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PULSATIONS OF MODEL RR LYRAE STARS

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ABSTRACT

Radial pulsations of models of RR Lyrae stars are studied in linearized nonadiabatic theory. Convection in the hydrogen and helium ionization zones is treated by use of a time-dependent generalization of mixing-length theory. In a series of models of constant mass, luminosity, and composition, effective temperature is varied as a parameter. It is found, in agreement with previous work, that models in the instability strip are pulsationally unstable. Cooler models, however, are stabilized by the convection. Very cool models with deep convective envelopes are nearly neutrally stable to pulsation.

Subject headings: convection — stars: interiors — stars: pulsation — stars: RR Lyrae

I. INTRODUCTION

Many pulsating variable stars have relatively low surface temperatures, placing them on the right-hand side of the Hertzsprung-Russell diagram. It has been clear for many years that these stars have extensive surface convection zones and that, since the usual seat of pulsational instability lies in the hydrogen and helium ionization regions where convection is also strongest, some way of modeling the interaction between pulsation and convection is needed.

In linear pulsation theory it has been usual either to ignore convection entirely (e.g., Cox 1963; Castor 1971) or to compute the underlying equilibrium model using some form of mixing-length theory, and then to assume that the fraction of the total energy flux that in the equilibrium model is transported by convection does not interact at all with the pulsation (Baker and Kippenhahn 1965; Iben 1971). In nonlinear pulsation theory convection has often been ignored (Christy 1966; Stobie 1969). Various simple recipes for taking some account of convection have been proposed (e.g., Cox *et al.* 1966) and have been used especially in models of long-period variables (Keeley 1970, 1977; Wood 1974). A more ambitious approach is that of Deupree (1977*a*), who describes convection by large two-dimensional eddies, the motion of which is limited by a time-varying eddy viscosity.

Convection in stars is characterized by very high Rayleigh numbers and very low Prandtl numbers. This is a highly turbulent situation, very far from laboratory conditions and extremely difficult to model in a satisfactory way. Some progress has been made on the problem (Latour *et al.* 1976; Toomre *et al.* 1976), but for the most part mixing-length theory, despite its well-known inadequacies, remains the only practical tool for constructing stellar models. The way of generalizing mixing-length theory so that it might be used to model convection in a pulsating atmosphere is not unique. Somewhat different formulations have been given by Gough (1965) and by Unno (1967), and these have been compared and discussed in detail by Gough (1977, hereafter G77).

Kamijo (1967) has analyzed the pulsational stability of a single Cepheid model, using Unno's version of the theory. Baker and Gough (1967) reported the computation of a series of Cepheid models using Gough's version, but they considered their quasi-adiabatic treatment to be unreliable. Aside from this, neither Unno's formulation nor Gough's has been applied to the construction of pulsating star models. It is the purpose of this paper to report such computations: the pulsation theory has been linearized, and convection has been treated by a simplified version of Gough's formulation.

One of the first problems to be considered is the question of the cool boundary of the instability strip. The usual instability mechanisms depend upon radiative energy transport, and if convection is completely ignored, the computed instability strip is very much wider than observations indicate. Already in the work of Baker and Kippenhahn (1965) it was seen that when the convective part of the energy flux is excluded from the pulsation calculations, the destabilizing mechanism gradually shuts down as lower effective temperatures are reached. Although the coolest models were only neutral, not stable, this suggested that convection is somehow responsible for setting the cool boundary of the instability strip. The nonlinear models of Deupree (1977*b*, *c*) also show instability in the strip and, moreover, a return to stability in the cooler models when convection becomes strong.

In this investigation, we have chosen to construct a series of RR Lyrae models. The well-defined edges of the RR Lyrae gap in globular clusters provide good points for comparing the models with observation. Eventually, it may be possible to use such observations to calibrate the theory, much as the solar radius is used to calibrate the mixing-length theory on and near the main sequence.

The effects of momentum transfer by the convection (turbulent pressure) have largely been ignored. For this reason we have made no extensive exploration of the parameter space defining these models. Although such effects can be included in the theory, they lead, as noted in G77, to a significant increase in the complexity of the problem. On the basis of a one-zone model it was reported (Gough 1967) that turbulent-pressure effects may be important for pulsational stability, but a consistent calculation, within the framework of mixing-length theory, has not yet been carried out. The computational difficulties can be overcome, however, and when this is done, we shall have a clearer idea of the importance of such effects. The results at hand must therefore be regarded as preliminary, but they do allow some tentative conclusions to be drawn.

II. CONSTRUCTION OF THE EQUILIBRIUM MODELS

The equations we have used to determine the structure and stability of the RR Lyrae models are given in G77. They may be written

$$\frac{\partial}{\partial m}(p + p_t) + (3 - \Phi)\frac{p_t}{4\pi r^3 \rho} = -\frac{1}{4\pi r^2}\left(\frac{Gm}{r^2} + \frac{\partial^2 r}{\partial t^2}\right), \quad (2.1)$$

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}, \quad (2.2)$$

$$c_p \frac{\partial T}{\partial t} - \frac{\delta}{\rho}\frac{\partial p}{\partial t} = -4\pi \frac{\partial}{\partial m}[r^2(F_r + F_c)], \quad (2.3)$$

where r is distance from the center of the star; m is the mass enclosed within the sphere r = constant; t is time; p, ρ, and T are gas pressure, density, and temperature; G is the gravitational constant; $\delta = -(\partial \ln \rho / \partial \ln T)_p$ is an expansion coefficient; F_r is the radiative flux; p_t and F_c are turbulent fluxes of momentum and heat given by

$$p_t = \rho\langle w^2\rangle, \quad (2.4)$$

$$F_c = \rho c_p \langle wT'\rangle; \quad (2.5)$$

and Φ measures the anisotropy of the turbulence:

$$\Phi \equiv \frac{\langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle}{\langle w^2\rangle}. \quad (2.6)$$

Note that in this formulation the Reynolds stresses have been computed directly from the turbulent velocities derived from mixing-length theory and have not been approximated further by the introduction of a turbulent viscosity. In equations (2.4)–(2.6), $\boldsymbol{u}$ is the turbulent velocity, with radial component w; T' is the temperature fluctuation associated with the convective motion; and c_p is the specific heat at constant pressure. The angular brackets denote an average over the spherical surface r = constant. In the remainder of this paper quantities that have been averaged in this way will be called mean quantities. This averaging has also been applied to the variables p, ρ, m, c_p, δ, and F_r, though for clarity the angular brackets have been omitted in equations (2.1)–(2.5) and will be omitted below from all other mean variables.

In G77 radiative transfer was treated by the Eddington approximation. Here we have simplified the treatment still further when dealing with the mean stratification of the star by setting the horizontally averaged integrated mean intensity J equal to the mean Planck function B. Thus the mean transfer equation reduces to the diffusion approximation:

$$F_r = -K\frac{\partial T}{\partial r}, \qquad K = \frac{4acT^3}{3\chi\rho}, \quad (2.7)$$

the partial derivative being at constant t, where χ is the Rosseland mean opacity, a is the radiation density constant, and c is the speed of light. We doubt that this approximation has introduced serious errors because above the level at which the mean optical depth is unity, the divergence of the radiative flux is small; when local mixing-length theory is employed, the convective flux is small in these regions and the radiative cooling time of the atmosphere is much shorter than the pulsation period.

These equations must, of course, be supplemented by formulae for F_c and p_t, an equation of state, and a means of determining the opacity.

The equations determining the structure of the equilibrium models are obtained by setting $\partial/\partial t = 0$ in equations (2.1) and (2.3). Equation (2.3) then has the first integral:

$$4\pi r^2(F_r + F_c) = L, \quad (2.8)$$

where L is the luminosity of the star. The mixing-length theory described in § III of G77 was used to obtain the fluxes p_t and F_c, giving

$$p_t = \tfrac{1}{4}\Phi^{-1}\eta^{-2}S^{-1}[(1 + \eta^2 S)^{1/2} - 1]^2 \times \left(\frac{Gm\rho}{r^2}\right)\left(\frac{\delta}{T}\right)l^2\beta, \quad (2.9)$$

$$F_c = \tfrac{1}{4}\Phi^{-1/2}\eta^{-3}S^{-1}[(1 + \eta^2 S)^{1/2} - 1]^3\phi K\beta, \quad (2.10)$$

where l is the mixing length;

$$\eta = 2\pi^{-2}\Phi^{-3/2}(\Phi - 1) \quad (2.11)$$

is a geometrical factor of order unity;

$$\beta = -\frac{dT}{dr} + \frac{\delta}{\rho c_p}\frac{dp}{dr} \quad (2.12)$$

is the superadiabatic lapse rate;

$$\phi = \left(1 + \frac{\Phi\Sigma}{\Phi - 1}\right)^{-1}, \quad \text{where } \Sigma = \frac{1}{3}\left(\frac{\pi}{\rho\chi l}\right)^2, \quad (2.13)$$

is a factor which differs from unity only when convective elements are optically thin; and

$$S = \frac{(Gm/r^2)(\delta/T)\beta l^4}{(\phi K/\rho c_p)^2} \quad (2.14)$$

is the product of the Prandtl number and a Rayleigh number based on the mixing length. S is the square of the ratio of the radiative diffusion time across a convective eddy to the characteristic growth time of an adiabatic disturbance with length scale l. In the equation defining Σ the photon scattering coefficient has been assumed to be small compared with χ. Note that the full Eddington approximation has been used to describe the fluctuations induced in the radiation field by the convection; the approximation $J = B$ has been employed only in the mean equations. The factor Φ was determined from the equations

$$\Phi = \frac{x^2}{x^2 - 1}, \quad (2.15)$$

$$x^2 = \frac{1}{2}\frac{1 + [3 + 2(4\pi^{-4}S + 9)^{1/2}]^{1/2}}{1 - \Sigma[\pi^{-1}S^{1/4}(1 - \Sigma)^{3/4} + 1]} \quad \text{if } S < \pi^4\Sigma^{-4}(1 - \Sigma)$$

$$= \infty \quad \text{otherwise}, \quad (2.16)$$

which approximates the shape of the most rapidly growing eddy of given vertical extent.

As noted in § VI of G77, inclusion of the turbulent pressure complicates the problem significantly, because not only p_t but also its gradient appear in the hydrostatic equation. The order of the system of ordinary differential equations, which is usually three when equation (2.8) is used in place of (2.3), is increased by one when the p_t gradient is included, and the new system has singularities at both boundaries of each convection zone. To avoid these difficulties, we constructed a sequence (A) of equilibrium models in which we ignored turbulent pressure entirely. The differential system remains third order in this case, and the temperature gradient in the presence of convection is determined by an algebraic equation, which was solved by Newton-Raphson iteration. The mixing-length formulation is very similar to that of Böhm-Vitense (1958), the only difference being the relatively minor change in the way in which radiative transfer is approximated and a small change in the way of specifying the quantities Φ and η that characterize the shape of an eddy.

We have computed a second sequence (B) of equilibrium models, taking some account of the turbulent pressure p_t. Equations (2.1), (2.2), and (2.4)–(2.16) were used, with the principal exception that the gas pressure gradient in the formula for the superadiabatic temperature gradient was replaced by the gradient of total pressure, changing equation (2.12) to

$$\beta = -\frac{dT}{dr} - \frac{p}{p + p_t}\frac{Gm\delta}{r^2 c_p}. \quad (2.17)$$

By so doing the order of the differential system is artificially reduced from four to three and the singular points at the edges of convection zones are removed. This procedure is apparently similar to that adopted by Henyey, Vardya, and Bodenheimer (1965). As argued in G77, the p_t gradient may play an important role in determining the pulsational stability, even if it does not influence the structure of the static models appreciably. It should therefore be incorporated into the static models consistently. Thus the results of our series B are tentative.

The equations for the static models were integrated, as usual, from the surface inward, starting from an optical depth τ of 0.05. Each model was specified by its mass M, its luminosity L, and its effective temperature T_e. The remaining surface boundary condition was determined by assuming that the atmosphere above $\tau = 0.05$ is isothermal and in hydrostatic and radiative equilibrium, having the opacity law $\chi \propto \rho^\lambda$ that is continuous at $\tau = 0.05$. Here λ is the value of $(\partial \ln \chi/\partial \ln \rho)_T$ at $\tau = 0.05$. A predictor-corrector scheme of second-order accuracy was used. Opacities were obtained from the formula fitted by Stellingwerf (1975) to the Los Alamos "King" series of opacity tables. The equation of state included hydrogen and helium in the manner described by Baker and Kippenhahn (1962).

In the range $0.05 \leq \tau \leq 0.3$ the independent variable was τ, uniformly divided into intervals typically of magnitude 10^{-2}. In this region the variation of the mass variable m was ignored. The deepest extent $\tau = 0.3$ was so chosen to be high enough in the atmosphere to ensure that it was well clear of the convection zone. For $\tau \geq 0.3$, the common logarithm of total pressure $P = p + p_t$ was used as the independent variable. The mesh spacing was determined by restricting changes in the dependent and independent variables from one mesh point to the next to lie between certain limits, as is common in initial value integrations. The resulting spacing was normally 0.04 in $\log P$, but often became as small as 0.005, particularly in the outer part of the convection zone. Care was taken to ensure that the mesh deep in the convection zones of the cooler models was fine enough for spatial oscillations of the type illustrated in Figure 4 to be resolved adequately. The integration continued until one-half of the star's mass had been included; this required, typically, 400–500 integration steps and always included the entire convection zone. At each step a number of quantities were stored, to be used subsequently in constructing coefficients for the linearized pulsation equations.

III. LINEARIZED STABILITY COMPUTATIONS

The linearization of the equations of motion (2.1)–(2.3) and (2.7) and the boundary conditions was accomplished in the usual way (see, e.g., Baker and

Kippenhahn 1965). The only difference between the equations solved here and those solved by Baker and Kippenhahn was the inclusion of extra terms involving the turbulent fluxes. The boundary conditions were unaltered because our boundaries never lay within a convection zone. Centered difference equations of second-order accuracy were constructed to represent this system in the region $\tau \geq 0.3$, using the same mesh as that used in the equilibrium model. The difference equations were solved by Newton-Raphson iteration, iterating the eigenfunctions and eigenvalues simultaneously in a manner similar to that described by Baker, Moore, and Spiegel (1971). The procedure is essentially the same as that which had been used previously by Baker and Kippenhahn (1965), the sole difference being that in the earlier computations the frequency was not regarded as an eigenvalue: the real part was estimated from an adiabatic analysis, and the growth rate was computed subsequently from a work integral. The addition of the extra eigenvalue means, of course, that a further boundary condition is needed. This was provided in the present computation either by insisting that the displacement vanish at the lower boundary, in which case both the period and the growth rate of free oscillations were determined as eigenvalues, or by forcing the frequency to be real and setting only the real part of the displacement to zero at the lower boundary. In the latter case only the period was obtained as an eigenvalue, and the stability of the envelope was computed, as it was by Baker and Kippenhahn (1965), from the work on the lower boundary required to maintain the forced neutral oscillations. The results of the two methods differed by less than the truncation error of the difference equations.

The fluctuation in F_c (and, for series B, that in p_t) was computed according to the detailed prescription given in § V of G77. The formulae are summarized in Appendix B. Consistent with the static models, in series A the turbulent-pressure fluctuation was neglected entirely, so the resulting system of linear equations was fourth order in complex quantities. In series B turbulent pressure was included in the momentum equation but not in the perturbation of β; the perturbation in β was obtained by formally perturbing the expression (2.17). Thus once again the pulsation equations were fourth order and were consistent with the equations for the equilibrium models.

IV. RESULTS

The series of RR Lyrae models has the following parameters: $M = 0.7\, M_\odot$, $L = 44.5\, L_\odot$, composition $X/Y/Z = 0.7/0.297/0.003$, and ratio of mixing length to pressure scale height of 1.5. The range of effective temperatures covered was 5500–8000 K, in steps of 100 or 200 K, except near the red edge of the instability strip, where some additional models were computed. Only the fundamental mode of radial pulsation was considered. For each equilibrium model, the pulsation was computed in two ways, without the convective interaction and with it. The former is identical to the scheme used by Baker and Kippenhahn (1965) and provides a basis for comparison. The results are summarized in Table 1, in which are listed the computed period and the stability coefficient $-\omega_I/\omega_R$ for each model, where ω_R and ω_I are the real and imaginary parts of the frequency. The stability coefficient is defined in such a way that positive values imply instability.

We direct attention first to series A (turbulent pressure ignored). As can be seen from Table 1, convection exerts a negligible influence on the pulsation periods,

TABLE 1

PERIODS AND STABILITY COEFFICIENTS

	Series A				Series B			
	Convective Interaction Ignored		Convective Interaction Included		Convective Interaction Ignored		Convective Interaction Included	
$T_{\rm eff}$ (K)	Period (days)	$-\frac{\omega_I}{\omega_R} \times 10^3$	Period (days)	$-\frac{\omega_I}{\omega_R} \times 10^3$	Period (days)	$-\frac{\omega_I}{\omega_R} \times 10^3$	Period (days)	$-\frac{\omega_I}{\omega_R} \times 10^3$
5500	1.100	+0.004	1.100	−0.009	1.093	+0.004	1.093	−0.011
5600	1.025	+0.008	1.024	−0.015	1.016	+0.009	1.016	−0.012
5700	0.951	+0.021	0.951	−0.18	0.940	+0.027	0.940	−0.16
5800	0.880	+0.060	0.880	−0.60	0.866	+0.094	0.867	−0.85
5850	...	...	0.847	−1.21	...	...	0.834	−0.88
5900	0.814	+0.197	0.815	−0.81	0.794	+0.385	0.796	+1.26
5950	...	...	0.781	+0.72	...	...	...	...
6000	0.742	+0.587	0.743	+1.18	0.714	+0.680	0.716	+3.01
6100	0.676	+1.16	0.676	+2.28	0.661	+1.76	0.662	+3.63
6200	0.629	+1.67	0.630	+2.38	0.619	+1.81	0.620	+3.76
6400	0.557	+1.61	0.558	+2.34				
6600	0.499	+1.43	0.499	+2.23				
6800	0.449	+1.26	0.449	+2.02				
7000	0.407	+1.11	0.407	+1.63				
7200	0.370	+0.922	0.369	+1.15				
7400	0.337	+0.652	0.337	+0.72				
7600	0.308	+0.358	0.308	+0.38				
7800	0.283	−0.504	0.283	−0.51				
8000	0.260	−0.620	0.260	−0.62				

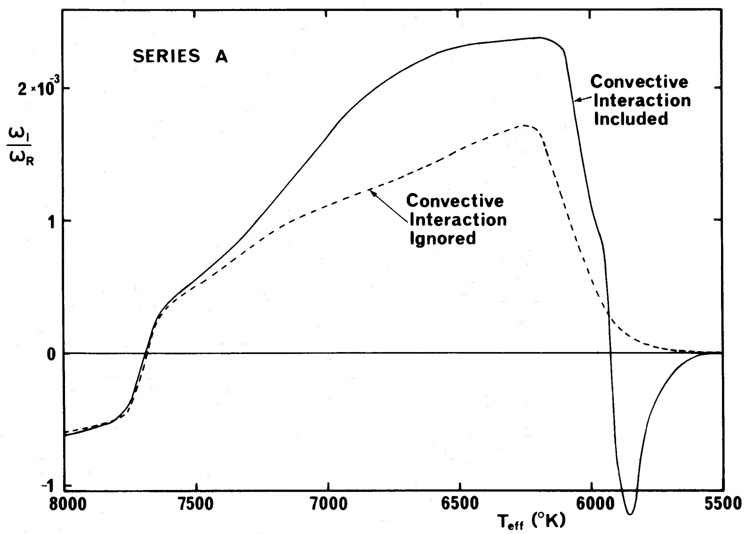

FIG. 1.—Stability coefficient $-\omega_I/\omega_R$ as a function of effective temperature for series A (turbulent pressure ignored). The dashed curve shows the results from the old theory (convective interaction neglected). The solid line (new theory) shows the stabilizing effect of convection at lower effective temperatures when the pulsation-convection interaction is included.

but it does affect the growth rates. In Figure 1 the stability coefficient is plotted against effective temperature. At the higher temperatures the stability coefficients in the two cases are nearly identical, as one would expect. At intermediate temperatures ($T_e \approx 6500$ K) the interaction with convection has added somewhat to the destabilization of models that are already quite unstable. At temperatures below about 6000 K the difference is more profound. In the series without convective interaction, the growth rate falls slowly to zero as T_e decreases from about 6200 K. As is well known, this results from the decrease in the proportion of the energy flux that is carried by radiation. When the interaction of convection with the pulsation is included, however, the decline in the growth rate is steeper and the models become stable below $T_e \approx 5900$ K; at yet lower effective temperatures the decay rate decreases until, at $T_e \approx 5500$ K, the models are nearly neutrally stable, though they are still slightly damped.

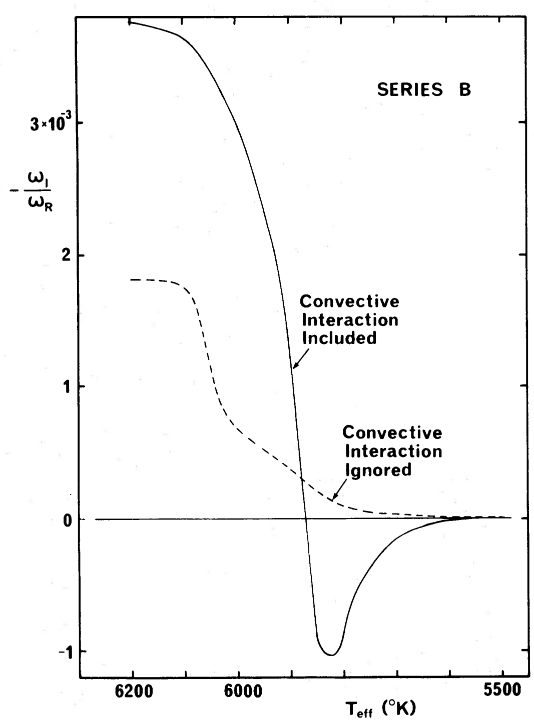

FIG. 2.—As in Fig. 1, but for series B (turbulent pressure partially included). The results are qualitatively the same as for series A. Hotter models for series B were not computed.

In Figure 2 the data are plotted for series B (turbulent pressure included). For the cooler models the results are very similar to those of series A. The hotter models have much higher growth rates when convection is included. We have not carried the series to effective temperatures higher than 6200 K because in the equilibrium models with very shallow convection zones the turbulent-pressure gradient is unbelievably large near the outer edge of the zone. When this gradient is properly taken into account by using the definition (2.12) for β instead of equation (2.17), we find that this feature of the solutions is removed; in the present series, into which the p_t gradient has not been fully incorporated, the hotter equilibrium models are surely unreliable. It is conceivable that when the full set of equations is solved, some change in both boundaries of the instability strip will be found.

The contribution of a particular layer of the star to the stability coefficient may be seen from the quantity ϖ, which is a measure of the mechanical energy contributed at a given level of the envelope (see eq. [A8] of Appendix A) when the star executes free oscillations. Regions having positive ϖ contribute to destabilization. In Figures 3 and 4 this quantity is plotted for two models of series A having effective

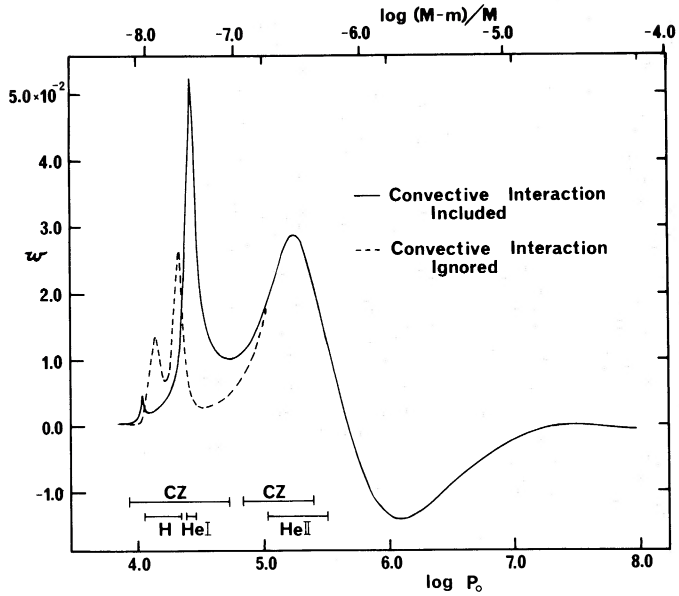

FIG. 3.—Normalized mechanical work contribution w for the model with T_e = 6400 K (series A). Solid and dashed lines have meanings similar to those in Figs. 1 and 2. Both theories predict instability. The lines marked "CZ" indicate the extents of the convection zones. The extents of the zones of ionization of H and He, from 10% to 90% ionization, are similarly indicated.

temperatures of 6400 and 5800 K. In the former there are two convection zones which are quite thin and ineffectual, but convection has already changed the magnitude and depth of two of the principal regions of excitation, these being nearer to the edges of the upper convection zone when the convective interaction is included. The net amount of destabilization, as reflected in the stability coefficient, is also different, as can be seen in Figure 1. The effect of the new theory is even more striking in the cooler models that are stabilized by convection. With the old procedure, as shown by the dashed curve in Figure 4, w is very small; indeed, it is negligible everywhere within the convection zone except near the edges, where some fraction of the energy flux is radiative. When the convective interaction is included, large-amplitude spatial oscillations in the fluctuation of the convective flux appear, and this is reflected as oscillations in w. When w is integrated over the entire envelope, the net contribution from the convection is relatively small, but definitely stabilizing.

The oscillatory behavior is typical of all models with deep convection zones. In such zones the stratification becomes nearly adiabatic and the superadiabatic

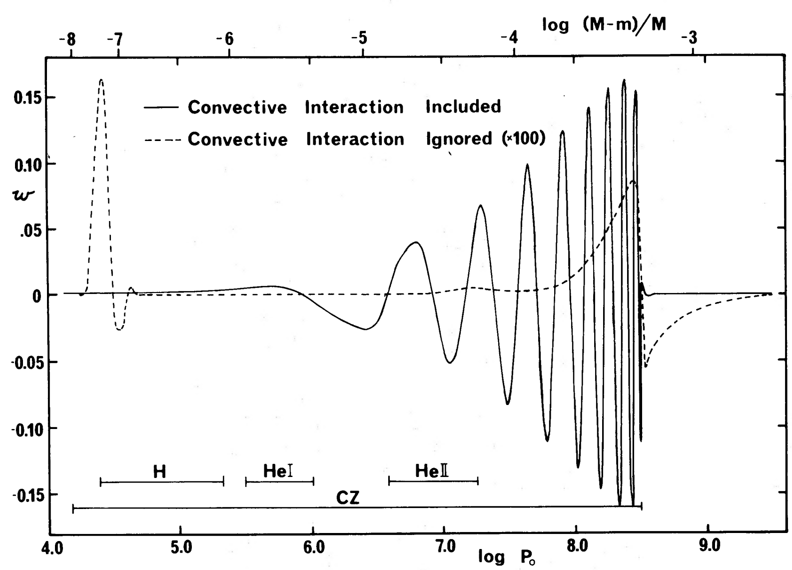

FIG. 4.—As in Fig. 3, for the model with T_e = 5800 K (series A). In the case in which the convective interaction was ignored, w has been multiplied by 100. This case is unstable; the one in which the convective interaction was included is stable.

temperature gradient is very small. In consequence the ratio of the pulsation frequency to the convective growth rate becomes large, as does the relative perturbation β_1 of the superadiabatic temperature gradient. As we show in Appendix C, the fluctuation in the convective flux is then large and out of phase with β_1, leading to spatial oscillations of short wavelength. As this wavelength can be small compared with the mixing length, the oscillations cannot represent true variations in the convective heat flux. They can arise in the computations because the local mixing-length approach used here takes into account at any point in the envelope only that convective eddy which gives the maximum contribution to the heat flux. In this respect our results are artificial. We present them because, having set up our equations to describe a clearly defined physical picture, we aimed to solve them properly and to discover the implications of the theory. One can eliminate the oscillations by artificially averaging various quantities over a number of mesh points, but this is just a numerical device and is not meaningful physically. Within the mixing-length framework a nonlocal theory that properly averages over all the eddies that contribute to the flux would smooth out most of the variation, and this is the approach that we intend to take in future work.

V. CONCLUSIONS

Although our results must be regarded as preliminary, they are in tolerable agreement with observations. The blue edge of the instability strip is, according to Figure 1, between 7600 and 7700 K, which is several hundred kelvins too hot. This boundary, however, is essentially the same for the old theory and the new, and it is well known that it can be adjusted by changing the composition parameters (see, e.g., Iben 1971; Iben and Huchra 1971). The red edge of the instability strip, occurring at an effective temperature of slightly less than 6000 K and at a period of approximately 0ᵈ75, is at about the right place (cf. discussion in Baker 1965). Since parameters like mass, luminosity, and composition can be varied within certain limits (van Albada and Baker 1971), as indeed can the mixing length, there is reason to expect that a good fit to observations can eventually be made. Our concern is that it may be too easy to fit the data and that the theory may therefore have no predictive power. Before attempting to adjust the parameters, however, we intend first to incorporate the turbulent pressure and its gradient in the manner prescribed by the theory.

With regard to the position of the red edge of the instability strip, our results are comparable to those of Deupree (1977*b*, *c*), but this is about the only point of similarity. It is not our purpose here to compare the two types of models, but it should be pointed out that not only is the treatment of pulsation different in the two cases (nonlinear in Deupree, linear here), but also the complexities of convection at high Rayleigh number are dealt with by means of fundamentally different approximations.

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APPENDIX A

WORK INTEGRALS WHEN ω IS COMPLEX

The method we have adopted to solve the linearized pulsation equations is essentially that of Baker and Kippenhahn (1965). It differs, however, in that here we have determined the imaginary part of the frequency ω_I as an eigenvalue of the problem rather than estimating it from the work integral. The frequency is thus complex. This modifies somewhat the relationship between the stability coefficient and the work integrals. We sketch here the treatment we have used.

We begin by neglecting the Reynolds stresses with the equations

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial p}{\partial m} - \frac{Gm}{r^2}, \tag{A1}$$

$$\frac{\partial v}{\partial m} = \frac{1}{\rho}, \tag{A2}$$

where for later convenience we have introduced the variable $v = 4\pi r^3/3$. These equations are linearized in the usual way. To accomplish this, we introduce Lagrangian perturbation amplitudes δr, δv, $\delta\rho$, δp, such that $r = r_0(m) + \delta r(m)e^{i\omega t}$, etc., and set $\omega = \omega_R + i\omega_I$. The linearized equations are

$$\left(\omega^2 + \frac{4Gm}{r_0^3}\right)\delta r = 4\pi r_0^2 \frac{d\delta p}{dm}, \tag{A3}$$

$$\frac{d\delta v}{dm} = -\frac{\delta\rho}{\rho_0^2}. \tag{A4}$$

We multiply equation (A3) by δr^* and use the fact that $\delta v = 4\pi r_0^2 \delta r$. The asterisk denotes complex conjugate. Substituting equation (A4) into the right-hand side of the resulting equation, we obtain

$$\left(\omega^2 + \frac{4Gm}{r_0^3}\right)|\delta r|^2 = \frac{d}{dm}(\delta v^* \delta p) + \frac{\delta \rho^* \delta p}{\rho_0^2}\,.$$

We now take the imaginary part of this expression and introduce the notation $\delta r = r_0 r_1$, $\delta v = v_0 v_1$, $\delta \rho = \rho_0 \rho_1$, $\delta p = p_0 p_1$. After some rearrangement the result is

$$4\pi \frac{\omega_I}{\omega_R}(\tfrac{1}{2}\omega_R^2 r_0^2 |r_1|^2) = -\pi \left\{\frac{d}{dm}[p_0 v_0 \operatorname{Im}(p_1^* v_1)] + \frac{p_0}{\rho_0}\operatorname{Im}(p_1^* \rho_1)\right\}\,. \tag{A5}$$

To compare this with previous work, we note that the right-hand side of equation (A5) can be written

$$-\left\{\frac{d}{dm}[4\pi^2 p_0 r_0^3 \operatorname{Im}(p_1^* r_1)] + \pi \frac{p_0}{\rho_0}\operatorname{Im}(p_1^* \rho_1)\right\}\,. \tag{A6}$$

The quantity in square brackets in the expression (A6) is just the work integral called W in Appendix C of Baker and Kippenhahn (1962) and is discussed in detail there. It is easily shown from equation (C8) of that paper that when $\omega_I = 0$ (the case considered there),

$$\frac{dW}{dm} = -\pi \frac{p_0}{\rho_0}\operatorname{Im}(p_1^* \rho_1)\,,$$

so that the right-hand side of equation (A5) vanishes when $\omega_I = 0$, as it must.

In the limit $\omega_I \ll \omega_R$ the left-hand side of equation (A5) is just the increment of kinetic energy of a unit mass of material in a period $2\pi/\omega_R$; the right-hand side comprises two terms, the first being a kind of "flux divergence" of mechanical work, the second a "source term." It can be shown that the source term is just the work done per period by a unit mass of material. Upon integrating equation (A5) from the bottom of the envelope ($m = M_B$) to the surface ($m = M$), we obtain

$$\frac{\omega_I}{\omega_R} = -\frac{[\pi p_0 v_0 \operatorname{Im}(p_1^* v_1)]_{M_B}^{M} + \int_{M_B}^{M} \pi (p_0/\rho_0)\operatorname{Im}(p_1^* \rho_1)\,dm}{4\pi \int_{M_B}^{M} \tfrac{1}{2}\omega_R^2 r_0^2 |r_1|^2\,dm}\,. \tag{A7}$$

The first term in the numerator is negligible because $p_0 p_1 \ll (GM_B^2/r_0^5) r_1$ at $m = M$ and $v_1(= 3r_1) = 0$ at $m = M_B$. Then equation (A7) is essentially equivalent to the expression given by Castor (1971).

The quantity

$$\varpi = -\left[\pi \frac{p_0}{\rho_0}\operatorname{Im}(p_1^* \rho_1)\left(m \frac{d \ln m}{d \ln p_0}\right)\right] \Big/ \left(\int_{M_B}^{M} \tfrac{1}{2}\omega_R^2 r_0^2 |r_1|^2\,dm\right) \tag{A8}$$

is that which is plotted in Figures 3 and 4 of this paper. Thus it should be true that

$$\frac{\omega_I}{\omega_R} = -\frac{1}{4\pi}\int_{\ln p_S}^{\ln p_B} \varpi\, d \ln p_0\,, \tag{A9}$$

where p_S and p_B are the equilibrium pressures at the stellar surface and at the base of the region of integration. We have verified numerically that this is indeed the case.

The above considerations apply when turbulent pressure is neglected. When $p_t \neq 0$, the formulae above must be modified by replacing p wherever it occurs by $P = p + p_t$. There will also be additional contributions from the divergence of the nonisotropic part of the Reynolds stress, which is represented by the second term on the left-hand side of equation (2.1). The net effect is that the expression (A6) becomes

$$-\frac{d}{dm}[4\pi^2 P_0 r_0^3 \operatorname{Im}(P_1^* r_1)] - \pi \frac{P_0}{\rho_0}\operatorname{Im}(P_1^* \rho_1)$$
$$+ \pi \frac{p_{t0}}{\rho_0}\{(3 - \Phi_0)[\operatorname{Im}(p_{t1} r_1^*) - \operatorname{Im}(\rho_1 r_1^*)] - \Phi_0 \operatorname{Im}(\Phi_1 r_1^*)\}\,. \tag{A10}$$

The subsequent equations (A7) and (A8) are correspondingly modified. It can be seen that the contribution of turbulent pressure to the work integral may not be negligible, particularly if p_{t1} is out of phase with r_1 and ρ_1.

APPENDIX B

LINEARIZED PERTURBATIONS OF THE CONVECTIVE FLUXES AND ANISOTROPY PARAMETER

The equations determining the Lagrangian mean perturbations to the heat flux, turbulent pressure, and anisotropy factor Φ are derived in G77 and are summarized below. If for any variable f the Lagrangian perturbation is written $\delta f = f_0(m)f_1(m)e^{i\omega t}$, where f_0 corresponds to the equilibrium model, then

$$F_{c1} = U + c_{p1} + (W_{10} + \Theta_{10})\mathscr{F} + W_{11} + \Theta_{11} + (W_{12} + \Theta_{12})\mathscr{F}\mathscr{G}, \quad \text{(B1)}$$

$$p_{t1} = U + 2(W_{10}\mathscr{F} + W_{11} + W_{12}\mathscr{F}\mathscr{G}), \quad \text{(B2)}$$

$$\Phi_1 = -\Phi_{10}\mathscr{F} + \Phi_{11}, \quad \text{(B3)}$$

where

$$U = (2r_1 - \rho_1)(\mathscr{F} - 1) + (m_1 + n_1)\mathscr{F} - (2 - i\sigma)(W_{10} - W_{12})\mathscr{F} - 2(1 + i\sigma)^{-1}(W_{11} - l_1) - W_{12}\mathscr{F}[1 + (2 - i\sigma)\mathscr{G}], \quad \text{(B4)}$$

$$\mathscr{F} = \mathscr{I}\Gamma(2 - i\sigma), \quad \text{(B5)}$$

$$\mathscr{G} = \mathscr{J}/\mathscr{I} + \psi(2 - i\sigma), \quad \text{(B6)}$$

$$\mathscr{I} = 107\{E_1[2.88(1 + i\sigma)] - 320E_1[2.88(3 + i\sigma)]\}, \quad \text{(B7)}$$

$$\mathscr{J} = i\frac{d\mathscr{J}}{d\sigma} = \frac{12}{(1 + i\sigma)(3 + i\sigma)}\left(\frac{5^{1/2}s}{2}\right)^{i\sigma}, \quad \text{(B8)}$$

where Γ, ψ, and E_1 are the gamma function, digamma function, and exponential integral of order unity, $s = 0.05$, and

$$\sigma = \frac{\omega}{\nu - \kappa} \quad \text{(B9)}$$

is the ratio of the complex pulsation frequency to the growth rate of convective elements;

$$\nu^2 = \mu^2 + \kappa^2, \qquad 2\kappa = \frac{K}{\rho c_p}k^2\phi, \qquad \mu^2 = \frac{Gm\beta\delta}{r^2T\Phi}, \quad \text{(B10)}$$

where $k^2 = \Phi(\Phi - 1)^{-1}(\pi/l)^2$ is the square of a total wavenumber characterizing a convective eddy. As in the text, the subscripts zero have been omitted from equilibrium variables.

The remaining quantities appearing in equations (B1)–(B4) are given by

$$W_{10} = g_1 + \delta_1 + \Delta_1 - \epsilon^2\kappa_{10} - (1 - \epsilon^2)\mu_{10} + \frac{2i}{\sigma}\frac{(1 + \epsilon)\mu_{11} - \epsilon(1 - i\sigma)\kappa_{11}}{2 - i\sigma(1 - \epsilon)} + 2(\Phi - 1)x_1, \quad \text{(B11)}$$

$$W_{11} = -\frac{2i}{\sigma}\frac{(1 + \epsilon)\mu_{11} - \epsilon\kappa_{11}}{2 + i\sigma(1 - \epsilon)}, \quad \text{(B12)}$$

$$W_{12} = (1 + \epsilon)\mu_{10} - \epsilon\kappa_{10}, \quad \text{(B13)}$$

$$\Theta_{10} = W_{10} + W_{12} + \Phi_{10}, \quad \text{(B14)}$$

$$\Theta_{11} = (1 + i\sigma)W_{11} + \Phi_{11} + T_1 - g_1 - \delta_1 - i\sigma\Phi^{-1}(\rho_1 + 2r_1), \quad \text{(B15)}$$

$$\Theta_{12} = W_{12}, \quad \text{(B16)}$$

$$\Phi_{10} = -2(1 - \Phi^{-1})(3r_1 + \rho_1 + \Phi x_1), \quad \text{(B17)}$$

$$\Phi_{11} = 2(1 - \Phi^{-1})(3r_1 + \rho_1), \quad \text{(B18)}$$

where

$$g_1 = -\left(2 + \frac{\omega^2r^3}{Gm}\right)r_1, \quad \text{(B19)}$$

$$2\mu_{10} = -\Phi_{10}, \quad \text{(B20)}$$

$$2\mu_{11} = \delta_p p_1 + (\delta_T - 1)T_1 - \Phi^{-1}\sigma(1+\epsilon)^{-1}[\sigma(1-\epsilon) - 2i\epsilon](\rho_1 + 2r_1)$$
$$- 2(1-\Phi^{-1})(\rho_1 + 3r_1) + \beta_1 + g_1 , \quad \text{(B21)}$$

$$\kappa_{10} = -2\phi[(2 - 3\Phi^{-1})r_1 + (1-\Phi^{-1})\rho_1 + H_1 - x_1] , \quad \text{(B22)}$$

$$\kappa_{11} = 2\phi[(2-3\Phi^{-1})r_1 - \Phi^{-1}\rho_1] + 3T_1 + (1-2\phi)\chi_1 - c_{p1} + \tfrac{1}{2}i\sigma(\epsilon^{-1} - 1)$$
$$\times [(1 - \delta + c_{pT})T_1 - C^{-1}\delta_T p_1 - g_1 - \delta_1 + 2(3 - 4\Phi^{-1})r_1 + (2 - 3\Phi^{-1})\rho_1] , \quad \text{(B23)}$$

$$\Delta_1 = l_1 + \beta_1 - T_1 + \epsilon(1-\epsilon)(\mu_{12} - \kappa_{12}) , \quad \text{(B24)}$$

$$m_1 + n_1 = \rho_1 + (1+\epsilon)\mu_{12} - \epsilon\kappa_{12} , \quad \text{(B25)}$$

$$2\mu_{12} = g_1 + \delta_1 + \beta_1 - T_1 + 2(\Phi - 1)x_1 , \quad \text{(B26)}$$

$$\kappa_{12} = 3T_1 - 2\phi(l_1 + \rho_1 - x_1) + (1 - 2\phi)\chi_1 - c_{p1} , \quad \text{(B27)}$$

and where $\epsilon = \kappa/\nu$, $C = \rho c_p T/(p\delta)$, H is the pressure scale height, and subscripts p and T denote partial logarithmic derivatives with respect to p and T at constant T and p; e.g., $f_p \equiv (\partial \ln f/\partial \ln p)_T$. The quantities β_1 and x_1 are obtained by linearizing the perturbations to equations (2.17) and (2.16), the second of equations (2.13), and equation (2.14).

As the edge of a convection zone is approached, the factors $\mathscr{F}$ and $\mathscr{F}\mathscr{G}$ at first become extremely small. In stars that are pulsationally unstable or neutral this trend continues up to the edge of the zone, where $\mathscr{F}$ and $\mathscr{F}\mathscr{G}$ vanish, but when $\omega_\mathrm{I} > 0$, the trend reverses at a point very close to the boundary, where the ratio of the lifetime of convective eddies to the pulsation period is about exp $(\pi\omega_\mathrm{R}/2\omega_\mathrm{I})$. Within the boundary layer between this point and the edge of the zone, the pulsational perturbations to the convective eddies are dominated by the initial conditions since the subsequent development of the convective motion takes place predominantly after the pulsation amplitude has decayed to comparatively low values. The lifetimes of convective eddies and the relative perturbations in the convective fluxes formally diverge at the boundaries of the convection zones, an artificial consequence of using a local convection theory. It is possible to analyze the pulsational stability of the star in this case by the method that forces the frequency to be real, but strictly speaking the free oscillations must be studied as an initial value problem. In the latter case, the lower limits of the integrals in equations (4.13), (4.15), and (4.16) of G77 become finite, and the divergence disappears. We have mimicked such an analysis by simply considering $\mathscr{F}$ and $\mathscr{F}\mathscr{G}$ to vanish in these boundary layers of pulsationally stable stars, with results which agree with the method of forced neutral oscillations.

APPENDIX C

NATURE OF THE SPATIAL OSCILLATIONS OF ϖ

The spatial oscillations in F_{c1}, and consequently in ϖ, arise whenever there is a deep convection zone in which the superadiabatic temperature gradient is small. In this limit the ratio σ of the pulsation frequency to the convective growth rate is large. Thus the response of the convective motion to the pulsation is only weakly dependent on the initial conditions and the destruction rate of the eddies, and is determined almost entirely by the equations governing the growth of the convective instability. Because convection is so efficient as to have forced the difference between the mean temperature gradient and its adiabatic value to be small, any nonadiabatic perturbation of either gradient that does not preserve the balance between the two will produce a relative perturbation of their difference that is very much larger. This leads to large-amplitude flux perturbations. The following argument, which is an extension of a discussion given by Gough (1976) concerning flux modulations in the supergranular layer of the Sun induced by long-period temporal oscillations, illustrates how such modulations are predicted to oscillate spatially when local convection theories are employed.

Deep in the adiabatic part of the convection zone, $\sigma \gg 1$ and, as our computations show, β_1 is large compared with σr_1, σp_1, and σT_1. In this region radiative losses from convective elements may be ignored. Under these circumstances, as one can readily see either by examining equations (B1) and (B4)–(B27) or more directly by approximating the equations of motion (4.1) and (4.2) of G77 (with or without the nonlinear fluctuation interaction terms), the dominant contribution to the perturbation in the convective heat flux comes from that part of the temperature fluctuation which arises from the advection of the pulsational perturbation β_1 of the mean superadiabatic temperature gradient. Thus

$$F_{c1} \approx \theta_1 \approx \frac{-i}{\omega}\left(\frac{\beta w}{\theta}\right)\beta_1 , \quad \text{(C1)}$$

where w and θ are rms convective velocity and temperature fluctuations in the equilibrium model. These amplitudes are related by

$$\theta \approx \frac{\beta l}{2} \approx \frac{\beta w}{\mu}, \tag{C2}$$

as can be seen either from equation (4.8) of G77 or from the order-of-magnitude estimates (3.13) and (3.14) of G77, where μ approximates the growth rate of the convective motion and is defined in equations (B10). Hence

$$F_{c1} \approx -\frac{i}{\sigma}\beta_1: \tag{C3}$$

the perturbed heat flux is proportional to the perturbed superadiabatic temperature gradient but lags behind by $\pi/2$. Thus heat is transported by a diffusion-like equation with an imaginary diffusivity. The characteristic exponential decay in the amplitude of spatial temperature oscillations that one normally expects from diffusion when an oscillating temperature perturbation is applied at a boundary is replaced by nondecaying spatial oscillations, whose wavelength is $2^{1/2}\pi$ times the usual skin depth. Deep in the star the thermal capacity is large, principally because the density is large, and the wavelength of the spatial oscillations is small. Its value is easy to estimate. Since it is principally the temperature fluctuation that is modified by this nonadiabatic phenomenon,

$$\delta\beta \approx \frac{\partial}{\partial r}\delta T; \tag{C4}$$

then the dominant terms in the energy equation may be crudely approximated by

$$\rho c_p \frac{\partial}{\partial t}\delta T \approx \frac{\partial}{\partial r}\delta F \approx \frac{\partial}{\partial r}\delta F_c \approx -i\frac{\partial}{\partial r}\frac{F_c}{\sigma\beta}\frac{\partial}{\partial r}\delta T; \tag{C5}$$

whence, granted that the solution oscillates rapidly,

$$\frac{\partial^2}{\partial r^2}\delta T + \left(\frac{\rho c_p \sigma\omega\beta}{F_c}\right)\delta T \approx 0, \tag{C6}$$

which has a sinusoidal solution with wavelength

$$2\pi\left(\frac{F_c}{\rho c_p \sigma\omega\beta}\right)^{1/2} \approx \frac{\pi l}{\sigma}. \tag{C7}$$

Equation (C6) is actually the leading order equation of a JWKB analysis of the full linearized pulsation equations in the limit of large σ. The value of σ increases with depth in the adiabatic part of the convection zone, and the wavelength of the spatial oscillations decreases. At about one wavelength above the base of the convection zone in the model illustrated in Figure 4, for example, $\sigma \approx 30$, and the wavelength predicted by the expression (C7) is 0.16 pressure scale heights, a value which agrees well with that yielded by the numerical computation. The oscillations become more severe in the cooler models which have even deeper convection zones.

Because the superadiabatic temperature gradient is so small, the heat flux perturbation is much greater than the mean temperature perturbation: from the analysis presented above

$$|F_{c1}| \approx \frac{2T}{\beta l}|T_{1n}| \gg |T_{1n}|, \tag{C8}$$

where by T_{1n} we mean the "nonadiabatic" part of T_1 that shows oscillations. We cannot predict the actual magnitudes of F_{c1} and T_{1n} in terms of r_1 from this analysis since to do so would involve matching the rapidly oscillating part of the solution onto the smoothly varying parts elsewhere in the domain. Nevertheless, we have checked that the ratio of F_{c1} to T_{1n} implied by equation (C8) agrees well with the numerical solutions, and have also isolated the terms that dominate the equations in regions where $\sigma \gg 1$ and confirmed that the analysis presented above does indeed represent the essential physics in such regions.

We emphasize that the arguments presented above arise in a quite general way from equations of motion of a convective eddy such as equations (4.1) and (4.2) of G77, and do not depend on the details of the assumptions in the mixing-length theory. Thus one should expect to encounter such spatial oscillations in F_{c1}, and in quantities such as ϖ that depend on it, using any local convection theory that takes cognizance of the dynamics of the turbulence. That is not to say that the oscillations are real. Local theories do not take into account the fact that the mean turbulent fluxes at any level are sums of contributions from eddies at different levels; when this is done, the fluxes depend on an average over a region whose thickness is comparable to the eddy scale. Thus any spatial variation on a scale much less than the mixing length must be suspect.

It is worth noting that other simple recipes for time-dependent convection also give rise to oscillations of the type we have encountered. In particular, that suggested by Cox *et al.* (1966) is given by

$$\frac{dF_c}{dt} = \frac{F_{c0} - F_c}{\tau_c}, \tag{C9}$$

where F_{c0} is the instantaneous value of the convective heat flux computed from the usual formula for the static envelope and τ_c is a time characteristic of the eddy lifetime. Cox *et al.* suggest taking

$$\tau_c = \frac{l}{w}. \tag{C10}$$

Because in the limit $\sigma \to \infty$ the perturbation of β dominates that of F_{c0}, the perturbed equation (C9) reduces approximately to

$$i\omega F_{c1} \approx \frac{3}{2\tau_c}\beta_1. \tag{C11}$$

This is identical to (C3) if $\tau_c = \frac{3}{2}\mu^{-1} \approx \frac{3}{4}l/w$. Cox *et al.* encountered no problems because in their paper they discuss only thin convective envelopes, but according to J. P. Cox (private communication) convergence difficulties were encountered when an attempt was made to compute pulsations of deep convective envelopes. Such difficulties presumably arose in part because the grid was too coarse to resolve the oscillations. Wood (1974) also reports similar difficulties with model Mira envelopes, and circumvents the problem by using an equation of the type (C9) for the convective velocities and by determining the convective temperature fluctuation from the time-independent formula. This simply removes the dominant physics from the fluctuating energy equation. Keeley (1970) presumably also encountered this difficulty because when using (C9) and (C10) he replaced w by 1 km s^{-1} whenever it fell below that value, thus artificially preventing σ from becoming large. The arbitrary averaging procedure over two mesh intervals adopted by Keeley and by Wood is insufficient to remove these difficulties when the grid is too coarse to resolve the oscillations. In subsequent computations Keeley (1977) has allowed σ to take more realistic values and has introduced a diffusion term as a mathematical device for removing the spatial oscillations. This has some similarity to the procedure adopted by Kruskal, Schwarzschild, and Härm (1977), who used the Eddington approximation to Spiegel's nonlocal mixing-length formalism (Spiegel 1963; Ulrich 1976) to remove an instability that develops if the instantaneous value of the time-independent local mixing-length formula for the convective heat flux is used in modeling time-dependent stellar envelopes.

APPENDIX D

G77 CORRIGENDA

Several misprints and omissions in the formulae quoted in G77 are corrected below.

The quantity J appearing in equations (2.10) and (2.11) is the integrated mean intensity.

In the definition (3.17), k should be replaced by K.

The coefficient of $\partial \ln T/\partial t$ in the definition (4.7) should be $1 - \delta + c_{pT}$.

In equation (4.28), τ should be replaced by T.

The upper limit of integration in equation (4.30) should be $5^{1/2}s$.

The final term on the right-hand side of equation (5.14) should be $(W_{12} + \Theta_{12})FG$.

There is a misprint in equation (5.23) which is corrected in equation (B8) of this paper.

The sign of the coefficient of r_1 in equation (B1) should be reversed.

Equation (B5) should be

$$\begin{aligned}\kappa_{11} = {} & \frac{(\chi_T - \delta)(1 - J/B)}{4 + (\chi_T - \delta)(1 - J/B)}\left[\frac{\chi_T\chi_{T1} - \delta\delta_1}{\chi_T - \delta} + \frac{J}{B - J}(4T_1 - J_1)\right] \\ & - 2k^2\frac{(2 - 3\Phi^{-1})r_1 - \Phi^{-1}\rho_1 - (\chi + \frac{1}{2}\varsigma)\chi_1/(\chi + \varsigma)}{k^2 + 3\rho^2\chi(\chi + \varsigma)} + 2(2 - 3\Phi^{-1})r_1 + 3T_1 - 2\Phi^{-1}\rho_1 - \frac{\chi\chi_1}{\chi + \varsigma} - c_{p1} \\ & + \tfrac{1}{2}i\sigma(\epsilon^{-1} - 1)[(1 - \delta + c_{pT})T_1 - C^{-1}\delta_T p_1 - g_1 - \delta_1 + 2(3 - 4\Phi^{-1})r_1 + (2 - 3\Phi^{-1})\rho_1].\end{aligned} \tag{D1}$$

Setting $\varsigma = 0$ and $J = B$ for both the mean equilibrium state and the pulsations reduces equation (D1) to equation (B23) of this paper.

There is a term missing from the coefficient of r_1 in equation (B7) of G77, and the coefficient of ρ_1 has the wrong sign. The correct equation, after some rearrangement, is equation (B21) of this paper.

The coefficient a in equation (C2) is 8 when l is proportional to distance from the boundary.

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