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A REINVESTIGATION OF THE STANDARD MODEL FOR THE DYNAMICS OF A MASSIVE BLACK HOLE IN A GLOBULAR CLUSTER

D. N. C. LIN
Institute of Astronomy, Cambridge; and Board of Studies in Astronomy and Astrophysics, University of California, Santa Cruz

AND

SCOTT TREMAINE
Institute for Advanced Study, Princeton, New Jersey
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ABSTRACT

We investigate the validity of two of the standard assumptions used in calculating the dynamics of a massive black hole in the center of a dense stellar system such as a globular cluster. First, the motion of the hole due to perturbations from its associated cusp of bound stars is usually neglected. We find that this assumption introduces substantial errors in the relaxation rate in the cusp; however, these errors are less than the statistical uncertainties due to the small number of cusp stars. The contribution of cusp star perturbations to the overall Brownian motion of the hole is negligible. Second, the effects of close encounters between stars are usually neglected. We confirm that close encounters have only a small effect on the stellar density distribution in the cusp, but we find that ejection by close encounters is the dominant hole-induced source of mass loss from the cluster core: a star entering the cusp is more likely to be ejected from the core than to be swallowed by the hole. A centrally condensed cluster with a massive hole may lose stars by ejection faster than by evaporation. Both ejection from the cusp and diffusion into the cusp pump energy into the cluster and slow core collapse. The diffusive flux alone is not sufficient to halt core collapse before the core disappears within the cusp; however, a complete understanding of the fate of the core will require numerical calculations which include the effects of ejection as well. These results show that close encounters play a major role in the evolution of a hole plus cluster system.

Subject headings: black holes — clusters: globular — stars: stellar dynamics

I. INTRODUCTION

Bahcall and Ostriker (1975) and Silk and Arons (1975) have argued that the X-ray sources seen in globular clusters are the result of accretion onto a massive black hole in the cluster center. Whether or not they emit X-rays, there is some reason to suppose that a massive black hole may form in the cluster center during core collapse (e.g., Lightman and Shapiro 1978). By now there have been a number of studies of the dynamical interaction of such an object with the cluster core (Peebles 1972*a*, *b*; Bahcall and Wolf 1976, 1977; Frank and Rees 1976; Shapiro and Lightman 1976; Lightman and Shapiro 1977; Dokuchaev and Ozernoy 1977; Young 1977*a*, *b*; Cohn and Kulsrud 1978; Ipser 1978; Shapiro and Marchant 1978; Marchant and Shapiro 1979, 1980). These studies have focused on the relaxation of stars through two-body encounters. The stars diffuse from unbound orbits in the cluster core into a cusp of more and more tightly bound orbits around the hole, until they are finally disrupted by tides or star-star collisions.

All of these calculations are based on a number of common assumptions. Two of the most important are:

1. *The only perturbing forces on the hole come from unbound stars.* In reality, of course, the hole moves in response to perturbations from both the unbound stars in the cluster core and the bound stars in the density cusp around the hole. The effects of perturbations from unbound stars are relatively easy to estimate: arguments due to Bahcall and Wolf (1976, henceforth BW), which are reproduced in § III, show that the hole oscillates around the cluster center with an amplitude given by equation (17). This Brownian motion has little effect on the rate of consumption of stars (Young 1977*a*); it also has little effect on the evolution of the cusp, since the periods of stellar orbits in the cusp are generally much shorter than the period of the hole's oscillation (Cohn and Kulsrud 1978).

The perturbations due to bound stars are more difficult to handle, and so far they appear to have been ignored in the literature. In § II we present an estimate of their effect on the evolution of the density cusp. In § III we discuss their contribution to the amplitude of the hole's motion about the cluster center.

2. *The only important two-body encounters in the cusp are distant ones, involving small fractional changes in the velocity of the participating stars.* The usual justification of this assumption (see BW) is that the cumulative effects of distant encounters dominate the effects of close encounters by a factor $\sim \ln \Lambda$, where $\Lambda = M/m$ is the ratio of hole mass to stellar mass. This result is correct only if the parameters describing successive encounters between two stars are completely

uncorrelated, and this is not likely to be the case for distant encounters in a Keplerian field. Thus the proper value for Λ, and the extent to which distant encounters dominate, are not well known. The resolution of this problem is beyond the scope of this paper, and we will generally leave the value of $\ln \Lambda$ in the cusp as a free parameter.

However, no matter what the proper value of $\ln \Lambda$ may be, close encounters have important consequences for the evolution of the hole-cusp-cluster system. In particular, in § IV we show that the rate of ejection of stars from the core by close encounters generally exceeds the rate of consumption by the hole, and the rate of ejection from the cluster can be comparable to the evaporation rate from the cluster.

Although our results can be applied to other stellar systems, for simplicity we will work only with globular clusters in this paper. Numerical results will be expressed relative to a "standard" cluster (closely resembling M15 = NGC 7078) with core radius $r_c = 0.5$ pc and line-of-sight velocity dispersion $\sigma = 10$ km s^{-1}. The central density is

$$\rho = \frac{9\sigma^2}{4\pi G r_c^2} = 6.6 \times 10^4\, M_\odot\, \mathrm{pc}^{-3} \left(\frac{\sigma}{10\ \mathrm{km\ s^{-1}}}\right)^2 \left(\frac{0.5\ \mathrm{pc}}{r_c}\right)^2 . \tag{1}$$

The core mass is $M_{\mathrm{core}} \approx \frac{4}{3}\pi \rho r_c^3 \approx 1.17 \times 10^4\, M_\odot (\sigma/10\ \mathrm{km\ s^{-1}})^2 (r_c/0.5\ \mathrm{pc})$. We assume that the mass of a typical core star is $1\, M_\odot$. At a typical cluster distance $D = 10$ kpc, $1'' \approx 0.05$ pc.

II. THE EVOLUTION OF THE DENSITY CUSP

In this section we are concerned with the interactions between the hole and the stars bound to it. Thus, we neglect the influence of the cluster core (i.e., the unbound stars) and consider an isolated hole of mass M at position $\boldsymbol{R}_0$ surrounded by a cloud of N bound stars of mass m and positions $\boldsymbol{R}_i$, $i = 1, \ldots, N$. The equations of motion are

$$\frac{d^2\boldsymbol{R}_0}{dt^2} = Gm \sum_{j=1}^{N} \frac{\boldsymbol{R}_j - \boldsymbol{R}_0}{|\boldsymbol{R}_j - \boldsymbol{R}_0|^3}, \tag{2a}$$

$$\frac{d^2\boldsymbol{R}_i}{dt^2} = -GM \frac{\boldsymbol{R}_i - \boldsymbol{R}_0}{|\boldsymbol{R}_i - \boldsymbol{R}_0|^3} + Gm \sum_{j=1, j\neq i}^{N} \frac{\boldsymbol{R}_j - \boldsymbol{R}_i}{|\boldsymbol{R}_j - \boldsymbol{R}_i|^3}, \qquad i = 1, \ldots, N . \tag{2b}$$

Let $\boldsymbol{r}_i = \boldsymbol{R}_i - \boldsymbol{R}_0$. Subtracting the two equations, we find

$$\frac{d^2\boldsymbol{r}_i}{dt^2} = -G(M + m)\frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|^3} + GM \sum_{j=1, j\neq 1}^{N} \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{|\boldsymbol{r}_j - \boldsymbol{r}_i|^3} - Gm \sum_{j=1}^{N} \frac{\boldsymbol{r}_j}{|\boldsymbol{r}_j|^3}, \qquad i = 1, \ldots, N . \tag{3}$$

The first term describes the Keplerian motion of star i around the hole. The second or "direct" term arises because the other stars perturb the motion of the test star, and the third or "indirect" term arises because the other stars perturb the hole, thus changing the separation between the hole and test star. In all existing studies of the evolution of the cusp, the hole is assumed to be fixed at the cluster center. In this approximation the indirect terms do not appear in the equations of motion (cf. eq. [4] below). Since the direct and indirect terms in (3) have comparable magnitudes, this is a serious oversimplification.

We have been unable to find any simple analytic technique for determining the effect of the indirect term on the relaxation rate. Therefore, we have resorted to N-body simulations. We numerically followed the evolution of three systems with the same initial conditions but different equations of motion:

1. System (D + I): the equations of motion are (2a) and (2b). Both direct and indirect terms are present.
2. System (D): this is like system (D + I) except that the hole is fixed at the origin. Equation (2b) is unchanged, but equation (2a) is replaced by $\boldsymbol{R}_0 = 0$. The analog to equation (3) is:

$$\frac{d^2\boldsymbol{r}_i}{dt^2} = -GM \frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|^3} + Gm \sum_{j=1, j\neq 1}^{N} \frac{\boldsymbol{r}_j - \boldsymbol{r}_i}{|\boldsymbol{r}_j - \boldsymbol{r}_i|^3}, \qquad i = 1, \ldots, N . \tag{4}$$

Only the direct term is present. This is the system examined by previous investigators.

3. System (I): in this system the gravitational interactions between the stars have been removed (the stars attract the hole but not each other). The equations of motion are:

$$\frac{d^2\boldsymbol{R}_0}{dt^2} = Gm \sum_{i=1}^{N} \frac{\boldsymbol{R}_i - \boldsymbol{R}_0}{|\boldsymbol{R}_i - \boldsymbol{R}_0|^3}, \tag{5a}$$

$$\frac{d^2\boldsymbol{R}_i}{dt^2} = -GM \frac{\boldsymbol{R}_i - \boldsymbol{R}_0}{|\boldsymbol{R}_i - \boldsymbol{R}_0|^3}, \qquad i = 1, \ldots, N , \tag{5b}$$

and

$$\frac{d^2\boldsymbol{r}_i}{dt^2} = -G(M + m)\frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|^3} - Gm \sum_{j=1, j\neq i}^{N} \frac{\boldsymbol{r}_j}{|\boldsymbol{r}_j|^3}, \qquad i = 1, \ldots, N . \tag{6}$$

Only the indirect term is present. This system is designed to isolate the effects of the indirect perturbations.

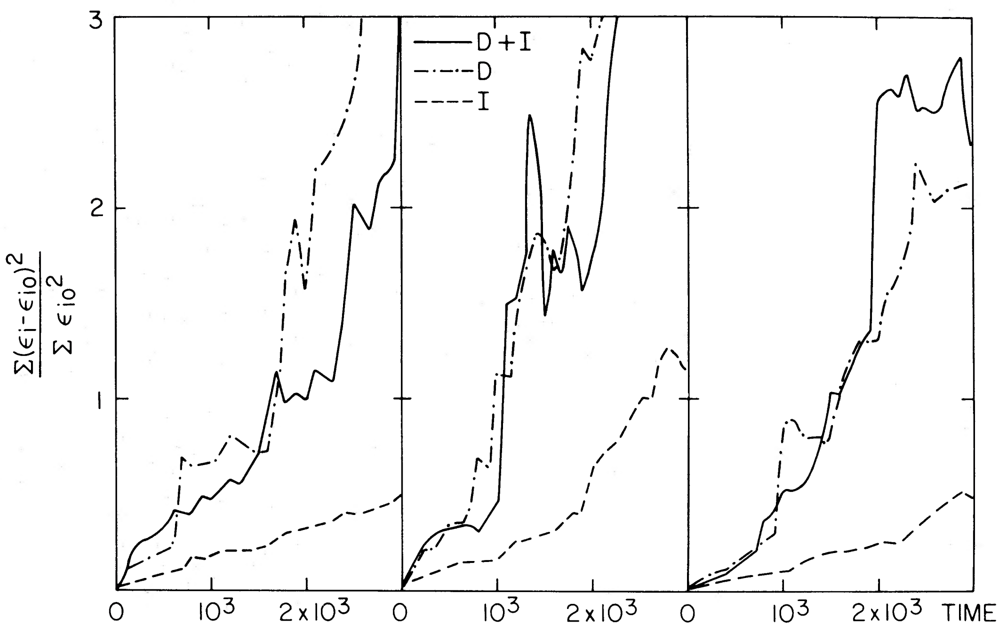

FIG. 1.—Plot of $\sum (\epsilon_i - \epsilon_{i0})^2/\sum \epsilon_{i0}^2$ versus time for three runs each of system (D + I), system (D), and system (I).

The initial conditions were chosen to resemble those expected in real globular clusters. The expected number of stars in the cluster cusp is (Young 1977*a*; Bahcall and Wolf 1977)

$$N_{\rm cusp} \approx 70\ (M/10^3\ M_\odot)^3(10\ {\rm km\ s^{-1}}/\sigma)^4(0.5\ {\rm pc}/r_c)^2\ . \tag{7}$$

Thus, we expect $N \approx 100$ for $M/m \approx 10^3$; in fact, to reduce computer expenses we chose $N \approx 100$, $M = 1$, and $m = \frac{1}{300}$ (in units where $G = 1$). The hole is initially at rest; the positions and velocities of the stars are chosen at random from a phase-space density distribution

$$\begin{aligned} f(\boldsymbol{r}_i, \boldsymbol{v}_i) &\propto \epsilon_i^p\ , \qquad \epsilon_0 < \epsilon_i < \epsilon_1\ , \\ &= 0\ , \qquad \epsilon_i < \epsilon_0 \quad \text{or} \quad \epsilon_i > \epsilon_1\ , \end{aligned} \tag{8}$$

where $\epsilon_i = M/|\boldsymbol{r}_i| - \frac{1}{2}v_i^2$ and $\boldsymbol{v}_i = d\boldsymbol{r}_i/dt$. We used $\epsilon_1 = 10\epsilon_0 = 1$. Following BW, we set $p = \frac{1}{4}$. The integrations were carried out with an N-body routine originally written by Sverre J. Aarseth, using a small softening parameter $b = 10^{-4}$ (i.e., the term $|\boldsymbol{r}_j - \boldsymbol{r}_i|^{-1}$ in the interparticle potential is replaced by $[(\boldsymbol{r}_j - \boldsymbol{r}_i)^2 + b^2]^{-1/2}$) to speed up the integrations.

We evolved all three systems for 3000 time units comparison, the orbital period of an individual star in the field of the hole lies between 2.22 and 70 time units). We follow the evolution of the quantities $\epsilon_i(t) = M/|\boldsymbol{r}_i(t)| - \frac{1}{2}v_i(t)^2$. The most useful measure of the amount of relaxation is the mean square change in the ϵ_i's from their initial values $\epsilon_i(t = 0) = \epsilon_{i0}$. We expect

$$\sum_{i=1}^{N} (\epsilon_i - \epsilon_{i0})^2 = \frac{t}{t_r} \sum_{i=1}^{N} \epsilon_{i0}^2\ , \tag{9}$$

where the relaxation time t_r will be different for each system.

The results from three typical runs are shown in Figure 1. In each case we plot $y = \sum (\epsilon_i - \epsilon_{i0})^2/\sum \epsilon_{i0}^2$ versus t. From the figure we estimate t_r from the time when $y = 1$. We find $t_r \approx 1500 \pm 500$ for system (D + I) and system (D), and $t_r \approx 5000 \pm 2000$ for system (I). The differences between system (D + I) and system (D) are not significant since they are smaller than the variations between different runs of the same system. For comparison, the relaxation time obtained analytically by solving the Fokker-Planck equation is (see Appendix) $t_r = 2.4 \times 10^3 (\ln \Lambda)^{-1}$. Thus, in the simulation the effective value of $\ln \Lambda \approx 2$ compared with (cf. § I) the standard estimate $\ln \Lambda \approx \ln M/m \approx 6$ (BW). The analytic estimates are slightly too large, but we will not pursue the source on the discrepancy here.

Figure 1 shows that the indirect terms cause relaxation at a substantial rate. Thus, it is disturbing that this effect is not included in the standard Fokker-Planck treatment of relaxation in the cusp. Moreover, there does not appear to be any simple modification of the Fokker-Planck equation which can include the indirect terms. On the other hand, since the number of stars in our simulated cusp is comparable to the number in a real cusp, Figure 1 also shows that the additional relaxation due to the indirect terms is smaller than the statistical fluctuations in the relaxation rate. Thus, it seems likely that no serious error is made by calculating the evolution of the cusp on the assumption that the hole is fixed.

III. BROWNIAN MOTION OF THE HOLE

It is important to understand the causes of the excursions of the hole from the cluster center because the present displacement of the hole may be an observable quantity. The High Resolution Imager on the *Einstein* (*HEAO 2*) X-ray Observatory has the capability of determining the position of a source to better than several seconds of arc (Giacconi *et al.* 1979). By the mid 1980s, the Wide Field/Planetary Camera on the Space Telescope will be able to locate a massive black hole to within $\sim 0\rlap{.}''1$ if it can detect either radiation from accreting gas or the cusp of stars bound to the hole.

The amplitude of the excursions expected due to interactions with core (i.e., unbound) stars was calculated by BW. This calculation is reproduced with some added detail in § III*a*. The amplitude expected from interactions with cusp (i.e., bound) stars is calculated in § III*b*. An additional contribution comes from recoil as stars are ejected from the cusp; this amplitude is calculated in § III*c*.

Note that the position of the cluster center can never be determined to an accuracy better than the statistical uncertainty $\Delta R \approx N_{\text{core}}^{-1/2} r_c$, where $N_{\text{core}} \approx \frac{4}{3}\pi\rho r_c^3/m$ is the number of stars in the core. Numerically,

$$\Delta R \approx 4 \times 10^{-3} \text{ pc } (r_c/0.5 \text{ pc})^{1/2} (10 \text{ km s}^{-1}/\sigma) , \tag{10}$$

or about $0\rlap{.}''1$ at 10 kpc.

a) *Interactions with Unbound Stars*

This is the process investigated by BW. Gravitational perturbations from passing stars tend to bring the hole's kinetic energy into equipartition with the kinetic energy of the stars. The time require to reach equipartition is[1]

$$t_{\text{eq}} = \frac{1.2 \times 10^5 \text{ yr}}{\ln \Lambda} \left(\frac{10^3\, M_\odot}{M}\right) \left(\frac{\sigma}{10 \text{ km s}^{-1}}\right) \left(\frac{r_c}{0.5 \text{ pc}}\right)^2 . \tag{11}$$

In the derivation of this equation it is assumed that the momentum transfers are uncorrelated in successive encounters of a given star with the hole (cf. § I). This is certainly not true for very distant encounters if the hole is near the cluster center. Thus, only encounters which are sufficiently "close" (say, impact parameter $b \gtrsim b_{\max}$) contribute to the relaxation. The value of $\ln \Lambda$ corresponding to a given $b_{\max}$ is

$$\ln \Lambda \approx \ln \left[1 + \left(\frac{b_{\max}\sigma^2}{GM}\right)^2\right] . \tag{12}$$

As we stated in § I when discussing the value of $\ln \Lambda$ in the cusp, determining the proper choice of Λ or $b_{\max}$ is beyond the scope of this paper. However, our only purpose here is to show that the hole will be in equipartition, so a conservative lower limit to $\ln \Lambda$ is sufficient.

If the hole is at rest at the cluster center, the change in a star's impact parameter between successive encounters with the hole is $\Delta b \approx r_c(t_d/t_r)^{1/2}$, where t_d is the radial orbital period and t_r is the star-star relaxation time (Spitzer and Hart 1971). For $b < \Delta b$, successive encounters are certainly uncorrelated. Thus, $b_{\max} \gtrsim r_c(t_d/t_r)^{1/2}$ or

$$\frac{b_{\max}\sigma^2}{GM} \gtrsim 0.6 \left(\frac{r_c}{0.5 \text{ pc}}\right)^{1/2} \left(\frac{\sigma}{10 \text{ km s}^{-1}}\right) \left(\frac{10^3\, M_\odot}{M}\right) . \tag{13}$$

Consider the standard cluster. Combining equations (11)–(13), we see that if $M \lesssim 600\, M_\odot$ then $t_{\text{eq}} \lesssim 1.2 \times 10^5$ yr $(10^3\, M_\odot/M)$, and if $M \gtrsim 600\, M_\odot$ then $t_{\text{eq}} \lesssim 3 \times 10^5$ yr $(M/10^3\, M_\odot)$. These times are much shorter than the cluster age.

We conclude that in a typical globular cluster there will always be energy equipartition between the core stars and the hole. The probability that the hole's position and velocity lie in an interval $d\boldsymbol{r}_h d\boldsymbol{v}_h$ is thus $p(\boldsymbol{r}_h, \boldsymbol{v}_h) d\boldsymbol{r}_h d\boldsymbol{v}_h \propto \exp(-ME/m\sigma^2) d\boldsymbol{r}_h d\boldsymbol{v}_h$, where $E = \frac{1}{2}v_h^2 + \phi(r_h)$ is the hole's energy. The potential ϕ is just the cluster's unperturbed gravitational potential. We do not have to consider any contributions to ϕ from the hole's perturbation of the stellar distribution since the average effect of these perturbations on the hole is zero when equipartition exists. Substituting $\phi(r) = \frac{2}{3}\pi G\rho r^2$ and integrating over $\boldsymbol{v}_h$, we find the probability that the hole's position lies in a volume $d\boldsymbol{r}_h$:

$$p(\boldsymbol{r}_h) d\boldsymbol{r}_h = \left(\frac{3m\sigma^2}{2MG\rho}\right)^{-3/2} \exp\left(-\frac{2\pi MG\rho r_h^2}{3m\sigma^2}\right) d\boldsymbol{r}_h . \tag{14}$$

[1] We use $t_{\text{eq}} = \langle v_h^2 \rangle / \langle (\Delta v_\parallel)^2 \rangle_{x=0}$, where $\langle v_h^2 \rangle = m\sigma^2/M$ is the hole's equipartition velocity dispersion and $\langle (\Delta v_\parallel)^2 \rangle$ and x are defined by Spitzer and Hart (1971).

The probability distribution in projected radius R is

$$p(R)dR = 4\pi RdR \int_0^\infty p(R^2 + z^2)^{1/2}dz$$

$$= \frac{4\pi MG\rho RdR}{3m\sigma^2} \exp\left[-\frac{2\pi MG\rho R^2}{3m\sigma^2}\right] . \tag{15}$$

Using equation (1), we find

$$p(R)dR = 3\frac{MRdR}{m{r_c}^2} \exp\left[-\frac{3MR^2}{2m{r_c}^2}\right] . \tag{16}$$

The mean value of R is $\langle R\rangle = (\pi m/6M)^{1/2}r_c$. This result can also be obtained from BW: they find $\langle r_h\rangle = r_c(8m/3\pi M)^{1/2} = 0.92r_c(m/M)^{1/2}$; and if θ is the angle between r_h and the line of sight, we have $\langle R\rangle = \langle r_h\rangle\langle\sin\theta\rangle = \pi/4\langle r_h\rangle = r_c(\pi m/6M)^{1/2}$ just as before. Numerically,

$$\langle R\rangle = 0.01 \text{ pc } (10^3\ M_\odot/M)^{1/2}(r_c/0.5 \text{ pc}) . \tag{17}$$

b) *Interactions with Bound Stars*

In this section we present analytic and numerical estimates of the amplitude of the hole displacements due to stars bound to it.

Consider a bound, isolated system consisting of a hole of mass M and N_{cusp} stars of mass m. The mass of the cusp is $M_{\text{cusp}} = mN_{\text{cusp}}$, and the mean square stellar velocity relative to the center of mass is $\langle v^2\rangle$. In the limit $M_{\text{cusp}} \gg M$, we expect that equipartition obtains, so that the rms hole velocity due to interactions with the cusp is

$$\langle {v_h}^2\rangle^{1/2} = (m/M)^{1/2}\langle v^2\rangle^{1/2} , \qquad M_{\text{cusp}} \gg M . \tag{18}$$

However, in a real cluster M_{cusp} is generally much less than M (see eq. [7]). In this regime equation (18) is not valid, because its derivation ignores the constraint that the total momentum of the hole and stars must be conserved.

A more accurate formula can be obtained by a simple argument due to Lynden-Bell (1979). Regard the stars as a single unit of mass M_{cusp} which interacts with the hole. The kinetic energy of relative motion of the hole and cusp is $\frac{1}{2}\mu {v_{\text{rel}}}^2$, where the reduced mass $\mu = M_{\text{cusp}}M/(M_{\text{cusp}} + M)$, $v_{\text{rel}} = v_h(M + M_{\text{cusp}})/M_{\text{cusp}}$, and v_h is the hole velocity in the center-of-mass frame. If we assume that there is equipartition between the relative kinetic energy and the kinetic energy of the individual stars, we find

$$\mu\langle {v_{\text{rel}}}^2\rangle = m\langle v^2\rangle$$

or

$$\langle {v_h}^2\rangle^{1/2} = \left[\frac{mN_{\text{cusp}}}{M(M + M_{\text{cusp}})}\right]^{1/2}\langle v^2\rangle^{1/2} . \tag{19}$$

In the limit $M_{\text{cusp}} \gg M$ we recover equation (18); in the limit $M_{\text{cusp}} \ll M$ we find

$$\langle {v_h}^2\rangle^{1/2} = (mM_{\text{cusp}}/M^2)^{1/2}\langle v^2\rangle^{1/2} , \qquad M_{\text{cusp}} \ll M ,$$
$$= {N_{\text{cusp}}}^{1/2}(m/M)\langle v^2\rangle^{1/2} . \tag{20}$$

This formula can be obtained by an independent argument: in the center-of-mass frame of a two-body system consisting of the hole and one cusp star, the hole velocity is $\boldsymbol{v}_h = -(m/M)\boldsymbol{v}$, where $\boldsymbol{v}$ is the stellar velocity. If we have N stars, with velocity vectors oriented at random, we expect

$$\langle {v_h}^2\rangle^{1/2} = \left(\frac{m}{M}\right)\left\langle\left(\sum_N \boldsymbol{v}\right)^2\right\rangle^{1/2} = N^{1/2}\left(\frac{m}{M}\right)\langle v^2\rangle^{1/2} ,$$

just as in equation (20).

Similarly, the rms displacement of the hole from the center of mass will be

$$\langle {r_h}^2\rangle^{1/2} = {N_{\text{cusp}}}^{1/2}(m/M)\langle r^2\rangle^{1/2} \qquad \text{if} \qquad M_{\text{cusp}} \ll M . \tag{21}$$

We have checked these results using the N-body experiments of § II. We computed the evolution of system (D + I) for six values of the stellar mass m and the number of stars N: $N = 10$, 30, 100, and $M_{\text{cusp}} = Nm = 0.02$, 0.333. In all cases $G = M = 1$ and the system was followed for 2000 time units. The initial conditions were the same as in § II. For each run we computed the ratio of the rms hole velocity to the rms stellar velocity in the center of mass frame. The results are given in Table 1. We have also listed rough error estimates, made by dividing each run into 400 time unit intervals, calculating

TABLE 1

RMS VELOCITY OF A BLACK HOLE IN A DENSITY CUSP

N	m	$M_{\rm cusp} = nM$	$\langle v_h^2\rangle^{1/2}/\langle v^2\rangle^{1/2}$	$[mM_{\rm cusp}/(1 + M_{\rm cusp})]^{1/2}$
100	2×10^{-4}	0.02	$(2.1 \pm 0.2) \times 10^{-3}$	2.0×10^{-3}
30	6.67×10^{-4}	0.02	$(3.6 \pm 0.1) \times 10^{-3}$	3.6×10^{-3}
10	2×10^{-3}	0.02	$(5.9 \pm 0.2) \times 10^{-3}$	6.3×10^{-3}
100	3.33×10^{-3}	0.333	$(3.1 \pm 0.2) \times 10^{-2}$	2.9×10^{-2}
30	1.11×10^{-2}	0.333	$(5.1 \pm 0.1) \times 10^{-2}$	5.3×10^{-2}
10	3.33×10^{-2}	0.333	0.101 ± 0.003	0.091

the rms velocities for each interval, and using the dispersion in these results to estimate the uncertainties in the rms velocities for the entire run. The results are in very good agreement with equation (19).

In principle, we could also use these simulations to check equation (21) for the rms hole displacement. Unfortunately, the statistical fluctuations in the displacement are much larger than in the velocity, and an accurate check is not possible at a reasonable cost in computer time.

We now apply our results to real clusters. To evaluate the rms displacement of the hole, we use equation (21) with $N_{\rm cusp}$ given by equation (7). For $\langle r^2\rangle^{1/2}$, the rms radius of the cusp stars, we use GM/σ^2, since this is the radius where the hole potential begins to dominate the cluster potential. Finally, we convert from the true displacement r_h to projected dispalcement R. We find

$$\langle R^2\rangle^{1/2} \sim 3 \times 10^{-4}\ {\rm pc} \left(\frac{M}{10^3\ M_\odot}\right)^{3/2} \left(\frac{10\ {\rm km\ s^{-1}}}{\sigma}\right)\left(\frac{0.5\ {\rm pc}}{r_c}\right). \tag{22}$$

The condition that $\langle R^2\rangle^{1/2}$ exceeds the statistical uncertainty ΔR (eq. [10]) is

$$M \gtrsim 0.3\ M_{\rm core} \tag{23}$$

($M_{\rm core}$ is defined after eq. [1]). However, our analysis is valid only for $M_{\rm cusp} \lesssim M$, and by equation (7) this constraint is satisfied only if $M \lesssim 0.2\ M_{\rm core}$ (this inequality was originally obtained by Bahcall and Wolf 1977, but with a typographical error in their eq. [4b]). Thus, it seems unlikely that the displacement of the hole due to stars bound to it can ever be reliably detected above statistical noise.

c) *Ejection of Bound Stars*

From time to time stars will be ejected from the cusp around the hole as a result of two-body encounters. If the encounters occur deep in the potential well of the hole, then the ejection can be quite violent: the escaping star may leave the cusp, the cluster, and even the Galaxy. The second star in the encounter will remain bound to the hole, so the hole and its associated cusp must recoil in the opposite direction. In this section we estimate the contribution of recoil effects from ejected stars to the rms displacement of the hole from the cluster center.

Consider an isolated hole-cusp system. If a star is ejected from the cusp with velocity v, the recoil velocity of the hole-cusp system is $v_h = -mv/M$ (we assume $M_{\rm cusp} \ll M$). If successive ejections are uncorrelated, we have

$$d/dt\langle v_h^2\rangle = (2m^2/M^2)(dE/dt)\,, \tag{24}$$

where $E = \frac{1}{2}v^2$ is the energy per unit mass and mdE/dt is the rate of energy loss from the hole-cusp system (we define a particle at rest at infinity to have $E = 0$). In reality the hole is embedded in the cluster potential, and an escaping ($E > 0$) star can return to the hole. Thus, to be conservative, we count stars as escapers only if $E > E_0 = \frac{1}{2}v_0^2$, where v_0 is the escape velocity from the cluster. In reality many escapers with $E > E_0$ will not be recaptured before two-body relaxation diffuses their orbits away from the hole, and so our estimate of E_0 is too large. However, we show below that our results do not depend strongly on E_0.

The rate of energy loss from escapers has been determined by Hénon (1969). It is straightforward to modify Hénon's equations (15) and (23) to the case we are considering, where all stars have mass m and only escapers with $E > E_0$ are counted. We find

$$\begin{aligned}\frac{dE}{dt} = 2^{15/2}\pi^4 G^2 m^2 \int_0^\infty r^2 dr \int_U^0 dE f(E) \int_{E_0+U-E}^0 dE' f(E') \\ \times \left\{\frac{2}{3}\frac{(E' + 2E_0 - 2E - U)(E' + E - E_0 - U)^{1/2}}{(E_0 - E)} - (E' - U)^{1/2} \ln\left[\frac{(E' - U)^{1/2} + (E' + E - E_0 - U)^{1/2}}{(E' - U)^{1/2} - (E' + E - E_0 - U)^{1/2}}\right]\right. \\ \left. + \frac{2}{3}\frac{E_0(E' + E - E_0 - U)^{3/2}}{(E_0 - E)^2}\right\}\end{aligned} \tag{25}$$

Here $f(E)$ is the phase space density distribution of cusp stars, $E = \frac{1}{2}v^2 + U$, and $U = -GM/r$ is the hole potential.

We interchange the order of integration and compute the integral over r. Equation (25) becomes

$$\frac{dE}{dt} = \frac{2^{9/2}}{3}\pi^5 G^5 m^2 M^3 \int_{-\infty}^{0} f(E)dE \int_{-\infty}^{0} f(E')dE' \times \left\{\frac{E_0}{(E_0 - E)^2(E_0 - E - E')^{3/2}} + \frac{3E_0 - 3E - E'}{|E'|^2(E_0 - E - E')^{1/2}(E_0 - E)} - \frac{3}{2|E'|^{5/2}} \ln\left[\frac{(E_0 - E - E')^{1/2} + |E'|^{1/2}}{(E_0 - E - E')^{1/2} - |E'|^{1/2}}\right]\right\}. \tag{26}$$

BW find that $f(E) \approx C|E|^p$, $|E| < E_{\max}$, and $f(E) = 0$, $|E| > E_{\max}$, where $C = 2^{-1/2}\pi^{-3/2}(\rho/m)\sigma^{-2p-3}$ and $E_{\max} \approx Gm/r_*$, where r_* is the stellar radius. With this approximation the integrals in equation (26) can be evaluated analytically for $p < \frac{1}{4}$ and $E_{\max} \gg E_0$, yielding

$$\frac{dE}{dt} = \frac{2^{11/2}}{3}\pi^{3/2}\frac{G^5\rho^2M^3}{\sigma^7}\left[\left(\frac{E_0}{\sigma^2}\right)^{2p-1/2}\frac{(3-4p)\Gamma(1+p)^2\Gamma(\frac{1}{2}-2p-2p)}{(3-2p)(1-2p)}\right], \quad -1 < p < \tfrac{1}{4}. \tag{27}$$

The apparent divergence in equation (27) as $p \to \frac{1}{4}$ is not real; it arises because, for $p \gtrsim \frac{1}{4}$, the integral depends on $E_{\max}$ and we have set $E_{\max} \to \infty$ in deriving (27). A similar expression for dE/dt is available for $p > \frac{1}{4}$; for $E_{\max} \gg E_0$ it is independent of E_0 and proportional to $(E_{\max}/\sigma^2)^{2p-1/2}$. BW estimate that $p \approx \frac{1}{4}$, and in this region dE/dt depends on both $E_{\max}$ and E_0 in a more complicated way; however, for our purposes it is sufficient to replace the brackets in (27) by a dimensionless factor Q which should not be very different from unity. Combining equations (24) and (27),

$$\frac{d}{dt}\langle v_h^2\rangle = \frac{2^{13/2}\pi^{3/2}}{3}\frac{G^5\rho^2Mm^2}{\sigma^7}Q. \tag{28}$$

The drag on the hole due to dynamical friction from the core stars is (Spitzer and Hart 1971)

$$\frac{dv_h}{dt} = -2(2\pi)^{1/2}\frac{G^2\rho M v_h}{\sigma^3} \times 0.376. \tag{29}$$

In equilibrium

$$\langle v_h^2\rangle = 25\frac{G^3\rho m^3}{\sigma^4}Q. \tag{30}$$

The resulting rms displacement is

$$\langle r_h^2\rangle^{1/2} = (3/4\pi G\rho)^{1/2}\langle v_h^2\rangle^{1/2} = (6Q)^{1/2}(Gm/\sigma^2). \tag{31}$$

Converting to projected displacement,

$$\langle R^2\rangle^{1/2} = 1 \times 10^{-4}\ \text{pc}\ Q^{1/2}(10\ \text{km s}^{-1}/\sigma)^2. \tag{32}$$

This exceeds the statistical uncertainty ΔR (eq. [10]) only if

$$M_{\text{core}} \lesssim 10Q\ M_\odot. \tag{33}$$

Since Q is of order unity, we conclude that the motion of the hole due to recoil from ejection cannot be detected above statistical noise in a typical globular cluster.

The overall conclusion from this section is that perturbations from unbound stars are the dominant cause of Brownian motion of the hole, and the expected amplitude of this motion is given by equation (17).

IV. THE IMPORTANCE OF CLOSE ENCOUNTERS

Considering only distant encounters, BW showed that the distribution of bound stars in the cusp around the hole would be close to a "zero-flow" solution; that is, the net flux in or out of the cusp would be much less than N_{cusp}/t_r, where t_r is the core relaxation time. This argument suggests that the net flow may, in fact, be dominated by close encounters: although they are less important by a factor $(\ln\Lambda)^{-1}$, they can expel a star from the cusp by a single encounter rather than a random walk.

To calculate this flux we again use formulae from Hénon (1969). Using the notation of § III, the rate at which stars are expelled from the cusp with energies exceeding E_0 is

$$\frac{dN_{\text{ej}}}{dt} = \frac{2^{17/2}}{3}\pi^4G^2m^2\int_0^\infty r^2dr\int_U^0 dEf(E)\int_{E_0+U-E}^0 dE'f(E')\frac{(E'+E-E_0-U)^{3/2}}{(E_0-E)^2}. \tag{34}$$

Computing the radial integral, we obtain[2]

$$\frac{dN_{\rm ej}}{dt} = \frac{2^{9/2}}{3}\pi^5 G^5 m^2 M^3 \int_{-\infty}^{0} \frac{f(E)dE}{(E_0 - E)^2} \int_{-\infty}^{0} \frac{f(E')dE'}{(E_0 - E - E')^{3/2}} . \tag{35}$$

Using the approximation of § III for $f(E)$, we find that, for $E_{\rm max} \gg E_0$,

$$\frac{dN_{\rm ej}}{dt} = \frac{2^{13/2}}{3}\pi^{3/2}\frac{G^5\rho^2 M^3}{\sigma^9}\left(\frac{E_0}{\sigma^2}\right)^{2p-3/2}\frac{\Gamma(1+p)^2\Gamma(\frac{3}{2}-2p)}{(3-2p)(1-2p)} , \qquad -1 < p < \tfrac{1}{2} . \tag{36}$$

For $p = \frac{1}{4}$,

$$\frac{dN_{\rm ej}}{dt} = 110.4\frac{G^5 M^3 \rho^2}{\sigma^9}\left(\frac{\sigma^2}{E_0}\right) . \tag{37}$$

This formula does not accurately represent the net flux for small E_0 because the compensating flux from the cluster into the cusp is not included. However, it can be used to estimate the rate at which cusp stars are ejected from the cluster, by setting $E_0 = \frac{1}{2}v_0^2$, where v_0 is the escape speed from the cluster, neglecting the attraction of the hole.

It is convenient to express M in units of $0.2M_{\rm core}$, since this is the largest hole mass for which our formulae are valid (see discussion after eq. [23]). Equation (37) becomes

$$\frac{dN_{\rm ej}}{dt} = 2.5\left(\frac{M}{0.2M_{\rm core}}\right)^3 \frac{(GM_{\rm core})^{1/2}}{r_c^{3/2}}\left(\frac{\sigma^2}{v_0^2}\right) . \tag{38}$$

For comparison, the evaporation rate is (Spitzer and Chevalier 1973)

$$\frac{dN_{\rm evap}}{dt} = 17\alpha\frac{(GM_{\rm cl})^{1/2}}{r_{\rm H}^{3/2}} \ln \Lambda , \tag{39}$$

where $M_{\rm cl}$ is the cluster mass, $r_{\rm H}$ is the radius containing half the mass, $\ln \Lambda \approx 10$ for most globular clusters, and α depends on the ratio of the cluster's tidal radius r_t to its core radius r_c(King 1966). The ratio of the ejection rate to the evaporation rate may be written as

$$\frac{dN_{\rm ej}/dt}{dN_{\rm evap}/dt} = \left(\frac{M}{0.2M_{\rm core}}\right)^3 f , \tag{40}$$

where

$$f = (0.015/\alpha)(M_{\rm core}/M_{\rm cl})^{1/2}(r_{\rm H}/r_c)^{3/2}(\sigma^2/v_0^2) . \tag{41}$$

For King's (1966) globular cluster models, the quantities $M_{\rm core}/M_{\rm cl}$, $r_{\rm H}/r_c$, and σ^2/v_0^2 are all known functions of r_t/r_c. Spitzer and Chevalier (1973) have measured α at two points: $\alpha = 0.05$ at $r_t/r_c = 5.2$ and $\alpha = 0.015$ at $r_t/r_c = 41$. For want of more complete data we will use the simple interpolation $\alpha = 0.13\,(r_t/r_c)^{-0.6}$. We have calculated f at the quartile points of Peterson and King's (1975) distribution of r_t/r_c for 43 clusters. The quartiles are at $r_t/r_c = 14.1$, 28.2, 60.3, and the corresponding values of f are 0.07, 0.13, and 0.23. Note that three of the four X-ray clusters with measured concentrations lie in the quartile where f is largest ($r_t/r_c = 53$, 90, 98, and 165, Jernigan and Clark 1979).

Our values of f show that the loss rate by ejection from the cusp can be comparable to the loss rate by evaporation in clusters with large central holes and high central concentration. Note that the ejected stars are lost from the center of the cluster core, while evaporated stars are lost from the halo. Thus, the two loss mechanisms will have very different effects on the cluster structure, and both must be included to understand the cluster evolution.

We can also compare the ejection rate to the rate of consumption of stars by the central hole. In this case the most convenient form of equation (37) is

$$\frac{dN_{\rm ej}}{dt} = 1.6 \times 10^{-6}\ {\rm yr}^{-1}\left(\frac{M}{10^3\ M_\odot}\right)^3\left(\frac{10\ {\rm km\ s^{-1}}}{\sigma}\right)\left(\frac{0.5\ {\rm pc}}{r_c}\right)^4\left(\frac{\sigma^2}{v_0^2}\right) \tag{42}$$

compared with the consumption rate (Bahcall and Wolf 1977; Shapiro and Marchant 1978; Cohn and Kulsrud 1978)

$$\frac{dN_{\rm eat}}{dt} = (1.5 \text{ to } 4.5) \times 10^{-7}\ {\rm yr}^{-1}\left(\frac{M}{10^3\ M_\odot}\right)^{61/27}\left(\frac{10\ {\rm km\ s^{-1}}}{\sigma}\right)^{21/9}\left(\frac{0.5\ {\rm pc}}{r_c}\right)^{28/9} . \tag{43}$$

For a cluster with the median concentration, $r_t/r_c = 28$, we have $\sigma^2/v_0^2 = 0.07$, and the probability that a cusp star will be ejected from a standard cluster is ~ 0.7 to 0.3 times the probability it is consumed. Possibly a more relevant number is the

[2] We have checked that equation (35) correctly predicts the escape rate for the numerical models of § II.

probability of ejection from the cluster core; setting $v_0 \sim \sigma$ in equation (42), we find that a cusp star is 3–10 times more likely to be ejected from the core than to be consumed by the hole.

Ejection also has important consequences for the cluster's energy balance. For our purposes we do not include stars bound to the cusp in determining the cluster energy $E_{\rm cl}$. Thus

$$E_{\rm cl} = \sum_i \tfrac{1}{2} m v_i^2 - \frac{1}{2} \sum_{i \neq j} \frac{Gm^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} - \sum_i \frac{Gm(M + M_{\rm cusp})}{|\boldsymbol{r}_i|}, \tag{44}$$

where the sum is over all stars not bound to the cusp. Each ejection reduces $M_{\rm cusp}$ by m so that every cluster star gains $\Delta E_i = Gm/|\boldsymbol{r}_i|$; it is easy to show that the total change is $E_{\rm cl} = \sum \Delta E_i = \frac{1}{2} m v_0^2$. The resulting energy flux from ejection is

$$\begin{aligned} \frac{dE_{\rm ej}}{dt} &= \tfrac{1}{2} m v_0^2 \frac{dN_{\rm ej}}{dt} \\ &= + m\sigma^2 \times 8 \times 10^{-7}\ {\rm yr}^{-1} \left(\frac{M}{10^3\ M_\odot}\right)^3 \left(\frac{10\ {\rm km\ s^{-1}}}{\sigma}\right)^5 \left(\frac{0.5\ {\rm pc}}{r_c}\right)^4 . \end{aligned} \tag{45}$$

The total cluster energy *increases* due to ejections. By contrast, the cluster energy changes very little from evaporation, since stars evaporate with nearly zero kinetic energy. Consumption of stars by the hole feeds energy into the cluster at a rate (Cohn and Kulsrud 1978; Shapiro and Marchant 1978)

$$\frac{dE_{\rm eat}}{dt} = + m\sigma^2 \times 5 \times 10^{-6}\ {\rm yr}^{-1} \left(\frac{M}{10^3\ M_\odot}\right)^3 \left(\frac{10\ {\rm km\ s^{-1}}}{\sigma}\right)^5 \left(\frac{0.5\ {\rm pc}}{r_c}\right)^4 . \tag{46}$$

This flux is larger than $dE_{\rm ej}/dt$ by a factor $\sim$ 6 or 7. However, our estimate of $dE_{\rm ej}/dt$ underestimates the true flux because there is an additional flux from stars diffusing down through the cusp to replace the stars lost by ejections. In any case, detailed comparisons of the energy flux from consumption and from ejection are not worthwile, since the two processes have very different effects: the consumption flux is deposted in stars at the cluster center, while the ejection flux is deposted more or less uniformly throughout the cluster (in each ejection a star at radius r gains $\Delta E = Gm/r$). Thus, ejection and consumption will have important but qualitatively different effects on the cluster's energy distribution. Stars which are expelled from the cusp but do not escape from the cluster also add energy to the cluster.

The importance of the energy flux from consumption was recognized by Shapiro (1977), who argued that the flux could halt the core collapse brought on by evaporation and eventually cause reexpansion of the core. He presented analytic calculations which showed that the central density reached a maximum and then decreased as the cluster evolved. However, Shapiro's arguments are inconclusive since he found that core collapse halted when $M \gtrsim 0.2 M_{\rm core}$, while his approximations, like ours, are only valid for $M \lesssim 0.2 M_{\rm core}$ (see discussion after eq. [23]). Core collapse has been followed past this point in Marchant and Shapiro's (1979, 1980) numerical models. These show no evidence that core collapse is halted; rather the collapse appears to continue until the core disappears within the cusp. At present there is no evidence that a central hole can halt core collapse, but the final answer must wait for numerical models which include the effects of close encounters.

Note that loss by ejections should not have a strong effect on the shape of the density cusp. Ejections lower the logarithmic slope $p = d \ln f / d \ln |E|$ below the "zero-flow" value of $\frac{1}{4}$; however, the change in p depends on the relative importance of close and distant encounters and should be $O([\ln \Lambda]^{-1})$. Thus, for $\ln \Lambda \gg 1$ we expect the change to be small. This justifies our use of $p = \frac{1}{4}$ in deriving equation (37).

Throughout this section (and this paper) we have assumed that all stars have the same mass. If a spectrum of stellar masses is present, there will be at least two major qualitative changes. First, the logarithmic slope p in the cusp will depend on the mass (Bahcall and Wolf 1977), second, the ejection rate of the lighter stars will be greatly enhanced (Hénon 1969). Both effects are important, and a realistic mass spectrum must be included in any comprehensive cluster evolution calculations.

In this section we have based our discussion of the effects of close encounters on one number which we can calculate fairly simply and accurately, the ejection rate (eq. [37]). These considerations are no substitute for numerical cluster models which include the effects of close encounters. In fact, our main purpose has been to motivate the construction of these models by demonstrating that close encounters play a major role in the evolution of the hole-cusp-cluster system.

V. CONCLUSIONS

In the last few years there has been rapid progress toward understanding the dynamical interactions of a massive central black hole with its parent cluster. The work so far has been based on two important simplifying assumptions: (1) the perturbing forces of bound stars on the hole have been neglected; and (2) close encounters have been neglected. This paper is an investigation of the validity of these assumptions.

We find that the perturbations of the hole due to bound stars do not affect the relaxation rate above statistical noise in a cusp of $\sim$ 100 stars or less. Their contribution to the overall displacement of the hole from the cluster center is undetectable. These results justify the neglect of bound star perturbations in previous studies.

We find that close encounters have only a small effect on the density distribution in the cusp (fractional change $\sim [\ln \Lambda]^{-1}$). However, the rate of ejection of cusp stars from the cluster core is larger than the rate of consumption of cusp stars; a star entering the cusp is more likely to end up in the halo than the hole. Ejection is also an important source of kinetic energy for the cluster core and halo, though whether a hole can ever halt core collapse is still in doubt.

These simple arguments are no substitute for numerical calculations. The logical next step is to extend the formalism of Bahcall and Wolf (1976, 1977), Shapiro and Marchant (1978) or Cohn and Kulsrud (1978) to include close encounters. The possibility of such an extension has already been discussed in the latter two papers.

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APPENDIX

We present an analytic estimate of the relaxation time (9). If the number of particles with energy between ϵ and $\epsilon + d\epsilon$ is $N(\epsilon)d\epsilon$, we have

$$\frac{1}{t_r} = \int_0^\infty d\epsilon N(\epsilon)\left\langle\frac{\Delta\epsilon^2}{\Delta t}\right\rangle \bigg/ \int_0^\infty d\epsilon N(\epsilon)\epsilon^2 , \tag{A1}$$

where the diffusion coefficient $\langle \Delta\epsilon^2/\Delta t\rangle$ is given by (BW, eq. [54])

$$\left\langle\frac{\Delta\epsilon^2}{\Delta t}\right\rangle = \frac{64\pi^2}{3} G^2 m^2 \ln \Lambda \epsilon^{5/2} \int_0^\infty d\epsilon' f(\epsilon')[\max(\epsilon, \epsilon')]^{-3/2} , \tag{A2}$$

and the phase space density $f(\epsilon)$ is related to $N(\epsilon)$ by (BW, eq. [42])

$$N(\epsilon) = [2\pi^3 G^3 M^3 \epsilon^{-5/2} f(\epsilon)]^{1/2} . \tag{A3}$$

In terms of the total number of stars N we have

$$\frac{1}{t_r} = \frac{2^{13/2} m^2 N \ln \Lambda}{3\pi G M^3} \frac{\int_0^\infty d\epsilon f(\epsilon)\epsilon^{-3/2} \int_0^\epsilon f(\epsilon')d\epsilon'}{\int_0^\infty d\epsilon f(\epsilon)\epsilon^{-1/2} \int_0^\infty d\epsilon' f(\epsilon')\epsilon'^{-5/2}} . \tag{A4}$$

For the distribution function of equation (8) we find

$$\frac{1}{t_r} = 0.3784 \frac{m^2 N \ln \Lambda}{G M^3} ; \tag{A5}$$

and for $G = M = 1$, $m = \frac{1}{300}$, $N = 100$,

$$t_r = 2378/\ln \Lambda . \tag{A6}$$

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D. N. C. Lin: Board of Studies in Astronomy and Astrophysics, University of California, Santa Cruz, CA 95064

Scott Tremaine: School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540