuum luminosity is proportional to the continuum luminosity exciting the lines. The radiation which excites the lines produced in the fully ionized part of the emission line gas, such as Ly α , C III] λ 1909, and C IV λ 1549, is in the EUV part of the spectrum (~ 1 –10 rydbergs). On the other hand, the Balmer lines, Mg II λ 2850, and the Fe II lines are thought to be produced in partially ionized gas excited by X-rays (Kwan and Krolik 1979; Weisheit, Shields, and Tarter 1981). Accurate continuum luminosities as a function of time are generally available only at longer wavelengths—typically in the visible region of the spectrum for low redshift objects such as Seyferts. We assume that variations in the EUV and X-ray regions of the spectrum are well correlated with the variations in the observed region. This assumption can be relaxed in the future when accurate monitoring of Seyferts and quasars in the UV and in soft X-rays is possible. As remarked above, it is not essential that the relation be linear, although we shall assume so here for simplicity.

There is another aspect to the continuum luminosity assumption: although it is not essential that L_c be isotropic, it is essential that the relative variability be isotropic in the sense that a doubling of the continuum emitted toward the observer must correspond to a doubling of the continuum emitted toward the line-emitting gas. This assumption may not be satisfied in optically violent variable QSOs in which the large amplitude variations have been attributed to beaming (e.g., Blandford and Rees 1978). However, in these objects measurements of the transfer function Ψ might allow one to infer the radiation pattern.

The fact that the transfer function $\Psi(v, t)$ must vanish for $t < 0$ (eq. [2.2]) is an expression of causality and provides a strong test of the assumptions underlying our model—namely, that variations in the line intensity are correlated primarily with prior variations in the observed continuum. The test can be carried out by Fourier transforming the data in time and then inverting the transform as in equation (2.4). In practice, the transforms must be evaluated using a finite span of data obtained at irregular intervals; they may be estimated numerically using standard techniques for smoothing and tapering the data (Bracewell 1965; Brault and White 1971). If after evaluating $\Psi(v, t)$ from a reasonably long, well sampled, and accurate span of data, one finds that $\Psi(v, t)$ is significantly nonzero for $t < 0$, then one of our assumptions must be in error. The assumption that L_l depends linearly on L_c was made for simplicity only, and it is possible that a different functional form would correct the problem. It may also be necessary to use UV or X-ray continuum fluxes rather than visible ones.

In practice, the full transfer function $\Psi(v, t)$ will probably be too noisy to be used, so we define a set of moment functions as follows:

$$L_l^{(n)}(t) \equiv \int_{-\infty}^{\infty} dv v^n L_l(v, t), \quad (2.5)$$

$$\Psi^{(n)}(t) \equiv \int_{-\infty}^{\infty} dv v^n \Psi(v, t), \quad (2.6)$$

for $n=0, 1, 2, \dots$. Note that $L_l^{(0)}(t)$ is just the total line luminosity, whereas $L_l^{(1)}(t)$ is a measure of the line asymmetry. The number of moments used depends on the quality of the data. In principle the full transfer function $\Psi(v, t)$ can be reconstructed from all of its moments in the usual manner. The moments $L_l^{(n)}$ and $\Psi^{(n)}$ are related by equation (2.1):

$$L_l^{(n)}(t) = \int_{-\infty}^{\infty} dt' L_c(t') \Psi^{(n)}(t - t'), \quad (2.7)$$

or, by virtue of equation (2.4),

$$\Psi^{(n)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{\tilde{L}_l^{(n)}(\omega)}{\tilde{L}_c(\omega)}. \quad (2.8)$$

In some models of the emission line region, the line profile is predicted to be symmetric. The odd moments therefore vanish. Since the higher moments emphasize the wings of lines where the uncertainty due to the ill-defined continuum level is the greatest, it is useful to introduce antisymmetric moments which are denoted by the subscript a :

$$L_{l,a}^{(n)}(t) = \int_{-\infty}^{\infty} dv v^n \operatorname{sgn}(v) L_l(v, t), \quad (2.9)$$

etc. We shall most often use $L_{l,a}^{(0)}$, which is proportional to the asymmetry, and $L_{l,a}^{(1)}$, which is proportional to the mean speed $\langle |v| \rangle$. The antisymmetric moments satisfy equations of the form (2.7) and (2.8) as well.

Also of interest is the response to the time-averaged continuum. Let $\bar{L}_c$ and $\bar{L}_l(v)$ be the time averages of $L_c(t)$ and $L_l(v, t)$, respectively. Then equation (2.1) implies

$$\bar{L}_l(v) = \Phi(v) \bar{L}_c, \quad (2.10)$$

where

$$\Phi(v) = \int_{-\infty}^{\infty} dt \Psi(v, t) = \tilde{L}_l(v, 0) / \tilde{L}_c(0). \quad (2.11)$$

These relations are simply extended to cover the moments $\Phi^{(n)}$.

b) Spatial and Velocity Structure

We now determine how the response function $\Psi(v, t)$, which can be evaluated from the observations in the manner described above, is related to the spatial and velocity structure of the emission line region. Let $j(\mathbf{r}, t')$ be the volume emissivity in $\text{ergs cm}^{-3} \text{ s}^{-1}$ for a given line at position $\mathbf{r}$ and time t' . Let $f(\mathbf{r}, \mathbf{w})$ be the velocity distribution at that point; the time dependence of f is negligible by virtue of our assumption that the observations span a time short compared to the dynamical time. Under the assumption that there is no absorption or scattering of the emitted line radiation, the observed line luminosity is

$$L_l(v, t) = \int d\mathbf{r} d\mathbf{w} dt' j(\mathbf{r}, t') f(\mathbf{r}, \mathbf{w}) \delta(v - \mathbf{w} \cdot \mathbf{n}) \delta(t' - t + \mathbf{r} \cdot \mathbf{n}), \quad (2.12)$$

where $\mathbf{n}$ is a unit vector pointing from the (distant) observer to the source. We have set $c=1$, so that distances are measured in units of light-seconds. The natural line width and thermal and pressure broadening are small compared to the velocity shift v and have been neglected.

We assume that the central ionizing source is small compared to the emission line region and that the recombination time ($\sim n_{e9}^{-1}$ hr, where n_{e9} is the electron density in units of 10^9 cm^{-3}) is negligible compared to the variability time. Then together with the assumptions discussed in § IIa above, we can express the emissivity as

$$j(\mathbf{r}, t') = \frac{\epsilon(\mathbf{r}) L_c(t' - r)}{4\pi r^2}, \quad (2.13)$$

when the reprocessing coefficient $\epsilon(\mathbf{r})$ has dimensions of inverse length. Since $j(\mathbf{r}, t)$ is limited by the blackbody

emissivity, we require $\epsilon(0)=0$. Now let $g(\mathbf{r}, v)$ be the projected one-dimensional velocity distribution function:

$$g(\mathbf{r}, v) = \int d\mathbf{w} f(\mathbf{r}, \mathbf{w}) \delta(v - \mathbf{w} \cdot \mathbf{n}). \quad (2.14)$$

If we set $\int f d\mathbf{w} = 1$, then we have $\int g dv = 1$ as well. Inserting relations (2.13) and (2.14) into equation (2.12) then yields the response function

$$\Psi(v, t) = \int d\mathbf{r} \frac{\epsilon(\mathbf{r}) g(\mathbf{r}, v)}{4\pi r^2} \delta[t - (r + \mathbf{r} \cdot \mathbf{n})]. \quad (2.15)$$

The time-integrated response $\Phi(v)$ defined in equation (2.11) is simply

$$\Phi(v) = \int d\mathbf{r} \frac{\epsilon(\mathbf{r}) g(\mathbf{r}, v)}{4\pi r^2}. \quad (2.16)$$

Our objective is to invert equation (2.15) in order to find $\epsilon(\mathbf{r})$ and $g(\mathbf{r}, v)$.

III. SPHERICAL CLOUD DISTRIBUTION

Most models of quasar emission line regions, such as radiation-pressure driven outflow (Mushotzky *et al.* 1972; Blumenthal and Mathews 1975) or spherical accretion (McCray 1979), assume spherical symmetry for the emitting gas. In the most general spherical case, the emissivity $\epsilon(r, \mu)$ may be anisotropic and depend on $\mu \equiv \hat{\mathbf{r}} \cdot \mathbf{n}$ as well as on r ; likewise, the velocity distribution $f(r, w, \mu')$ would depend on r , w , and $\mu' \equiv \hat{\mathbf{w}} \cdot \mathbf{n}$, which leads to a projected distribution $g(r, \mu, v)$. The transfer function (2.14) then reduces to

$$\Psi(v, t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} \frac{dr}{r} \epsilon\left(r, \frac{t}{r} - 1\right) g\left(r, \frac{t}{r} - 1, v\right), \quad (3.1)$$

where $H(t)$ is the unit step function. Unfortunately, this form is still too complicated to invert in general, and special cases must be considered.

Before doing so, however, it is worthwhile to show how (3.1) can be obtained directly if ϵ and g are independent of μ . At a given observer time t after a unit δ function flash from the central continuum source, the illuminated surface is a paraboloid of semi-latus rectum $t/2$ with focus at the continuum source and axis parallel to the observer direction (Couderc 1939; Morrison and Sartori 1969; Blandford and Rees 1972). Hence the observed emission can be considered as arising from portions of spherical shells with radii $r \geq t/2$. The observed luminosity from a given shell is proportional to the rate of increase of the volume of the shell with observer time, which is just $2\pi r dr$. Then, including the line profile g , we have $d\Psi = 2\pi r dr [\epsilon(r)/4\pi r^2] g(r, v)$, which integrates to give (3.1).

a) Isotropic Velocity Distribution

The simplest case to consider is one in which the velocity distribution is isotropic, so that $f(\mathbf{r}, \mathbf{w}) = f(r, w)$ and ϵ and g do not depend on μ . It is then simple to show that

$$f(r, v) = -\frac{1}{\pi} \frac{\partial g(r, v)}{\partial(v^2)} \quad (v \geq 0). \quad (3.2)$$

The moments of the three-dimensional distribution $f(r, w)$ are defined by

$$f^{(n)}(r) = 4\pi \int_0^{\infty} dw w^{n+2} f(r, w). \quad (3.3)$$

They are related to the symmetric moments $g^{(n)}(r)$ and the antisymmetric moments $g_a^{(n)}(r)$ (see eq. [2.9]) by

$$f^{(2n)}(r) = (2n+1) g^{(2n)}(r), \quad (3.4a)$$

$$f^{(2n+1)}(r) = 2(n+1) g_a^{(2n+1)}(r). \quad (3.4b)$$

We can now solve equation (3.1) by differentiating:

$$t \frac{\partial \Psi(v, t)}{\partial t} = -\frac{1}{2} \epsilon\left(\frac{t}{2}\right) g\left(\frac{t}{2}, v\right), \quad t > 0. \quad (3.5)$$

Since the zeroth moment of g is unity, we find

$$\epsilon(r) = -4r \frac{d\Psi^{(0)}}{dt}(2r) \quad (3.6a)$$

$$= \frac{2ir}{\pi} \int_{-\infty}^{\infty} d\omega \omega e^{-2i\omega r} \tilde{L}_l^{(0)}(\omega) / \tilde{L}_c(\omega). \quad (3.6b)$$

The line profile is symmetric, and so the odd symmetric moments of g are identically zero. However, we can use the first antisymmetric moment $g_a^{(1)}(r)$ to obtain

$$\langle |w| \rangle(r) = f^{(1)}(r) = 2 \left[\frac{d\Psi_a^{(1)}(2r)}{dt} \right] / \left[\frac{d\Psi^{(0)}(2r)}{dt} \right] \quad (3.7)$$

with the aid of equation (3.4b). The mean square velocity is

$$\langle w^2 \rangle(r) = f^{(2)}(r) = 3 \left[\frac{d\Psi^{(2)}(2r)}{dt} \right] / \left[\frac{d\Psi^{(0)}(2r)}{dt} \right]. \quad (3.8)$$

It is straightforward to express $\langle |w| \rangle$ and $\langle w^2 \rangle$ in terms of $\tilde{L}_l(\omega)$, as was done for $\epsilon(r)$ in equation (3.6b). In fact, given sufficiently accurate line profiles, the complete distribution function $f(r, v)$ can be determined from

$$f(r, v) = -\frac{1}{\pi} \left[\frac{\partial^2 \Psi(v, 2r)}{\partial t^2} \right] / \left[\frac{d\Psi^{(0)}(2r)}{dt} \right]. \quad (3.9)$$

b) Radial Velocity Distribution

An alternative case is that of purely radial flow with an arbitrary radial velocity dispersion at each point. Such a situation would be expected for either radiation-pressure driven outflow or accretion flow of clouds, provided the mean free path for cloud-cloud collisions is large enough that they are effectively collisionless. Estimates of the cloud covering factor (Oke 1974; Baldwin and Netzer 1978) are typically somewhat less than 0.1, so this condition is apparently satisfied in quasars.

For purely radial velocities w , the projected velocity v is just $w\mu$ and one can replace $f(r, w)dw$ with $f_r(r, w)dw$; hence,

$$g(r, v) = |\mu|^{-1} f_r(r, v\mu^{-1}). \quad (3.10)$$

The transfer function (3.1) becomes

$$\Psi(v, t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} dr \frac{\epsilon(r)}{r} \frac{f_r[r, v(t/r-1)^{-1}]}{|t/r-1|}, \quad (3.11)$$

and the moments are

$$\Psi^{(n)}(t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} dr \frac{\epsilon(r) f_r^{(n)}(r)}{r} \left(\frac{t}{r} - 1 \right)^n, \quad (3.12)$$

where

$$f_r^{(n)} = \int_{-\infty}^{\infty} dw f_r(r, w) w^n. \quad (3.13)$$

Since the time-dependent profile is not symmetric in this case, the antisymmetric moments are not needed.

Equation (3.12) can be inverted by using Fourier transforms, as described in Appendix A; the result is

$$\epsilon\left(\frac{t}{2}\right)f_r^{(n)}\left(\frac{t}{2}\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \Upsilon_n(\omega t) \frac{\tilde{L}_l^{(n)}(\omega)}{\tilde{L}_c(\omega)}, \quad (3.14)$$

where

$$\Upsilon_n(z) = \frac{e^{iz}}{2\pi} \int_{-\infty}^{\infty} dx e^{-izx} \frac{H(x)}{x} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega \ln x}}{e^{-i\omega \ln 2} \tilde{k}_n(\omega)} \quad (3.15)$$

and

$$e^{-i\omega \ln 2} \tilde{k}_n(\omega) = \sum_{j=0}^n \binom{n}{j} (-1)^{n+j} \frac{2^j}{i\omega + j}. \quad (3.16)$$

We have omitted a term proportional to $\delta(\omega)$ in equation (3.16) since it does not affect the value of Υ .

In general, $\epsilon f_r^{(n)}$ must be evaluated numerically. Since it is in the form of successive Fourier transforms, this can be done efficiently. However, the function $\Upsilon(z)$ can be evaluated analytically for the first few moments:

$$\begin{aligned} \Upsilon_0(z) &= iz, & \Upsilon_1(z) &= \frac{i}{z} (z^2 - 2iz - 2 + 2e^{iz}), \\ \Upsilon_2(z) &= \frac{e^{i(z+\pi/4)}}{(7z)^{1/2}} \left[(iz)^{i(\sqrt{7})/2} \gamma\left(\frac{7}{2} + \frac{1}{2}i\sqrt{7}, iz\right) - (iz)^{i(\sqrt{7})/2} \gamma\left(\frac{7}{2} - \frac{1}{2}i\sqrt{7}, iz\right) \right], \end{aligned} \quad (3.17)$$

where $\gamma(a, x)$ is an incomplete gamma function (see Appendix A for definition). If the right-hand side of equation (3.14) is denoted by $I_n(t)$, then $\epsilon(r) = I_0(2r)$, $\langle w \rangle(r) = I_1(2r)/I_0(2r)$, and $\langle w^2 \rangle(r) = I_2(2r)/I_0(2r)$.

The validity of this model can be checked by seeing whether $\epsilon(r)$ and $f_r(r)$ are positive definite. We have not obtained an explicit solution for f_r , but it is necessary that $f_r^{(2n)}(r)$ be positive definite.

c) Anisotropic Emissivity

There is substantial evidence that the clouds producing the emission lines in quasars and Seyferts are quite optically thick in the continuum. This appears necessary to account for the large Balmer/Ly α ratio and the strong Mg II and Fe II emission (Kwan and Krolik 1979, 1981; Weisheit, Shields, and Tarter 1981; Canfield and Puetter 1980). Such clouds are sufficiently opaque in the Lyman and Balmer lines that the emissivity in these lines is substantially larger on the side of the cloud facing the central ionizing source; Kwan and Krolik (1981) find an enhancement of a factor of ~ 100 for Ly α and ~ 4 for H α . This anisotropic emission leads to asymmetry in the time-averaged line profile. Since many quasars and Seyferts show remarkably symmetric lines, Ferland, Netzer, and Shields (1980) argued that the line profiles reflect turbulent motions rather than smooth outflow or inflow. On the other hand, Osterbrock (1977) has documented a number of broad-line radio galaxies with Balmer lines which are stronger on the red side of the line than on the blue side; some quasars show the opposite tendency for strong UV lines such as Ly α , C IV, and C III] (R. Weymann, private communication).

A simple representation of an anisotropic emissivity, which produces asymmetric line profiles, is provided by

$$\epsilon(r, \mu) = \epsilon(r)(1 + \alpha\mu), \quad -1 \leq \alpha \leq 1, \quad (3.18)$$

where α is constant.

First consider a radial velocity distribution. The symmetric moments of the transfer function are identical to those in equation (3.12) except for a factor $1 + \alpha[(t/r) - 1]$:

$$\Psi^{(n)}(t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} dr \frac{\epsilon(r) f_r^{(n)}(r)}{r} \left[1 + \alpha \left(\frac{t}{r} - 1 \right) \right] \left(\frac{t}{r} - 1 \right)^n. \quad (3.19)$$

Hence the solution for $\epsilon f_r^{(n)}$ is identical to that in equation (3.14), except that the factor $\tilde{k}_n(\omega)$ is

$$\tilde{k}_n(\omega) = \int_{-\infty}^{\infty} dx e^{i\omega x} (e^x - 1)^n [1 + \alpha(e^x - 1)] H(2 - e^x) \quad (3.20)$$

(cf. eq. [A7]).

Next, consider an isotropic velocity distribution. The transfer function is

$$\Psi(v, t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} dr \frac{\epsilon(r) g(r, v)}{r} \left[1 + \alpha \left(\frac{t}{r} - 1 \right) \right] \quad (3.21)$$

so the symmetric moments are

$$\Psi^{(n)}(t) = \frac{1}{2} H(t) \int_{t/2}^{\infty} dr \frac{\epsilon(r) g^{(n)}(r)}{r} \left[1 + \alpha \left(\frac{t}{r} - 1 \right) \right]. \quad (3.22)$$

This is identical in form with the moments (3.19) for the radial velocity distribution, except $f_r^{(n)}$ is replaced by $g^{(n)}(r)$ and the exponent n is zero. Hence the solution is again given by equation (3.14):

$$\epsilon(r) g^{(n)}(r) = I_n(2r), \quad (3.23)$$

where $I_n(t)$ is the right-hand side of equation (3.14) and the factor $\tilde{k}_n(\omega)$ entering I_n is given by equation (3.20) with $n=0$. The antisymmetric moments $g_a^{(n)}(r)$ are given by the same expression, but with $\tilde{L}_{l,a}^{(n)}(\omega)$ replacing $\tilde{L}_l^{(n)}(\omega)$ in I_n . The moments of the three-dimensional distribution function $f(r, w)$ are given in terms of $g^{(n)}(r)$ and $g_a^{(n)}(r)$ by equation (3.4).

IV. THIN ACCRETION DISK

An alternative model of the distribution of emission line gas is that it is confined to a thin disk orbiting a massive black hole (Lynden-Bell 1969; Hills 1975). Such a model has been criticized on a number of grounds: Osterbrock (1977) has shown that the distribution of line widths in Seyfert I galaxies disagrees with the distribution predicted from randomly oriented disks; however, the discrepancy can be removed by assuming some turbulence normal to the disk. Shields (1978) has shown that it is difficult to produce the line C III $\lambda 1909$ with a profile as broad as that of C IV $\lambda 1549$ without producing a broader [O III] $\lambda 5007$ line than observed; however, the discrepancy is only a factor 2. Capriotti, Foltz, and Byard (1980) have argued that it is difficult to produce the observed logarithmic line profiles, but, as we show below, any reasonable, symmetric profile can be modeled using a disk. Finally, Yee and Oke (1981) have observed an asymmetric variation in the profile of 3C 390.3. They point out that this rules out disk emission in this object since emission lines from disks must be symmetric. With the exception of this last argument, which requires observational confirmation, none of these arguments is conclusive.

Here we shall first consider steady emission from Keplerian disks, and then generalize our analysis to time-dependent emission from arbitrary thin disks. Note that if some of the continuum radiation is absorbed or beamed away from the disk, then the mean intensity of the ionizing radiation will be reduced and the radius of the emission region may be somewhat smaller than the $L_{e46}^{1/2}$ pc estimate in § I. Our assumption of a thin disk ensures that the flux normal to the disk falls off faster than r^{-2} , but this causes a reduction in the thickness of the emission region rather than a diminution of the excitation.

a) Steady Emission from a Keplerian Disk

We assume that the velocity dispersion is produced entirely by circular motion in the equatorial plane. If i is the angle of inclination of the disk and ϕ the azimuthal angle in the disk, then the projected velocity distribution is

$$g(r, v) = \delta[v + w(r) \sin \phi \sin i], \quad (4.1)$$

where without loss of generality we have assumed that the disk rotates in a positive sense about $\mathbf{n}$ and that $w(r) > 0$. Since the reprocessing of the incident continuum into lines results in reflection, we replace the reprocessing coefficient $\epsilon(r)$ by a reflection/reprocessing coefficient $\mathcal{R}(r)$ such that $\epsilon(r) = \mathcal{R}(r) \delta(z)$, where the z axis is normal to the disk. The steady line profile $\Phi(v)$ given in equation (2.16) becomes

$$\Phi(v) = \int_0^{\infty} dr \frac{\mathcal{R}(r) H[w(r) \sin i - |v|]}{2\pi r w(r) \sin i \left\{ 1 - v^2 / [w(r)^2 \sin^2 i] \right\}^{1/2}}. \quad (4.2)$$

Now focus on the case of a Keplerian disk in orbit around a central mass M , in which

$$w(r) = \left(\frac{GM}{r} \right)^{1/2} \equiv \frac{w_{n0}}{\sin i} \left(\frac{r_0}{r} \right)^{1/2}, \quad (4.3)$$

where the constant w_{n0} is the projected velocity at the fiducial radius r_0 . Equation (4.2) then assumes the form of the Abel integral equation

$$\frac{1}{\xi^{1/2}} \Phi \left(\frac{w_{n0} r_0^{1/2}}{\xi^{1/2}} \right) = \int_0^\xi \frac{dr F(r)}{(\xi - r)^{1/2}}, \quad (4.4)$$

with $\xi = w_{n0}^2 r_0 / v^2$ and $F(r) = \mathcal{R}(r) (2\pi w_{n0} r_0^{1/2} r^{1/2})^{-1}$. The possibility of inverting this equation has been pointed out by Fabian *et al.* (1979) for the case of stellar accretion disks. The solution is

$$F(r) = \frac{1}{\pi} \int_0^r \frac{d}{d\xi} \left[\frac{1}{\xi^{1/2}} \Phi \left(\frac{w_{n0} r_0^{1/2}}{\xi^{1/2}} \right) \right] \frac{d\xi}{(r - \xi)^{1/2}}. \quad (4.5)$$

This determines $\mathcal{R}$ as a function of the projected velocity $w \sin i$,

$$\mathcal{R}(v = w \sin i) = -2 \int_{v = w \sin i}^\infty \frac{d[v \Phi(v)]}{[1 - (w^2 \sin^2 i)/v^2]^{1/2}}. \quad (4.6)$$

Note that there is no length scale in equation (4.6), so it is impossible to infer the mass M or the inclination angle i from a steady profile alone. However, both can in principle be determined from an analysis of the variability.

As an application of this result, we find that the logarithmic profile $\Phi(v) = \Phi_0 \ln(\Delta v / |v|)$ will be produced by a reflection coefficient

$$\mathcal{R}(v = w \sin i) = 2\Phi_0 w \sin i \arccos \left(\frac{w \sin i}{\Delta v} \right) H(\Delta v - w \sin i). \quad (4.7)$$

for $w \sin i \lesssim \frac{1}{2} \Delta v$, this reduces to $\mathcal{R} \propto w \propto r^{-1/2}$, which corresponds to the approximate logarithmic profile found by Capriotti, Foltz, and Byard (1980).

b) Time-dependent Profiles

We return now to the general case in which $w(r)$ is arbitrary. If the central source emits a δ -function flash of continuum radiation, then the observer will see line emission from an expanding elliptical ring with eccentricity $\sin i$. The response function is

$$\Psi(v, t) = H(t) \int_0^\infty dr \int_0^{2\pi} d\phi \frac{\mathcal{R}(r)}{4\pi r} \delta[v + w(r) \sin \phi \sin i] \delta[t - r(1 + \cos \phi \sin i)]. \quad (4.8)$$

Again, we confine our attention to the moments. The symmetric moments are

$$\Psi^{(2n)}(t) = H(t) \int_\alpha^\beta dr \frac{\mathcal{R}(r) w^{2n}(r)}{2\pi r^2} \left(2\frac{t}{r} - \frac{t^2}{r^2} - \cos^2 i \right)^{n-1/2}, \quad (4.9a)$$

$$\Psi^{(2n+1)}(t) = 0, \quad (4.9b)$$

where $\alpha \equiv t/(1 + \sin i)$ and $\beta \equiv t/(1 - \sin i)$, whereas the antisymmetric moments are

$$\Psi_a^{(2n)}(t) = 0, \quad (4.10a)$$

$$\Psi_a^{(2n+1)}(t) = \int_\alpha^\beta dr \frac{\mathcal{R}(r) w^{2n+1}(r)}{2\pi r^2} \left(2\frac{t}{r} - \frac{t^2}{r^2} - \cos^2 i \right)^n. \quad (4.10b)$$

The odd symmetric and even antisymmetric moments vanish identically because the line must be symmetric at all times —i.e., $L_l(v, t) = L_l(-v, t)$ for all t .

These equations can be inverted by adapting the method of Appendix A. The general solution for a disk is

$$\mathcal{R}(r)w^n(r) = r \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega \ln r}}{\tilde{k}_n(\omega)} \int_{-\infty}^{\infty} d\tau e^{i\omega \tau} F_n(\tau), \quad (4.11)$$

where

$$\tilde{k}_n(\omega) = \sin^n i \int_{-1}^1 d\mu (1 + \mu \sin i)^{i\omega - 1} (1 - \mu^2)^{(n-1)/2}, \quad (4.12)$$

$$F_n(\tau) = \begin{cases} \Psi^{(n)}(e^\tau) & (n \text{ even}), \\ \Psi_a^{(n)}(e^\tau) & (n \text{ odd}). \end{cases} \quad (4.13a)$$

$$(4.13b)$$

For n even, $\tilde{k}_n(\omega)$ can be expressed as a hypergeometric function. For n odd, $\tilde{k}_n(\omega)$ can be expressed in terms of elementary functions, but the evaluation of $\mathcal{R}(r)w^n(r)$ is best done numerically.

A simple result can be obtained for the first antisymmetric moment. Differentiating equation (4.10b) for $n=0$ yields

$$t \frac{d\Psi_a^{(1)}(t)}{dt} = y \left(\frac{t}{1 - \sin i} \right) - y \left(\frac{t}{1 + \sin i} \right) \quad (4.14)$$

with

$$y(r) = \frac{\mathcal{R}(r)w(r)}{2\pi r}. \quad (4.15)$$

The solution (kindly supplied by N. Kylafis) is

$$\frac{\mathcal{R}(r)w(r)}{2\pi r} = - \sum_{n=0}^{\infty} \frac{d\Psi_a^{(1)}}{d \ln t} \left[t = \frac{(1 + \sin i)^{n+1}}{(1 - \sin i)^n} r \right]. \quad (4.16)$$

For example, if $\Psi_a^{(1)}(t) \propto t^{-m}$, which might be appropriate at late times, then one has

$$\frac{\mathcal{R}(r)w(r)}{2\pi r} = \frac{m\Psi_a^{(1)}(t=r)}{(1 + \sin i)^m - (1 - \sin i)^m}. \quad (4.17)$$

The results for the average line profile (eq. [4.6]) and for $\Psi_a^{(1)}$ (eq. [4.16]) can be applied straightforwardly to data to determine if the disk is approximately Keplerian and to give a relation between the unknown parameters M and $\sin i$. If the disk is approximately Keplerian, then the inferred expression for $\mathcal{R}$ can be inserted into the relation for the symmetric moments (eq. [4.9a]) to determine both M and $\sin i$. On the other hand, if the mass producing the gravitational field is not concentrated in a central point mass, then the disk will not be Keplerian and the full inversion procedure (eq. [4.11]) must be used.

V. STRAIGHT JET

The best known example of emission lines produced by a jet is provided by SS 433 (e.g., Abell and Margon 1979). In that case there are two oppositely directed jets, and the lines are clearly split into two components, in contrast to the single component lines in quasars. The kinematics of a single jet model have been discussed by Goldman and Bahcall (1981), but the resulting profiles do not agree well with observations of Seyferts and quasars.

Rather than apply our method to emission lines from jets, we briefly consider an application to optical and radio continuum emission. There have been many attempts to search for correlated optical and radio variability from active galactic nuclei. With the exception of AO 0235+164 and OJ 287, the correlation is at best weak. More recently, VLBI observations have revealed that the majority of compact radio sources are linear and it has been suggested that all the radio emission be produced in relativistic jets emanating from the nucleus. It may be that outbursts in the nucleus produce optical continuum emission directly and that simultaneously there is an increase in the discharge in the jet channel which is precessed into radio emission at a range of radii r along the jet. The ideas of § II are then applicable to this case too, and it might be possible to infer the reprocessing coefficient $\epsilon(r)$ of the jet using Fourier transforms of

the optical and radio continuum luminosity (cf. eq. [2.4]). Specifically, for an assumed linear relationship (and it is simple to generalize to the nonlinear case), the emissivity of a narrow jet is given by equation (2.13):

$$j(r, t) = \frac{\epsilon(r) \delta(\Omega) L_c(t - r/v)}{4\pi r^2}, \quad (5.1)$$

where $v < 1$ is the jet velocity, and the transfer function is

$$\Psi(t) = \frac{1}{4\pi} \frac{v}{(1 + v \cos i)} \epsilon\left(\frac{vt}{1 + v \cos i}\right), \quad (5.2)$$

where $\cos i = \mathbf{v} \cdot \mathbf{n}$. The function $\epsilon(r)$ can be obtained directly from equation (5.2) up to a scaling factor. Again, the transfer function $\Psi(t)$ must be positive for $t \geq 0$.

VI. ILLUSTRATIVE EXAMPLES

In § III and § IV, we have shown how to solve for the reprocessing or reflection coefficient, given adequate data on the line profiles and associated continuum luminosities and given specific geometrical assumptions. We now present some simple examples of calculated varieties of the moment functions for specific models. These can be used as a guide in deciding which geometrical assumptions are appropriate for a specific object. Our examples are chosen to give steady line profiles that have roughly logarithmic shape; such profiles occur in a variety of circumstances (Capriotti, Foltz, and Byard 1980). It should be emphasized that our method does not depend on this assumption. The three steady profiles we consider are

Logarithmic: $\Phi = \Phi_0 \ln(\Delta v/|v|),$ (6.1)

Exponential integral: $\Phi = \frac{1}{2} \Phi_0 E_1(v^2/2\sigma_0^2),$ (6.2)

Asymmetric: $\Phi = \Phi_0 \left[\ln \frac{\Delta v}{|v|} + \alpha \left(\operatorname{sgn} v - \frac{v}{\Delta v} \right) \right].$ (6.3)

As shown in Figure 1, these three profiles are very similar, except that the E_1 profile has wings which extend to infinity whereas the others terminate at $|v| = \Delta v$.

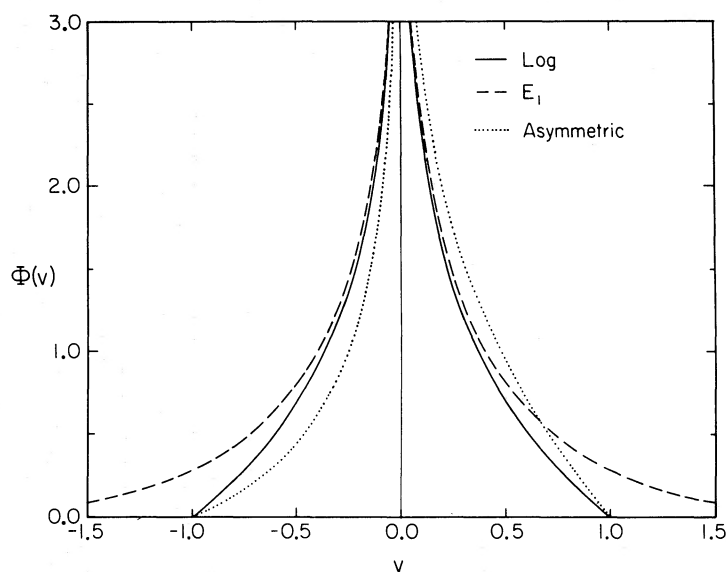


FIG. 1.—Time-averaged line profiles Φ as a function of velocity shift v . Logarithmic, exponential integral (E_1), and asymmetric logarithmic profiles (eqs. [6.1]–[6.3]) are shown. Of the models listed in Table 1, RAN has an asymmetric logarithmic profile, ISO has an E_1 profile, and the remainder have logarithmic profiles. The half-width at zero intensity Δv and, for the E_1 profile, the velocity dispersion σ_0 , have been set equal to unity.

TABLE 1
MODELS FOR THE KINEMATICS OF THE EMISSION LINE REGION

Model	Reprocessing Coefficient	Velocity Distribution
Radial Flow:		
RRA	$\epsilon = \frac{2\Phi_0 v_0}{r_0} [H(r-r_0) - H(r-2r_0)]$	$f_r = \frac{H(w) - H(w-v_0)}{v_0}$
RAN	$\epsilon = \epsilon_{\text{RRA}}(1 + \frac{1}{2}\mu)$	$f_r = \frac{H(w) - H(w-v_0)}{v_0}$
ROD	$\epsilon = \Phi_0 w_0 r_0^{1/2} \frac{H(r-r_0)}{r^{3/2}}$	$f_r = \delta[w - w_0(r_0/r)^{1/2}]$
ROA	$\epsilon = \Phi_0 w_0 r_0^{1/2} \frac{H(r-r_0)}{r^{3/2}}$	$f_r = \delta[w - w_0(1-r_0^{1/2}/r^{1/2})]$
Symmetric:		
ISO	$\epsilon = \left(\frac{\pi}{2} r_0\right)^{1/2} \frac{\Phi_0 \sigma_0 H(r-r_0)}{r^{3/2}}$	$f = \frac{\exp(-w^2/2\sigma^2)}{(2\pi\sigma^2)^{3/2}},$ $\sigma = \sigma_0(r_0/r)^{1/2}$
Disk	$\mathcal{R} = 2\Phi_0 w_{n0} \left(\frac{r_0}{r}\right)^{1/2} \cos^{-1}\left(\frac{r_0}{r}\right)^{1/2} H(r-r_0)$ $i = 30^\circ, 60^\circ, 75^\circ$	$w \sin i = w_{n0}(r_0/r)^{1/2}$ $\Delta v = w_{n0}$

NOTE.—For the calculations shown in Figures 1–7, we set $\Delta v = w_0 = w_{n0} = \sigma_0 = \Phi_0 = r_0 = 1$.

We have displayed eight models altogether, with the reprocessing or reflection coefficients and velocity distributions listed in Table 1. The first four of the models have asymmetric time-dependent line profiles, although only one has an asymmetric steady profile. These models are spherically symmetric and assume radial flow. The remaining four models have symmetric, time-dependent line profiles, and include one spherical model with an isotropic velocity distribution and three disk models at various angles of inclination. In all cases we have assumed that no line emission originates in the region $r < r_0$. Explicit expressions for the moments of the transfer equation are given in Appendix B.

a) Asymmetric Time-dependent Profiles

The time-dependent profiles of the radial flow models are asymmetric, even if the time-averaged profile $\Phi(v)$ is symmetric. This is because after a continuum flash we first see line emission predominantly from the near side and later from the far side of the emission line region, and in radial flow models the velocity distribution on the two sides is different.

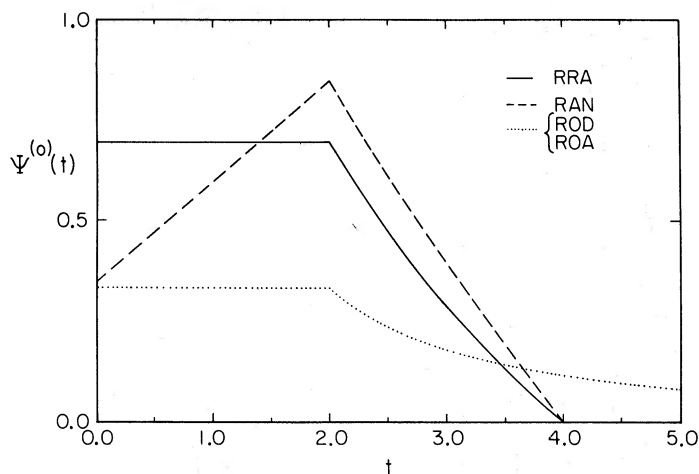


FIG. 2.—The zeroth moment of the transfer function $\Psi^{(0)}$ as a function of time t for the four radial flow models, which have asymmetric time-dependent profiles. $\Psi^{(0)}$ is proportional to the total line intensity following a delta-function continuum flash. The outward acceleration (ROA) and outward deceleration (ROD) models have identical zeroth moments.

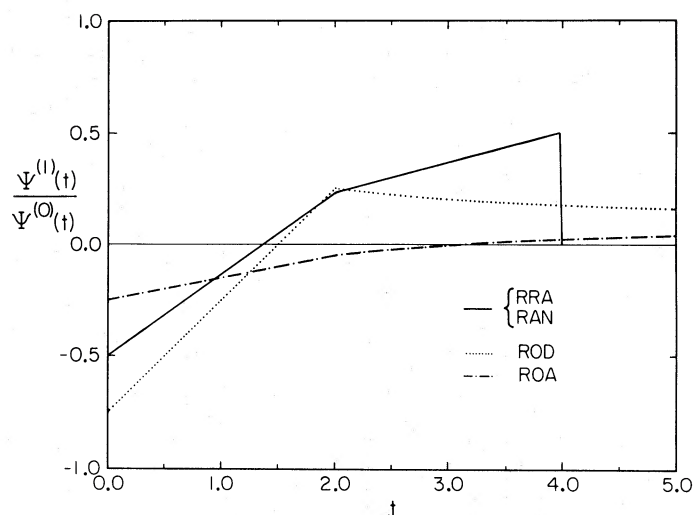


FIG. 3.—The ratio of the first and zeroth moments of the transfer function for the four radial flow models. $\Psi^{(1)}/\Psi^{(0)}$ is proportional to the flow velocity. Since the anisotropic emissivity model RAN has a velocity distribution identical to that of model RRA, $\Psi^{(1)}/\Psi^{(0)}$ is almost the same for the two models and only one curve has been plotted. The four models plotted all assume outflow of the emitting gas; if the gas were flowing inward, then the sign of $\Psi^{(1)}/\Psi^{(0)}$ would be reversed.

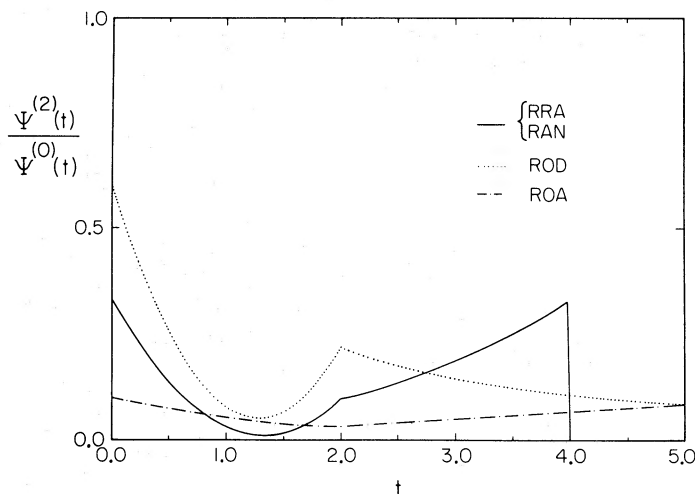


FIG. 4.—The ratio of the second and zeroth moments of the transfer function for the four radial flow models. $\Psi^{(2)}/\Psi^{(0)}$ is proportional to the velocity dispersion.

In model RRA (radial, rapid acceleration) the clouds are injected at a uniform rate per unit volume in the region $r_0 < r < 2r_0$ at zero velocity, then instantaneously and uniformly accelerated to a terminal velocity v_0 , at which point they are destroyed. This simple dynamical model roughly corresponds to the particle-driven outflow model of Scott *et al.* (1981, preprint). The assumption that the clouds are destroyed as soon as they reach their terminal velocity is necessary in order to obtain a logarithmic profile; its physical justification is unclear. The moments of the transfer function are plotted in Figures 2–4. In these and in subsequent figures, the large changes in slope are associated with the abrupt onset of emission at r_0 .

Model RAN (radial, anisotropic) is identical to model RRA except that the emissivity is anisotropic (§ IIIc) with $\epsilon \propto 1 + \frac{1}{2}\mu$. This is the only model considered in which the time-averaged line profile is asymmetric; it is given by equation (3.3). The zeroth moment of the transfer function is plotted in Figure 2. Since the velocity distribution is identical to that for model RRA, it turns out that $\Psi^{(1)}/\Psi^{(0)}$ and $\Psi^{(2)}/\Psi^{(0)}$ are almost, but not quite, identical for the two models, so only a single curve has been plotted in Figures 3 and 4.

Models ROD (radial, outward deceleration) and ROA (radial, outward acceleration) are examples of gradual velocity changes. In model ROD, the clouds are created at r_0 with a velocity $\Delta v = w_0$, and they subsequently move outward, with a velocity $w = w_0(r_0/r)^{1/2}$ which asymptotically approaches zero. Such a situation might be realized by tidal disruption of stars on plunge orbits by a massive black hole; the stellar debris produces the emission lines (Hills 1975). In model ROA the clouds are injected at r_0 with zero velocity, and they accelerate outward with $w = w_0[1 - (r_0/r)^{1/2}]$. Their terminal velocity is identical to the initial velocity of the clouds in model ROD. This velocity law is an approximation to that for radiative acceleration of optically thick clouds of constant cross section. In order to produce logarithmic profiles, the reprocessing coefficient must fall off as $r^{-3/2}$ in these two models. If the velocities were directed inward rather than outward, the even moments in Figures 2 and 4 would be unchanged, whereas $\Psi^{(1)}/\Psi^{(0)}$ would simply change sign.

b) Symmetric Time-dependent Profiles

We have considered two classes of models which produce time-dependent profiles which are symmetric. Model ISO is spherically symmetric and has a Maxwellian velocity distribution. The velocity dispersion is Keplerian, $\sigma = \sigma_0(r_0/r)^{1/2}$. The resulting time-averaged line profile has an exponential integral form (eq. [6.3]) provided the reprocessing coefficient falls off as $r^{-3/2}$. The moments of the transfer function are displayed in Figures 5–7. Because the line is always symmetric, $\Psi^{(1)}(t)$ vanishes identically, and we have plotted the first antisymmetric moment $\Psi_a^{(1)}(t)$ in its place (see eq. [2.9]).

Disks produce symmetric time-dependent profiles as well, if one ignores Doppler beaming effects of order v/c which tend to brighten the blue wing of the line. We display the moments of the transfer function for a Keplerian disk at three different angles of inclination, $i = 30^\circ$, 60° , and 75° . For a collection of randomly oriented disks, the probability that the inclination would be less than 30° or greater than 75° is 0.13 and 0.26, respectively. The central mass M is assumed to vary as $(\sin^2 i)^{-1}$ in order that the observed average line profile Φ be independent of i .

A weakness of Keplerian models, such as those considered here, is that in order to span a decade in velocity, the emitting clouds must span two decades in radius and (for a fixed ionization parameter) four decades in density. This is not easy to reconcile with the observations that C III] $\lambda 1909$, which is collisionally de-excited at $n_{\text{cr}} = 3 \times 10^9 \text{ cm}^{-3}$, has a profile virtually identical to that of C IV $\lambda 1549$, whereas [O III] $\lambda 5007$, with $n_{\text{cr}} = 10^6 \text{ cm}^{-3}$, is often 10 times narrower (Shields 1978).

c) Application

The results in Figures 2–7 can be used to obtain qualitative information about the kinematics of the emission line region. Assume that a set of high quality data on variations in the continuum and the line profiles of a given object has been obtained. Then, if the assumptions of our analysis are valid, the lowest moments of the transfer function should be calculable with reasonable accuracy. The characteristic size of the emission line region can be determined immediately by noting the point at which $\Psi^{(0)}$ drops to half its peak value.

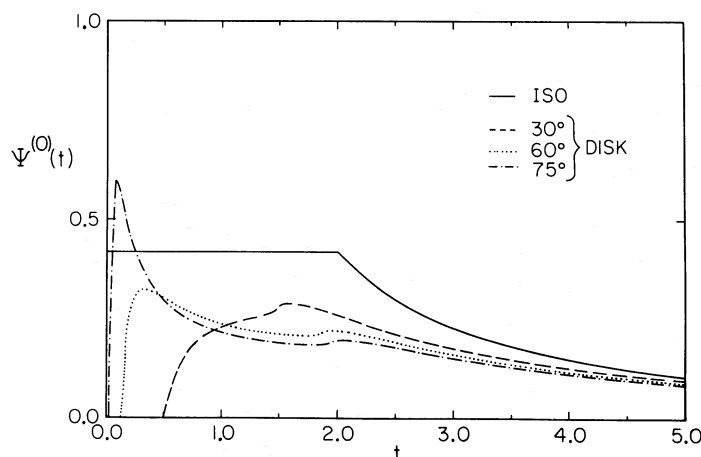


FIG. 5.—The zeroth moment of the transfer function as a function of time for the four models with symmetric time-dependent line profiles. The disk models, which are plotted at three angles of inclination, differ from model ISO, which is spherically symmetric and has an isotropic velocity distribution, in that there is a delay between the continuum flash and the onset of line emission.

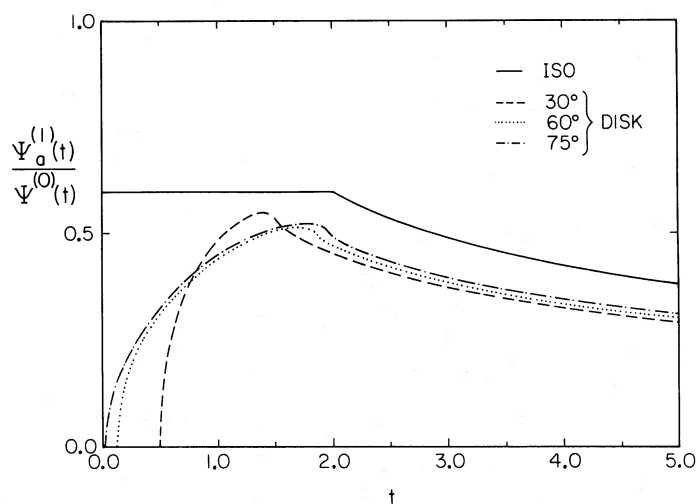


FIG. 6.—The ratio of the first antisymmetric moment to the zeroth moment of the transfer function for the four models with symmetric time-dependent line profiles. $\Psi_a^{(1)}/\Psi^{(0)}$ is proportional to the speed $|w|$ of the emitting gas. The symmetric moment $\Psi^{(1)}$ is identically zero.

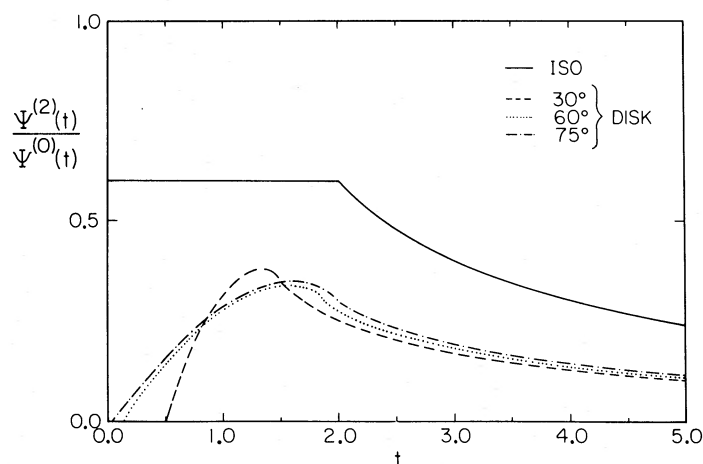


FIG. 7.—The ratio of the second and zeroth moments of the transfer function for the four models with symmetric time-dependent line profiles.

In order to determine the nature of the emission line region, one can proceed systematically as follows. First determine if the time-dependent profiles are symmetric ($\Psi^{(1)}=0$). Note that model ROA has a rather small value of $\Psi^{(1)}$ because large values of v occur at large r , where ϵ is small. Hence it would also be worthwhile to test the symmetry by evaluating $\Psi_a^{(0)}$, which depends on $\text{sgn } v$ rather than v . $\Psi_a^{(0)}$ also vanishes for symmetric lines.

If the time-dependent profiles are *asymmetric*, then the direction of the flow can be inferred from $\Psi^{(1)}$: for outflow, $\Psi^{(1)}$ is initially negative since the blueshifted emission from the near side of the emission line region dominates at early times; later, $\Psi^{(1)}$ becomes positive when the far side is dominant. The signs are reversed for inflow. The direction of the acceleration (i.e., whether the velocity is larger near the center of the emission line region or near the periphery) can be determined from $\Psi^{(2)}/\Psi^{(0)}$. If the velocities are largest near the center, as in model ROD, then $\Psi^{(2)}/\Psi^{(0)}$ is a maximum at $t=0$. On the other hand, if the velocities are largest at the periphery, then $\Psi^{(2)}/\Psi^{(0)}$ is larger at late times than it is initially. For example, in model ROA $\Psi^{(2)}/\Psi^{(0)} \rightarrow 0.31$ at late times, substantially larger than the maximum value shown in Figure 4, whereas in model ROD $\Psi^{(2)}/\Psi^{(0)} \rightarrow 0.44t^{-1} \rightarrow 0$ (see Appendix B). A similar, but somewhat weaker, effect occurs for the first moment: In model ROA $\Psi^{(1)}/\Psi^{(0)} \rightarrow 0.2$ at late times, whereas in model ROD $\Psi^{(1)}/\Psi^{(0)} \rightarrow (8t)^{-1/2} \rightarrow 0$. Models RRA and RAN represent an intermediate case in which the velocity distribution is independent of position. In this case the final value of $\Psi^{(2)}/\Psi^{(0)}$ is the same as the initial value. Finally, the isotropy of

the emissivity can be diagnosed by studying the line profile $\Phi(v)$ and the total line transfer function $\Psi^{(0)}(t)$. In the context of the models we are considering, an asymmetric *average* profile can be produced only by an anisotropic emissivity, since we are explicitly ignoring scattering and absorption of the radiation. The sense of the anisotropy is indicated by the sign of the asymmetry in Φ . For example, a red symmetry, such as we have considered, is characterized by having the peak in $\Psi^{(0)}$ displaced away from the origin (see Fig. 2) since the brightest clouds are on the far side of the emission line region.

Next, consider the case of *symmetric* time-dependent profiles. The main distinction between a disk with $\sin i \neq 1$ and a spherically symmetric distribution of clouds is that there is no emitting material along a line of sight in the case of a disk. Hence, the line emission from a disk is delayed for a time $r_0(1 - \sin i)$; prior to this time, $\Psi = 0$ (see Fig. 4). (The existence of the critical radius r_0 is guaranteed whenever the line profile is approximately logarithmic, since then most of the line emission arises outside the radius at which $w \sin i = \Delta v$, the half-width of the line.) Another difference between disk models and radial flow models is that for disks $\Psi^{(1)}/\Psi^{(0)}$ and $\Psi^{(2)}/\Psi^{(0)}$ are initially small since the velocity of the emitting gas is initially transverse to the line of sight (see Figs. 5, 6). If it is established that the emission is from a disk, then the angle of inclination can be estimated roughly from the shape of the curves. For example, disks at large angles of inclination have a sharp initial maximum in $\Psi^{(0)}$ shortly after $t = r_0(1 - \sin i)$ and a weak secondary maximum just after $t = r_0(1 + \sin i)$; at small angles of inclination there is only a broad maximum near $t = r_0(1 + \sin i)$. High time resolution would be necessary to measure the sharp initial maximum at large $\sin i$.

The examples discussed here would be useful in obtaining qualitative information about the emission line region provided it is approximated by one of the models in Table 1. To obtain more quantitative information in more general cases, the methods of §§ III–IV are required.

VII. DISCUSSION

In this paper, we have outlined an indirect method for mapping the emission line regions of those Seyfert galaxies and quasars that have variable line intensities and profiles. This method can work only if a galactic nucleus comprises a compact central continuum source surrounded by an extended distribution of line-emitting gas. Furthermore, the covering factor of the gas must be small and the whole region optically thin to Thomson scattering so that most continuum and line photons can escape directly from the nucleus. We have illustrated our results using models in which the line luminosity of an element of gas is directly proportional to the incident observed continuum flux, which we expect to be a good approximation, at least for Ly α . However, other possible functional forms may be appropriate for other resonance lines (cf. Davidson 1977). These functions can be modeled and their compatibility with a given set of data explored. Two further assumptions seem justified on physical grounds: rapid reprocessing and the stability of the gas distribution over several light crossing times. The latter assumption requires that the gas velocities be small compared to the speed of light, so that it is possible to use the nonrelativistic Doppler shift.

Under these conditions it is sensible to introduce a transfer function $\Psi(v, t)$ which represents the line profile that would be obtained if the ionizing continuum were a delta function in time. The transfer function must be zero for $t < 0$ and positive for $t \geq 0$. (This property can be used as a guide in selecting a possible nonlinear relationship between the line and continuum luminosities.) This transfer function can be calculated explicitly using equation (2.15) for any particular model of the gas distribution. Some explicit examples are given in § VI.

The transfer function can be used in different ways, depending upon the quality and quantity of the data. First, an observed continuum variation can be convolved with a model transfer function and the resulting line variations compared directly with what is observed. By using a range of models, like those given in Table 1 and Appendix B, and perhaps incorporating adjustable parameters, it should be possible to infer the gross distribution of the clouds.

A more powerful procedure is to derive the transfer function directly by Fourier transforming both the line and continuum fluxes. Note that as the data are real, the transfer function is guaranteed to be real. Furthermore, if there is a steady component of either the continuum or the line flux, then this will influence only the zero frequency part of the Fourier transform of the transfer function. Obviously, if a source undergoes a particularly violent outburst of short duration, then the postoutburst line response will provide a direct estimate of the transfer function, but for most sources the variability appears to be more gentle and Fourier methods seem indispensable.

How good must the data be in order to obtain a reliable transfer function? Obviously, the answer to this question depends upon the nature of the variations and the size of the emission region. Let us suppose that the temporal width of the transfer function is τ . If we make N observations of both the line and the continuum (with no intrinsic error) at a mean spacing of Δt , then the Fourier transform will extend up to the Nyquist frequency $\sim (\pi/\Delta t)$. We estimate that the derived transfer function will have a fractional accuracy $\sim 5/N$ provided that $\Delta t \lesssim 0.2\tau$. The effective spatial resolution of the map is $\sim c\Delta t$, and the dynamic range will be controlled by the duration $\sim N\Delta t$ of the observations. As the transfer function is anticipated to be fairly smoothly varying, there should be no difficulty in estimating it from $N \sim 50$ error-free observations spaced over 10τ , which is typically 1–3 years for Seyferts. Observations made more

frequently than the reprocessing time (~ 1 hour) will not contribute to $\Psi(v, t)$. If the cloud distribution is completely changed in a time $\sim c\tau/v \sim 30\tau$, then this constitutes an upper limit to the useful duration of an observing run, although if the cloud distribution is stationary in a statistical sense, then a longer run of observations will improve the signal-to-noise.

However, the observations will not be error-free. In addition to violations of the assumptions stated above, there will be an uncertainty introduced into the line luminosity arising from an error in the continuum level. If the uncertainty in the continuum flux is a specific fraction of the peak line flux, then there will be an associated fractional uncertainty of this magnitude in the line luminosity and the low moments of a line. High moments will be far more sensitive to the continuum level and so should be avoided.

If the transfer function is well determined, then it becomes a practical proposition to invert it directly using the methods of §§ III–V. In a sense this represents the very best that can be done with the data, and our understanding of the cloud distribution is then limited by the accuracy of the symmetry assumption that must be made. To obtain a sufficiently low signal-to-noise in the transfer function will probably require several more years of observation than is necessary to carry out simple model-fitting. If positive-definite reprocessing coefficient and velocity distribution cannot be produced in this manner, then we must probably conclude that the geometry of the emission line region is substantially more complex than we have assumed in this study. For example, the continuum source may be coextensive with the emitting clouds, or, equivalently, electron scattering may be important (Shields and McKee 1981). Either of these conclusions, which could be supported by studies of the variability and polarization of the continuum, would require modification of existing theoretical models of the line-emitting gas.

Until the advent of sensitive optical interferometry, the monitoring of variable line emission from Seyferts and quasars represents the best hope outside the radio band for resolving the spatial structure of the central few light-years of Seyferts and quasars. For these reasons, we urge that a regular program of spectroscopic observations for a few of the brightest Seyfert I nuclei be undertaken now. High quality spectra should be taken at sufficiently small time intervals that the change in line profiles from one observation to the next is not large. Ideally, the change in the continuum from one observation to the next should be small also; however, since the line profile is determined by an integral of the continuum, the constraint on the observation frequency of the continuum is somewhat weaker than for the lines. In the case of a typical luminous Seyfert, we estimate that biweekly observations for a period of 2 years would be satisfactory. A long-range observational program of some bright quasars should also be initiated. Since the amplitude of the profile variations is expected to be small, high accuracy is essential. Low luminosity quasars should have smaller emission line regions and show more rapid variability. Optically violently variable quasars may have a beamed continuum; if so, they are not good candidates for this study.

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APPENDIX A

SOLUTION OF INTEGRAL EQUATION

We wish to solve the equation

$$\Psi(t) = \int_{t/a}^{\infty} \frac{dr}{r} \zeta(r) K\left(\frac{t}{r}\right) \quad (t \geq 0) \quad (\text{A1})$$

for $\zeta(r)$, given $\Psi(t)$, $K(x)$, $\zeta(0)=0$, and the constant a . This equation can be solved using Mellin transforms, but we prefer to work with Fourier transforms, both for uniformity and because economical algorithms for taking numerical Fourier transforms are readily available. Define

$$\tau \equiv \ln t, \quad \rho \equiv \ln r, \quad k(x) \equiv K(e^x)H(a - e^x), \quad (\text{A2})$$

where t and r are measured in the same units and where $H(x)$ is the step function. Then equation (A1) becomes

$$\Psi(e^\tau) = \int_{-\infty}^{\infty} d\rho \zeta(e^\rho) k(\tau - \rho). \quad (\text{A3})$$

This equation may be inverted by using the convolution theorem for Fourier transforms:

$$\zeta(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega \ln r}}{\tilde{k}(\omega)} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \Psi(e^\tau), \quad (\text{A4})$$

where $\tilde{k}(\omega)$ is the Fourier transform of $k(x)$.

In some cases it is convenient to express $\zeta(r)$ as the inverse transform of $\tilde{\Psi}(\omega)\Upsilon(\omega t)$, where $\Upsilon(\omega t)$ is a function to be determined:

$$\zeta\left(r = \frac{t}{a}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{\Psi}(\omega) \Upsilon(\omega t). \quad (\text{A5})$$

Expressing $\Psi(e^\tau)$ in terms of $\tilde{\Psi}(\omega)$ and rearranging the order of integration in equation (A4), we find

$$\Upsilon(z) = \frac{e^{iz}}{2\pi} \int_{-\infty}^{\infty} dx e^{-izx} \frac{H(x)}{x} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega \ln x}}{e^{-i\omega \ln a} \tilde{k}(\omega)}. \quad (\text{A6})$$

As $\zeta(r)$ and $\Psi(t)$ are real, then $\Upsilon(z) = \Upsilon^*(-z)$.

The function Υ is determined by the kernel $K(t/r)$. In the case of spherically symmetric, radial flow, the kernel for the n th symmetric moment is $[(t/r) - 1]^n$ and $a=2$, so that

$$\tilde{k}_n(\omega) = \int_{-\infty}^{\infty} dx e^{i\omega x} (e^x - 1)^n H(2 - e^x) \quad (\text{A7})$$

$$= e^{i\omega \ln 2} \left[(-1)^n \pi \delta(\omega) + \sum_{j=0}^n \binom{n}{j} (-1)^{n+j} \frac{2^j}{i\omega + j} \right] \quad (\text{A8})$$

(see Lighthill 1958). For the asymmetric moments, $\tilde{k}^{(n)}(\omega)$ is more complicated and analytic progress is difficult. However, for the symmetric moments the function $\kappa(\omega) \equiv [\tilde{k}_n(\omega)]^{-1} e^{i\omega \ln 2}$ can be expanded by means of partial fractions into terms proportional to 1, to ω , and to $(\omega - c)^{-1}$, where $c \equiv c_r + ic_i$ is complex ($c_i \equiv \text{Im } c \neq 0$). If

$$\kappa(\omega) = 1, \quad \text{then } \Upsilon(z) = 1; \quad (\text{A9})$$

$$\kappa(\omega) = \omega, \quad \text{then } \Upsilon(z) = z; \quad (\text{A10})$$

$$\kappa(\omega) = \frac{1}{\omega - c} \begin{cases} c_i > 0, & \text{then } \Upsilon(z) = ie^{iz} \frac{\Gamma(ic, iz)}{(iz)^{ic}}; \\ c_i < 0, & \text{then } \Upsilon(z) = -ie^{iz} \frac{\gamma(ic, iz)}{(iz)^{ic}}; \end{cases} \quad (\text{A11})$$

$$(\text{A12})$$

where

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt \quad \text{and} \quad \gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$$

are incomplete gamma functions. With these results in hand, it is possible to determine Υ for different values of n ; the results for $n=0, 1, 2$ are given in equation (3.17) in the text [the result for $n=2$ was simplified by using the recursion relations for $\gamma(a, x)$].

For the particular case of this kernel, an alternative, more straightforward, technique exists for inverting the moments of the integral equation (A1). If

$$\Psi^{(n)}(t) = \int_{t/2}^{\infty} dr \frac{\xi_{n(r)}}{r} \left(\frac{t}{r} - 1 \right)^n, \quad (\text{A13})$$

then for $n=0$, differentiation gives

$$\xi_0\left(\frac{1}{2}t\right) = -t \frac{d\Psi^{(0)}}{dt}(t) \quad (\text{A14})$$

or

$$\xi_0(r) = -2r \frac{d\Psi^{(0)}}{dt}(2r). \quad (\text{A15})$$

For $n=1$, equation (A13) may be differentiated twice to give a first order differential equation for ξ_1 :

$$t \frac{d\xi_1\left(\frac{1}{2}t\right)}{dt} + 2\xi_1\left(\frac{1}{2}t\right) = -2t^2 \frac{d^2\Psi^{(1)}(t)}{dt^2}, \quad (\text{A16})$$

which has the solution

$$\xi_1(r) = -\frac{1}{2r} \int_0^{2r} dt \, t^2 \frac{d^2\Psi^{(1)}(t)}{dt^2} \quad (\text{A17})$$

Finally, for $n=2$, three successive differentiations lead to the equation

$$t^2 \frac{d^2}{dt^2} \xi_2\left(\frac{1}{2}t\right) + 4t \frac{d}{dt} \xi_2\left(\frac{1}{2}t\right) + 8\xi_2\left(\frac{1}{2}t\right) = -4t^3 \frac{d^3\Psi^{(2)}(t)}{dt^3}, \quad (\text{A18})$$

the solution of which is

$$\xi_2(r) = \left(\frac{2}{7r}\right)^{1/2} \int_0^{2r} dt \, t^{5/2} \sin\left[\frac{\sqrt{7}}{2} \ln\left(\frac{t}{2r}\right)\right] \frac{d^3\Psi^{(2)}(t)}{dt^3}. \quad (\text{A19})$$

Expressions (A15), (A17), and (A19) can be shown to be equivalent to equation (3.17) in the text.

APPENDIX B

SAMPLE TRANSFER FUNCTIONS

The moments of the transfer functions plotted in Figures 2–7 are listed here. These functions can be used in an elementary form of data analysis: If an emission-line region corresponds to one of the models listed in Table 1, then convolution of the continuum luminosity with a moment of the transfer function should yield the corresponding moment of the line luminosity (cf. eq. [2.7]).

Model RRA:

$$\Psi^{(n)}(t) = \frac{\Phi_0 v_0^{n+1}}{(n+1)r_0} \left\{ (-1)^n \min\left(\ln 2, \ln \frac{4r_0}{t}\right) + \sum_1^n \frac{(-1)^{n+j+1}}{j} \binom{n}{j} \left(\frac{t}{r_0}\right)^j \left[2^{-j} - \min\left(1, \frac{2r_0}{t}\right)^j \right] \right\};$$

$$0 \leq t \leq 4r_0. \quad (\text{B1})$$

Model RAN:

$$\Psi^{(0)}(t) = \frac{\Phi_0 v_0}{2r_0} \cdot \begin{cases} \frac{t}{2r_0} + \ln 2, & 0 < t < 2r_0 \\ 2 - \frac{t}{2r_0} + \ln \frac{4r_0}{t}, & 2r_0 < t < 4r_0; \end{cases} \quad (\text{B2})$$

$$\Psi^{(1)}(t) = \frac{\Phi_0 v_0^2}{2r_0} \cdot \begin{cases} \frac{3}{16} \frac{t^2}{r_0^2} - \frac{\ln 2}{2}, & 0 < t < 2r_0 \\ 1 - \frac{t^2}{16r_0^2} - \frac{1}{2} \ln \frac{4r_0}{t}, & 2r_0 < t < 4r_0; \end{cases} \quad (\text{B3})$$

$$\Psi^{(2)}(t) = \frac{\Phi_0 v_0^3}{6r_0} \cdot \begin{cases} \frac{7}{24} \left(\frac{t}{r_0}\right)^3 - \frac{3}{8} \left(\frac{t}{r_0}\right)^2 - \frac{1}{2} \frac{t}{r_0} + \ln 2, & 0 < t < 2r_0 \\ -\frac{1}{24} \left(\frac{t}{r_0}\right)^3 + \frac{1}{8} \left(\frac{t}{r_0}\right)^2 + \frac{1}{2} \frac{t}{r_0} + \ln \frac{4r_0}{t} - \frac{4}{3}, & 2r_0 < t < 4r_0. \end{cases} \quad (\text{B4})$$

Model ROD:

$$\Psi^{(0)}(t) = \frac{\Phi_0 w_0}{3r_0} \min \left[1, \left(\frac{2r_0}{t} \right)^{3/2} \right], \quad (\text{B5})$$

$$\Psi^{(1)}(t) = \frac{\Phi_0 w_0^2}{6r_0} \min \left[\frac{t}{r_0} - \frac{3}{2}, \frac{1}{2} \left(\frac{2r_0}{t} \right)^2 \right], \quad (\text{B6})$$

$$\Psi^{(2)}(t) = \frac{\Phi_0 w_0^3}{3r_0} \min \left[\frac{1}{3} \left(\frac{t}{r_0} \right)^2 - \frac{6}{7} \left(\frac{t}{r_0} \right) + \frac{3}{5}, 0.219 \left(\frac{2r_0}{t} \right)^{5/2} \right]. \quad (\text{B7})$$

Model ROA:

 $\Psi^{(0)}(t)$: same as ROD,

$$\Psi^{(1)}(t) = \frac{\Phi_0 w_0^2}{15r_0} \min \left[\frac{t}{2r_0} - \frac{5}{4}, \left(\frac{2r_0}{t} \right)^{3/2} - \frac{5}{4} \left(\frac{2r_0}{t} \right)^2 \right], \quad (\text{B8})$$

$$\Psi^{(2)}(t) = \frac{\Phi_0 w_0^3}{6r_0} \min \left[\frac{1}{42} \left(\frac{t}{r_0} \right)^2 - \frac{4}{35} \left(\frac{t}{r_0} \right) + \frac{1}{5}, 0.629 \left(\frac{2r_0}{t} \right)^{3/2} - \left(\frac{2r_0}{t} \right)^2 + 0.438 \left(\frac{2r_0}{t} \right)^{5/2} \right]. \quad (\text{B9})$$

Model ISO:

$$\Psi^{(2n)}(t) = \left(\frac{\pi}{2} \right)^{1/2} \frac{\Phi_0 \sigma_0^{2n+1}}{r_0} \frac{(2n)!}{(2n+3)2^n n!} \min \left[1, \left(\frac{2r_0}{t} \right)^{n+3/2} \right] \quad (\text{B10})$$

$$\Psi_a^{(2n+1)}(t) = \frac{1}{2} \frac{\Phi_0 \sigma_0^{2(n+1)}}{r_0} \frac{2^n n!}{(n+2)} \min \left[1, \left(\frac{2r_0}{t} \right)^{n+2} \right] \quad (\text{B11})$$

Disk:

$$\Psi^{(2n)} = \begin{cases} 0, & t/r_0 < 1-s \\ \frac{\Phi_0 w_{n0}^{2n+1}}{\pi t^{n+3/2}} r_0^{n+1/2} \int_0^{\arccos\{\max[-1, (1-t/r_0)/s]\}} d\phi \sin^{2n} \phi (1-s \cos \phi)^{n+1/2} \\ \cdot \arccos \left[\frac{r_0^{1/2}}{t^{1/2}} (1-s \cos \phi)^{1/2} \right], & t/r_0 > 1-s, \end{cases} \quad (\text{B12})$$

where $s = \sin i$; this is equivalent to equation (4.9a) with expression (4.7) for $\mathcal{R}(r)$. The first asymmetric moment is

$$\Psi_a^{(1)} = \frac{2\Phi_0 w_{n0}^2}{\pi s r_0} \cdot \begin{cases} 0, & t/r_0 < 1-s \\ F\left[\left(\frac{1-s}{t/r_0}\right)^{1/2}\right], & 1-s < t/r_0 < 1+s \\ F\left[\left(\frac{1-s}{t/r_0}\right)^{1/2}\right] - F\left[\left(\frac{1+s}{t/r_0}\right)^{1/2}\right], & 1+s < t/r_0, \end{cases} \quad (\text{B13})$$

where

$$F(x) = \frac{(2x^3 + 3x)}{32} (1-x^2)^{1/2} - \left(\frac{x^4}{4} - \frac{3}{32} \right) \arccos x. \quad (\text{B14})$$

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