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NONLINEAR ASTROPHYSICAL DYNAMOS: MULTIPLE-PERIOD DYNAMO WAVE OSCILLATIONS AND LONG-TERM MODULATIONS OF THE 22 YEAR SOLAR CYCLE

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ABSTRACT

Magnetic oscillations with multiple periods are discovered as natural solutions of the dynamo equation as a direct consequence of the assumption that a time-delay mechanism is intrinsic to the feedback action of a magnetic field on the dynamo process. This phenomenon can be regarded as a hysteretic effect of a nonlinear system. Numerical experiments are performed to study the excitation characteristics of the oscillatory modulations and the internal structure and behavior of the system. The observed long-term modulations of the solar cycle with their two principal characteristics, (i) the occasional occurrence of anomalous eras of no surface activity and (ii) periodicity of the modulations, are discussed and simulated as a guiding example of multiple-period oscillations. The multiplicity of periods of the magnetic oscillations may be universal and may be found in various magnetic phenomena of other astrophysical bodies.

Subject headings: hydromagnetics — Sun: activity

I. INTRODUCTION

In this series of studies of astrophysical dynamos, the solar cycle has been regarded as a typical example of the magnetic oscillations of rotating astrophysical bodies. The basic characteristics of one 22 year cycle can now be understood in terms of the linear properties of dynamo waves in the context of the mean magnetohydrodynamics (Parker 1955*b*, 1957, 1971; Lerche and Parker 1972; Babcock 1961; Leighton 1969; Steenbeck and Krause 1969; Yoshimura 1972, 1975*a*, *b*, 1976; see also review papers by Krause 1976; Stix 1976; and Parker 1977). Recently it was found that the nonlinear properties of the waves are evident in the observed long-term behavior of the solar cycle and that these can be studied within the same context of the mean magnetohydrodynamics (Yoshimura 1978*a*).

From the earliest study of sunspot statistics, it was noted that the envelope of the solar cycle has undergone a long-term modulation (Wolf 1856, 1862, 1868; Waldmeier 1961). Wolf had already noticed its two basic properties: (i) the existence of an anomalous era of extremely low activity during the 17th–18th centuries and (ii) a periodicity in the long-term modulation after that era (the 80 year modulation). These two properties constitute the basic problems for our understanding of the long-term variations of solar activity. The first property was studied later by Spörer and Maunder and was recently confirmed by Eddy (1976*a*). The second property has been studied by many subsequent researchers using various indices of solar activity (e.g., Gleissberg 1939, 1940, 1971; Hartmann 1971; Henkel 1972; Link 1963, 1977; Waldmeier 1957) and has remained a problem. Recent reanalysis of the sunspot frequency curve using the concepts developed in the present study suggests that the period of the modulation is not 80 years but rather 55 years, which is more regular than previously thought (Yoshimura 1978*c*). We call this modulation the second-period modulation whether its period is 80 years or 55 years. Besides this modulation, other longer-time-scale modulations have also been suspected (Schove 1955, 1962; Bray 1967, 1968; Henkel 1972; Eddy 1976*b*, 1977). In particular, the analysis of ^{14}C data over 7500 years by Eddy (1977) strongly suggests that there may be a regular and periodic modulation with a period on the order of 2500 years.

In the first formulation of the nonlinear dynamo to study these long-term behaviors, the suppressive effect of the magnetic field on the fluid motions and on their dynamo action was found to be the dominant mechanism which determines the amplitude of the oscillation (Yoshimura 1978*a*). The alternative mechanism of magnetic field eruption was thought to play a minor role in the process. It was assumed then that the dynamics of the fluid motions would not change if there were no magnetic field. It was expected initially that the nonlinearity of the dynamo wave, due solely to the magnetic influence on the dynamics, could bring about long-term modulations. However, it was found that stationary oscillatory states are achieved quickly for any oscillatory solutions. This was understood as an infinitesimal tendency of the solutions to limit cycles. This concept was developed in general theories of nonlinear oscillations (e.g., Lefschetz 1957;

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Yoshimura 1978*a*, *b*). According to this, the oscillation has an infinitesimal tendency toward a cyclic orbit (limit cycle) with one period. In other words, its amplitude should not undergo any periodic modulation. Thus there is an apparent incompatibility between the theory and the observed long-term modulations. The stationarity of oscillations is a consequence of the assumption that the modification of the dynamics of the fluid motions is strictly due to the oscillating magnetic field. Hence, in order to explain the occasional occurrence of an anomalous era of low surface activity, fluctuations of the dynamics due to its own cause were assumed and the consequences of this assumption were studied (Yoshimura 1978*a*).

However, the second property of the long-term modulation, the periodicity, is difficult to explain by a similar mechanism. If this is the case, there must be some long-term periodicity in the dynamics of the flow which is caused by its own (nonmagnetic) mechanism. Although we cannot rule out this possibility at present, the possibility of periodic modulations by some magnetic mechanism should be examined thoroughly since the magnetic influence on the dynamo process is now regarded as an important factor in determining the level of the solar cycle oscillation.

The purpose of the present paper is to investigate this possibility thoroughly and to demonstrate that multiple-period modulations naturally result in oscillatory solutions if a certain mechanism, the time-delayed process, is included in the feedback mechanism. The same mechanism can provide an explanation even for the first property, the occasional occurrence of an anomalous era, when more than one periodicity appears in the modulations (§ IV).

II. THE TIME-DELAYED FEEDBACK ACTION OF THE MAGNETIC FIELD ON THE DYNAMO PROCESS

Before we proceed to describe the results of the numerical simulation, it is heuristically worthwhile to examine the feedback process in order to understand the diverse and complex aspects of the interaction between the magnetic and velocity fields. This is especially important in understanding why we need more than one delay-time parameter to represent the feedback process, as will be discussed later. The first and primary part of the interaction, the action of the velocity field on the magnetic field, is termed the dynamo process. The reverse process, the magnetic influence on the velocity field, is termed the magnetic reaction. The magnetic influence on the first and primary dynamo process is called the feedback. We should notice that there is a distinct difference between the last two concepts.

There are two aspects of the feedback process which were ignored in the previous formulation (Yoshimura 1978*a*). One is that the magnitude of the feedback process is a function not only of the magnetic field strength but also of its configuration. The other is that the presence of the magnetic field does not affect the dynamo process directly, but rather indirectly, through modifying various factors of the dynamics which determine the velocity field. First of all, the Lorentz force of the magnetic field can work on the fluid motions as an acceleration (deceleration) without any time delay. This is why the feedback process was assumed to take place instantaneously in the first formulation. The dynamo process, however, is a function of the velocity field and not of the acceleration. To provide the dependence of the dynamo process on the magnetic field in the framework of mean magnetohydrodynamics, one step of integration to get the velocity field is necessary. During this integration, which corresponds to adjustment of the velocity field to the presence of the magnetic field, some time lapse is necessary. For example, when some magnetic field is put into the system, the dynamo process can *begin* to adjust itself instantaneously, but it cannot be modulated instantly to the level at which the system is expected to be adjusted. The feedback process is in fact a time-delayed process. This mechanism could be represented by one delay-time parameter.

Second, the modification of the velocity field not only affects the dynamo process but also changes the basic thermal field since the velocity field transports heat. The convective instability of this basic thermal field is the driving force of the velocity field. Thus the modification of the thermal field by the magnetic field can eventually affect the dynamo process. The thermal adjustment of the system to the presence of the magnetic field and the velocity field adjustment associated with the modified thermal field also need some time lapse. This mechanism is also a time-delayed process, which should require a different delay-time parameter from the first one.

Moreover, the first aspect of the feedback process—that it is also a function of the magnetic configuration—can bring about time-delayed interaction between the velocity and magnetic fields. The magnetic field in the dynamo system behaves as dynamo waves propagating along isorotation surfaces (Yoshimura 1975*a*, *b*). The dynamo waves in the deep part can affect the dynamo process of the upper part after the field in the deep part propagates to the upper part. This time-delayed interaction would also require another delay-time parameter.

Thus the feedback process is not a simple function of the magnetic field in the past. It takes place through at least three channels in the dynamics of the system. Hence it should require at least three distinctive delay-time parameters. This would be important in relation to higher-order modulations of the magnetic oscillations (§ IV).

In order to study these complex phenomena, we start from a simple formulation of the dynamo process with one delay-time parameter. After we understand the implication and behavior of the simple dynamo models fully and after we reexamine the observed behavior of the Sun with concepts developed in the theory in mind, we can proceed to much finer models. An advantage of the present formulation of the dynamo problem is that it can be used as a diagnostic tool to explore the dynamics of the interior, revealing what kinds of mechanisms or parameters of the mechanisms are important in the dynamics. Even in a much finer model, some parametrization of basic processes is necessary since fluid motions and magnetic fields of greatly different scales play roles in complex phenomena in nature. We can reveal and clarify each process step by step by such a parametrization technique.

III. MULTIPLE-PERIOD NONLINEAR OSCILLATIONS

Figure 1 shows a typical example of the multiple-period dynamo-wave magnetic oscillations which were studied extensively by solving the nonlinear dynamo equation numerically. The dynamo equation governing the oscillations is the same as in Yoshimura (1972, 1975*a*, 1978*a*, *b*), except that the feedback action of the magnetic field is a function of the field intensity at an earlier time, t_d:

$$\frac{\partial \Psi}{\partial t} = \left[\frac{(1-\mu^2)}{r^2}\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial r^2}\right]\Psi + N_R R \Phi\,, \quad \text{(III-1)}$$

$$\frac{\partial \Phi}{\partial t} = \left[\frac{(1-\mu^2)}{r^2}\frac{\partial^2}{\partial \mu^2} + \frac{\partial^2}{\partial r^2}\right]\Phi + N_G G \Psi\,, \quad \text{(III-2)}$$

$$N_R = N_R{}^0 \exp\left(-a_N |\Phi(t - t_d)|_{\max}^{N_f}\right)\,, \quad \text{(III-3)}$$

$$N_G = N_G{}^0 \exp\left(-a_N |\Phi(t - t_d)|_{\max}^{N_f}\right)\,, \quad \text{(III-4)}$$

where $\Psi = Ar\cos\theta$ and $\Phi = Br\cos\theta$ represent the poloidal and toroidal fields, respectively; A is the longitudinal component of the vector potential of the general magnetic field, and B is the longitudinal component of the field itself; $(t, \phi, \mu = \sin\theta, r)$ are coordinates of the spherical system; R and $G = (1-\mu^2)[(\partial\Omega/\partial\mu)\partial/\partial r - (\partial\Omega/\partial r)\partial/\partial\mu]$ are the regeneration and generation operators describing the MHD induction processes in three-dimensional space due to the flows of the differential rotation and the nonaxisymmetric global convective waves (Yoshimura 1972); N_R and N_G represent the strength of the regeneration and generation processes, which are suppressed by the presence of the magnetic field to the expected level, corresponding to the field under consideration as described in equations (III-3) and (III-4) after delay time t_d; $|\Phi|_{\max}$ is the maximum value of the toroidal field at a fixed time as an index of the strength of the magnetic field. The nonlinear process is now described by three parameters: a_N, N_f, and t_d.

The structure of the differential rotation and the regeneration factor, which is related to the structure of global convection (Yoshimura 1972; 1976), for this case (Fig. 1) are the same as those of the standard case of a deeper upper zone (Fig. 3*b* of Yoshimura 1978*a*). The parameters which describe the structure of rotation and convection determine the behavior of the dynamo waves in the linear domain, which was studied in detail by Yoshimura (1975*a*), and the parameter a_N determines the amplitude of the nonlinear oscillation (Yoshimura 1978*a*); $a_N = 0.001$. The multiple-period behavior of the oscillation is controlled by the parameters N_f and t_d; $N_f = 5$, $t_d = 0.03$ (29 years for the case of the Sun [Yoshimura 1975*a*]). The period of the long-term modulation (second period) depends on t_d: the larger t_d is, the longer the

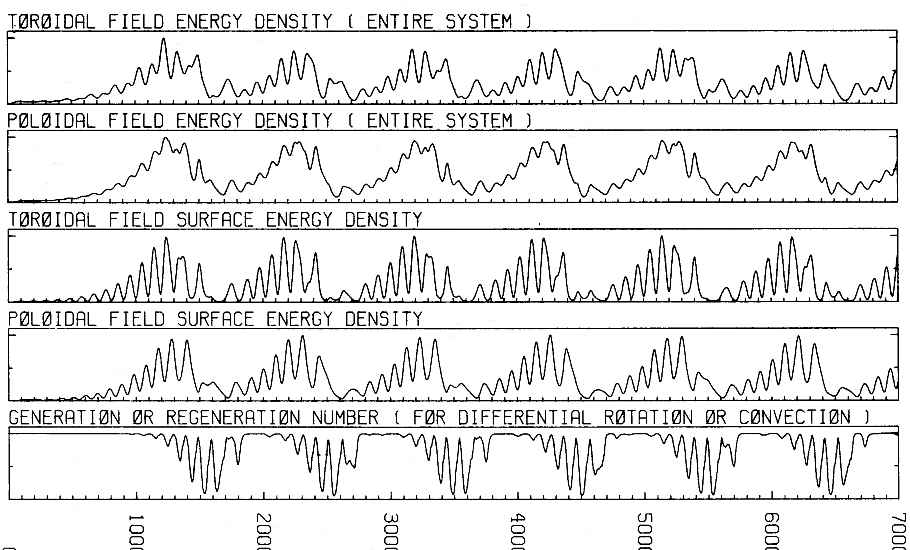

FIG. 1.—The evolution of the energy indices of the nonlinear magnetic oscillation with a second-period modulation which results from the delayed nonlinear reaction process. The basic period of the oscillation corresponds to 22 years of the solar cycle, and the second period corresponds to its 80 year modulation. The integration time span is approximately 670 years. The meanings of the energy indices described at the head of the curves are explained in Yoshimura (1978*a*). Each peak has its own spatial structure corresponding to the 11 year solar cycle (Fig. 4). The values of the nonlinear parameters are: $a_N = 0.001$, $N_f = 5$, and $t_d = 0.03$. Notice the possibility of double peaks during one (11 year) cycle around step 6300.

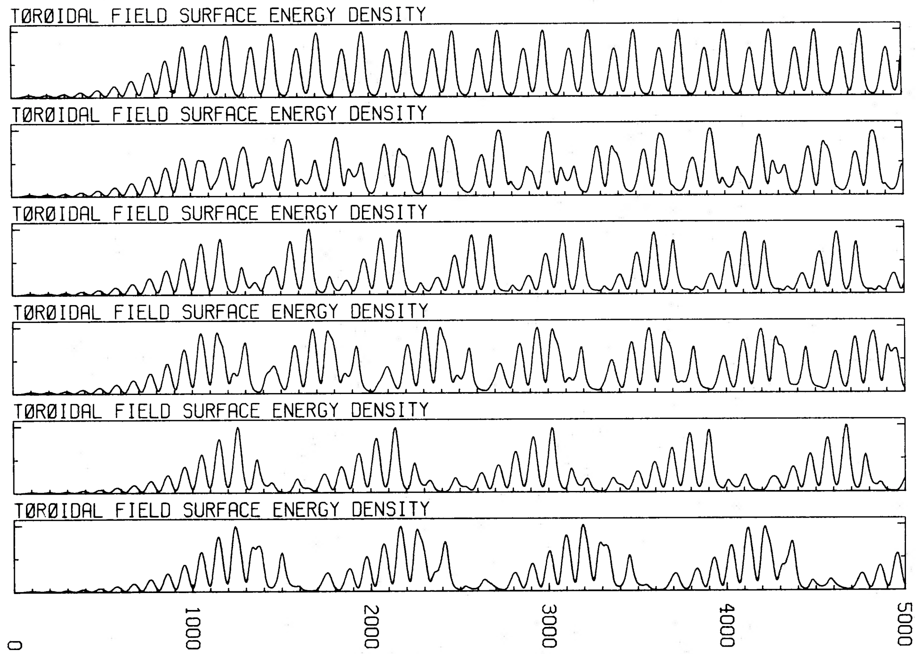

FIG. 2.—Six cases of the nonlinear magnetic oscillation with various values of the delay time parameter t_d, showing the dependence of the length of the second-period modulation on t_d. From top to bottom, $t_d = 0.005$ (5 year), 0.01 (10 year), 0.015 (15 year), 0.02 (20 year), 0.025 (25 year), and 0.03 (30 year). The larger the value of t_d, the longer the period of the modulation. The values of N_f are, from top to bottom, $N_f = 15$, 8.25, 8, 7, 6, and 5. These are chosen since the effects of N_f become strong as t_d increases. Other parameters are the same as in Fig. 1.

period (Fig. 2). If we assume a null time delay, no long-term modulation can appear. The value of $t_d = 0.03$ was chosen so that the second period is seven cycles (~80 years for the case of the Sun). If we assume a smaller value of t_d, a 55 year modulation can be reproduced (Yoshimura 1978*c*). The value of $N_f = 5$ makes the degree of modulation of the oscillation of the internal field similar to that of the observed sunspot relative number curve; the larger the value of N_f, the greater the depth of the grand minimum of the long-term modulation. However, this makes the modulation of differential rotation and global convection too large, as the lowermost curve of Figure 1 shows. This is inevitable as long as we regard the sunspot relative number curve as a true (and proportional) indicator of the magnetic field in the interior of the Sun.

Figure 3 shows a case of magnetic oscillation similar to the case of Figure 1, except that $N_f = 3$ and eruption of the magnetic field from the upper dynamo zone is taken into account (Parker 1955*a*; Yoshimura 1975*a*). As was studied by Yoshimura (1978*a*), this does not affect the behavior of the nonlinear oscillation of the internal field very much. However, the amount of erupted energy is so sensitive to a slight degree of modulation of the internal field that the degree of modulation of the erupted energy curve is more pronounced than that of the internal field. If we regard the erupted energy curve as corresponding to the sunspot frequency curve, then the frequency curve can show a conspicuous long-term modulation while the internal magnetic field does not undergo a drastic modulation. For case A of Figure 3, $B_{\rm crit} = 2.0$; for case B, $B_{\rm crit} = 1.5$, where $B_{\rm crit}$ is the critical strength of the magnetic field above which the eruption of the internal magnetic field takes place.

As case A of Figure 3 demonstrates, if $B_{\rm crit}$ is high, there can be extended eras when there is no eruption of the internal magnetic field in the form of active regions and sunspots while the internal magnetic field is undergoing ordinary dynamo wave oscillations. If $B_{\rm crit}$ is low, field eruption can take place even in the grand minima of the long-term modulation, as case B shows. If this is the case, a slight change in the structure of the convection zone, which changes the value of $B_{\rm crit}$ slightly, could be the cause of the deeper grand minimum of the 80 year modulation. We should keep in mind that this could be a partial explanation of the anomalous era of no surface activity (§ I).

In order to study how the second-period modulation takes place in the basic-period oscillation, the evolution of the general magnetic field was investigated. Figure 4 shows the evolution of the field in the interior (Fig. 4*a*) and near the surface (Fig. 4*b*) for the case of Figure 1. Notice, especially, the role of the lower dynamo zone. Substantial concentration of magnetic field takes place in some phases of the solar cycle. This phenomenon is characteristic of the nonlinear oscillation with delayed feedback process and is not seen in the oscillation without it (Fig. 8 of Yoshimura 1978*a*). The delayed feedback process

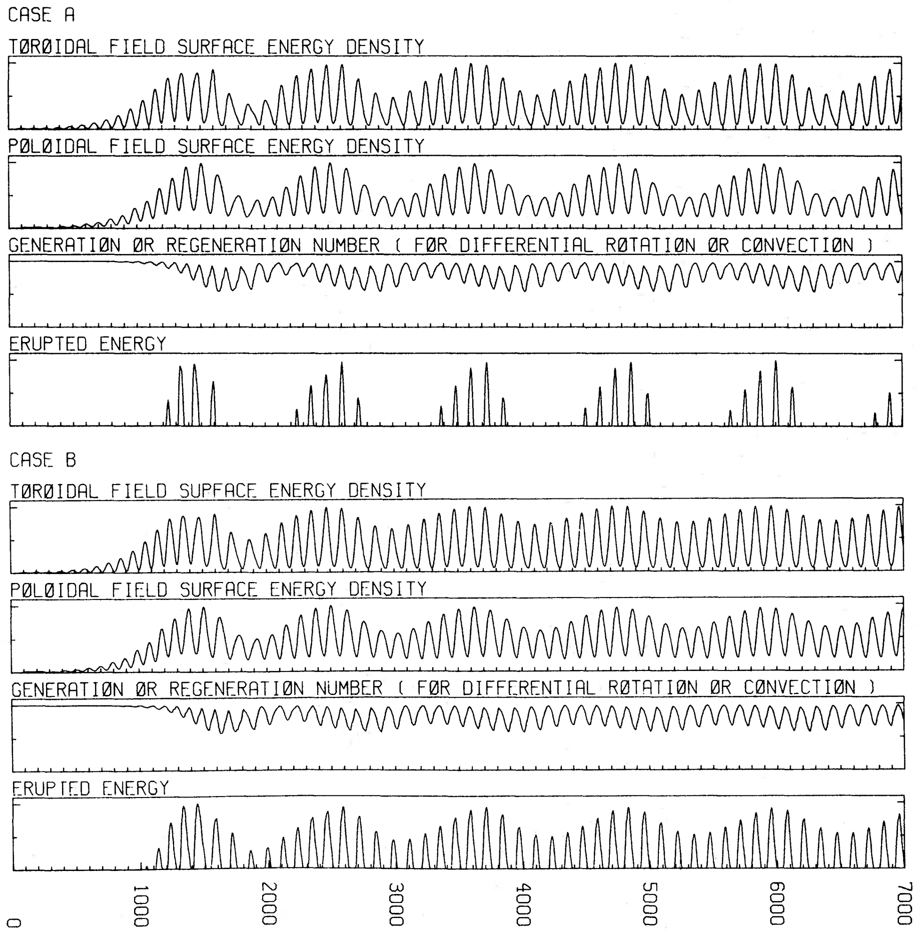

FIG. 3.—Case A (*Upper diagram*), the evolution of the indices of the nonlinear oscillation with a second-period modulation similar to Fig. 1, but with magnetic field eruption process from the upper layer ($B_{crit} = 2.0$, $E_x = 0.001$) and with $N_f = 3$. Notice that there is no surface eruption during grand minima of the 80 year modulation, although internal fields are undergoing normal oscillations corresponding to the 22 year solar cycle. Case B (*Lower diagram*), similar to case A but with $B_{crit} = 1.5$. The erupted energy index, amplifying slight changes of the internal field, shows a conspicuous second-period modulation. This should be a better index of the surface magnetic activity including sunspots.

causes an interaction between the lower and the upper zones. Since there is only one growing mode in the solution of the dynamo equation in the linear domain, these two waves could not be independent linear modes of the dynamo system. But they should be due to the delayed interaction between different parts of the same nonlinear wave trains, which have periods slightly different from each other depending on the position within the wave trains. Thus long-term modulation can be understood as a beat phenomenon between different parts of the same wave trains (the self-beat).

The structure of magnetic fields such as those shown in Figure 4 is especially important from an observational point of view. Active regions whose bipolar axes did not obey Hale's polarity rule were sometimes observed. These active regions may have originated within a deeper part, where toroidal flux tubes, progenitors of active regions, have a polarity opposite to that of the main toroidal flux tubes near the surface.

IV. EXCITATION AND THE HYSTERETIC NATURE OF THE SECOND-PERIOD MODULATION

In order to study how the second-period modulation of the nonlinear oscillation is excited, some numerical experiments were performed. In a case shown in Figure 5, the value of t_d was set at 0.03 from step 1 to step 5000 and then at 0 after step 5000 (other parameters are the same as in Fig. 1). The second-period modulation is established quickly after structural adjustment from a given initial condition. Then, immediately after step 5000, the normal nonlinear oscillation with one period takes place, reaching a limit cycle. This shows that the second-period modulation is not a transient phenomenon, characteristic of the field structural adjustment phase, but rather is a phenomenon entirely due to the time-delayed nature of the feedback process.

Figure 6 shows an opposite case; t_d was set to be 0 from step 1 to step t_s around 5000 and then to be

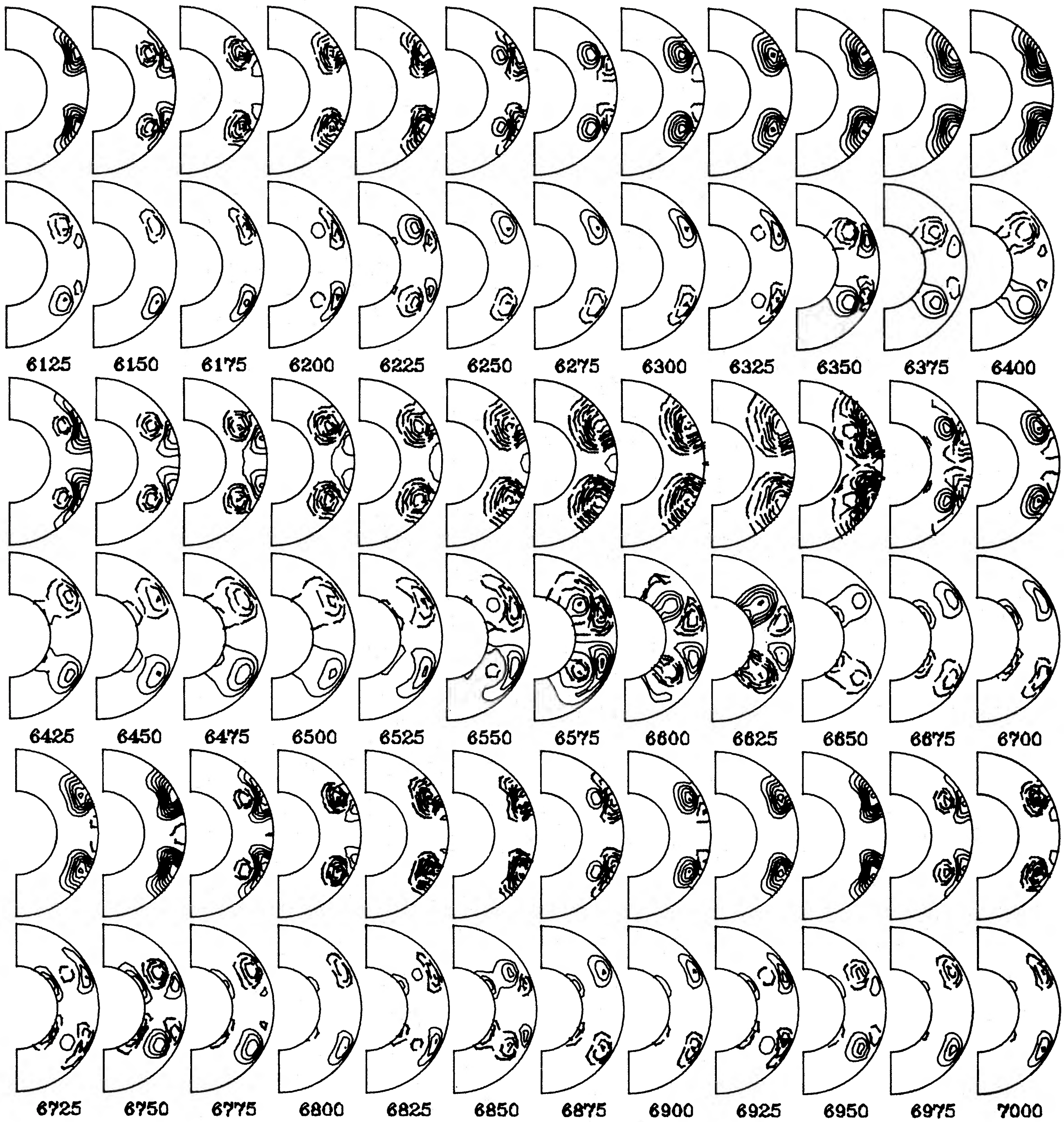

FIG. 4*a*.—The evolution of the internal magnetic field of the multiple-period nonlinear oscillation for the case of Fig. 1. The upper diagram shows field lines of the general (axisymmetric) poloidal field; the lower diagram, the contours of the general toroidal field. The structure of the differential rotation and convection (the regeneration factor) driving the dynamo for this case is the same as for the deeper upper zone case (Fig. 3 of Yoshimura 1978*a*).

0.03. In the initial part, normal stationary nonlinear oscillation was achieved. This part was given as a set of initial conditions for the evolution after step 5000. Then, the second-period modulation was excited. This shows that the second-period modulation is a self-exciting phenomenon. Notice an interesting phenomenon, viz., that the amplitude of the modulation can depend on the initial condition. In the lowermost diagram of Figure 6, the delayed feedback process was incorporated at step 5040. The modulation is not as large as in other cases. Figure 7 shows another extreme case of this phenomenon. As the uppermost diagram shows, this case of $t_d = 0.025$ (250 steps) has a second-period modulation if the solution is

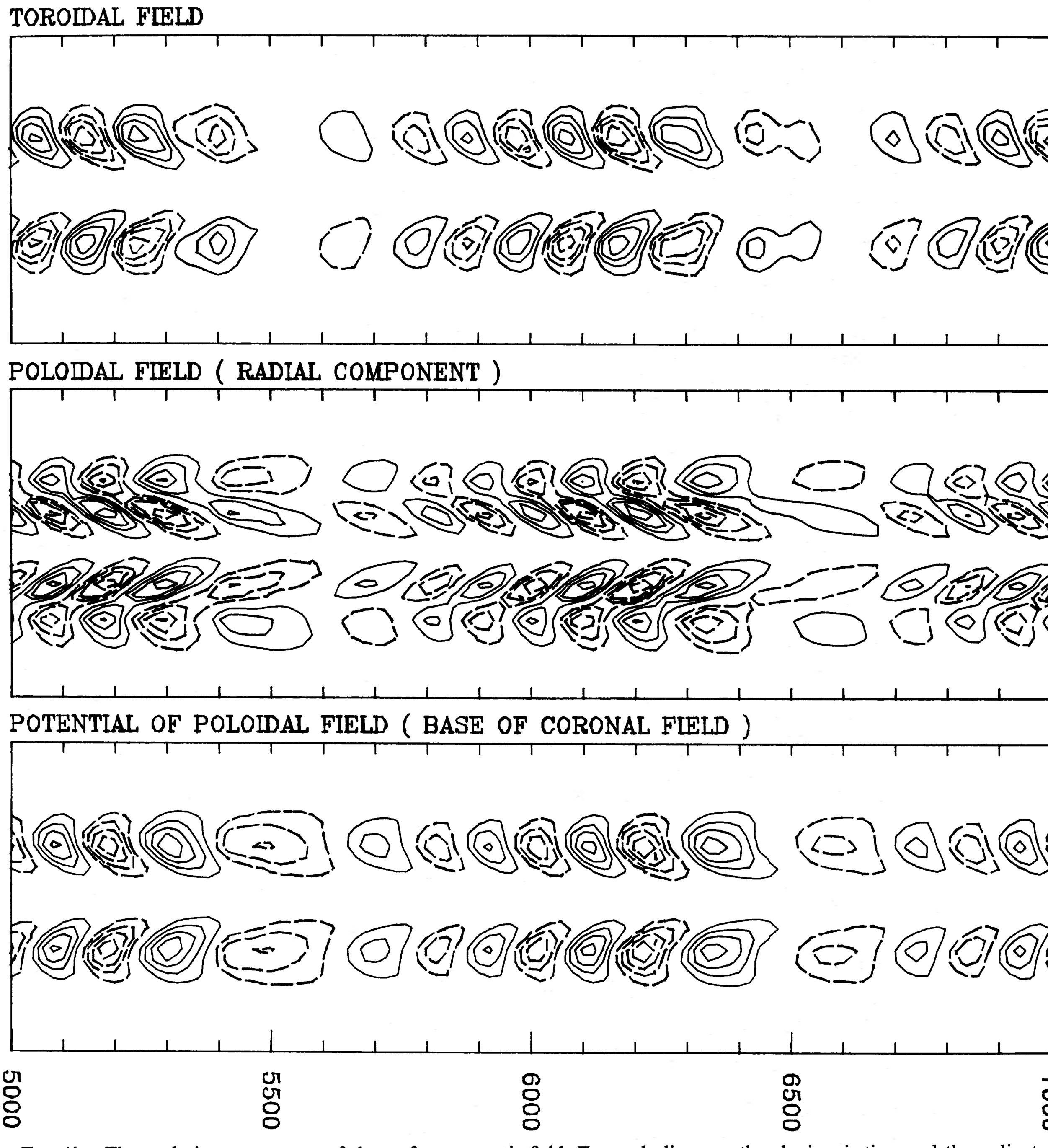

FIG. 4*b*.—The evolutionary patterns of the surface magnetic field. For each diagram, the abscissa is time and the ordinate is latitude from +90° (North Pole) to −90° (South Pole). Each peak of Fig. 1 has such structure of surface (Fig. 4*b*) and internal (Fig. 4*a*) magnetic field corresponding to the 11 year solar cycle.

integrated from a negligible level of field. However, if a series of initial conditions of stationary oscillatory state with one period is given as in Figure 6, no conspicuous second-period modulation occurs for any case of recovery time. The adjustment of the system to the two-period oscillatory state is a delicate process in this case. However, even in such a system, if the set of initial conditions is such that the amplitude of the oscillation is increasing (or decreasing) as in Figures 1, 2, and 3, a second-period modulation can be excited. In this case, it could be said that the nonlinear system has a long memory of an initial phase in which the amplitude of the oscillation is increasing (decreasing). How the system evolves after a certain epoch depends not only on the physical conditions at the epoch but also on how the system

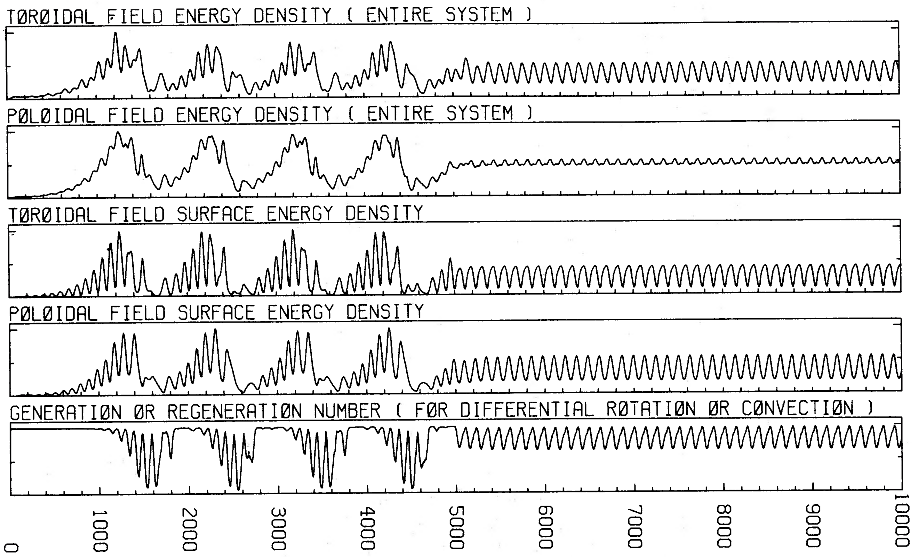

FIG. 5.—A numerical experiment of the nonlinear dynamo system with delayed feedback process showing that the second-period modulation is not a transient phenomenon but an intrinsic property of the nonlinear system. The value of t_d was set to be 0.03 from step 0 to step 5000 and to be 0 after step 5000. Other parameters are the same as in Fig. 1.

has evolved before that epoch (hysteretic effect). This is a characteristic of a system in which time-delayed interaction takes place.

V. HIGHER-ORDER MODULATIONS OF THE NONLINEAR MAGNETIC OSCILLATIONS

If the time-delayed feedback process is described by one delay-time parameter, the long-term modulation itself is periodic and has only one period no matter how long the integration of the nonlinear equation is performed. Figure 8 shows such a case of integration over a long time span. The second-period modulation, as well as the 22 year basic oscillation, is stable, and no other long-term modulation appears.

In order to investigate possibilities of third-order or higher periodic modulation, many cases of delayed

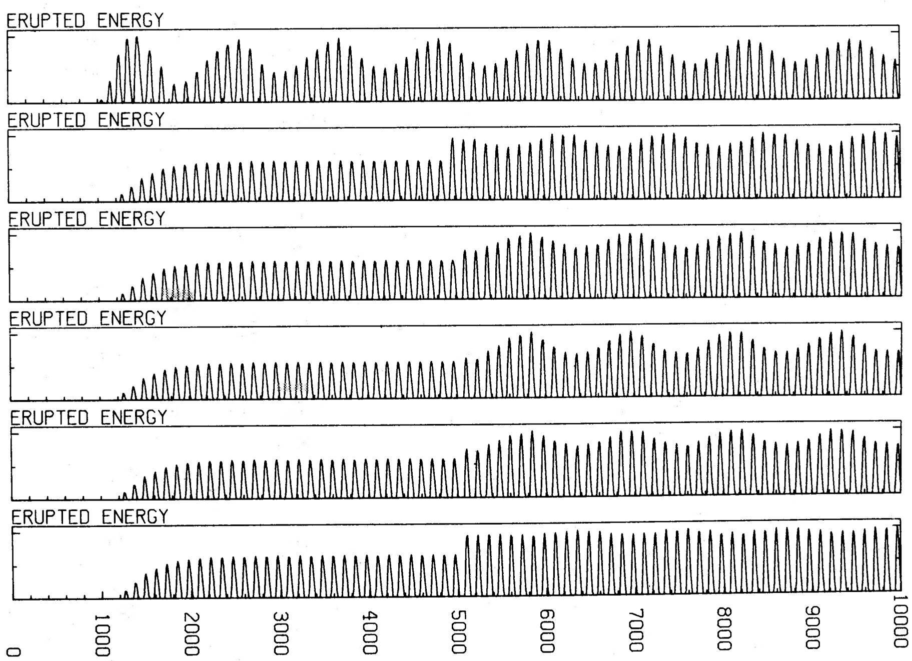

FIG. 6.—Another numerical experiment to show that the second-period modulation can be excited from a series of initial conditions of its stationary oscillation. The excitation is autonomous, and the growth rate seems to be positive. The value of t_d was set to be 0 from step 0 to step t_s around 5000 and then to be 0.03. The values of t_s are, from top to bottom, 0, 4960, 4980, 5000, 5020, and 5040. Other parameters are the same as in Fig. 3*a*.

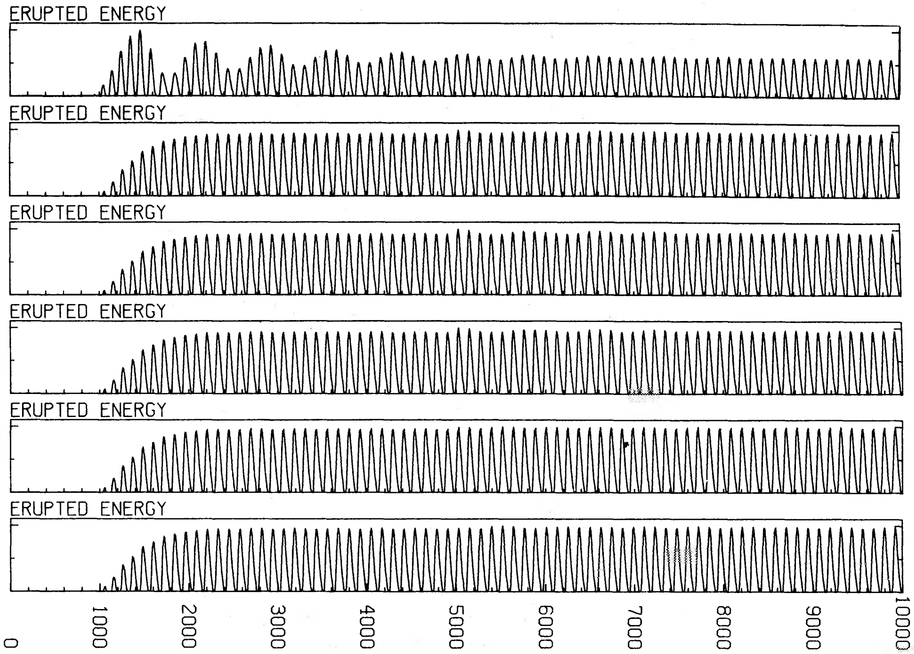

FIG. 7.—Similar to Fig. 6, but with $t_d = 0.025$. The values of t_s are, from top to bottom, 0, 4960, 4980, 5000, 5020, and 5040. The topmost diagram shows that the modulation can be excited from an initial condition of negligible field; but, once excited, it damps slowly. The excitation is a transient phenomenon, and the growth rate seems to be negative. This situation is reflected in the behavior of the system in the subsequent diagrams. No conspicuous modulation is excited when a series of initial conditions of stationary oscillation is given.

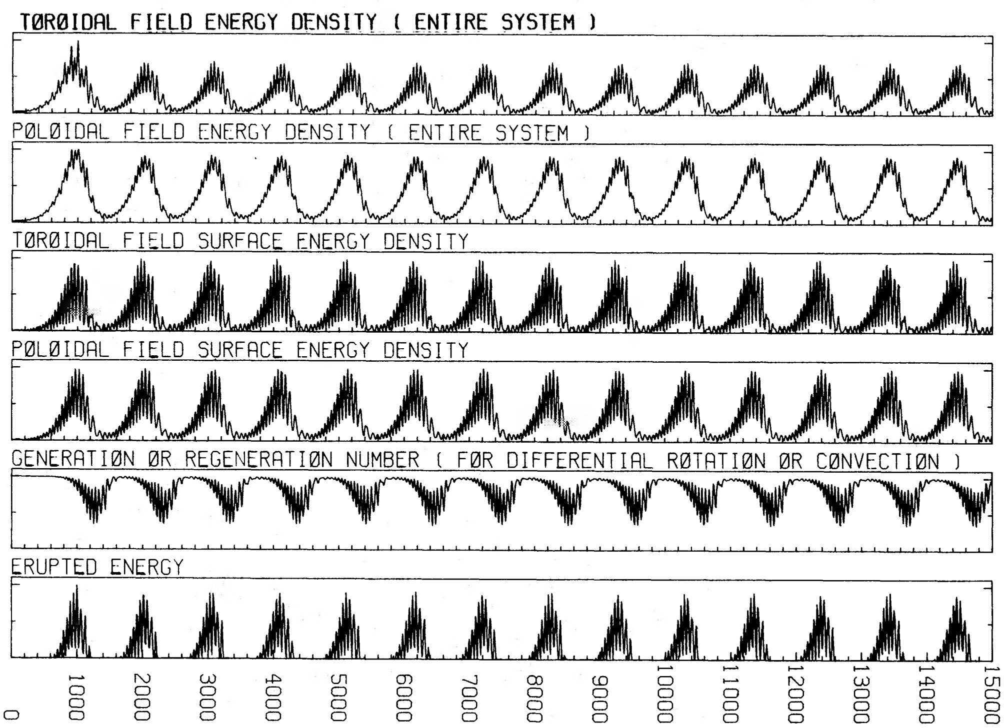

FIG. 8.—Integration of the nonlinear dynamo equation over a long time span (3000 years for the solar parameters of Fig. 3, except that the time step is now 0.0002 as in Figs. 9 and 10 and $B_{\text{crit}} = 1.25$). The second-period modulation itself is stable, and no other modulation occurs if, as in this case, the delayed feedback process is described by one time parameter.

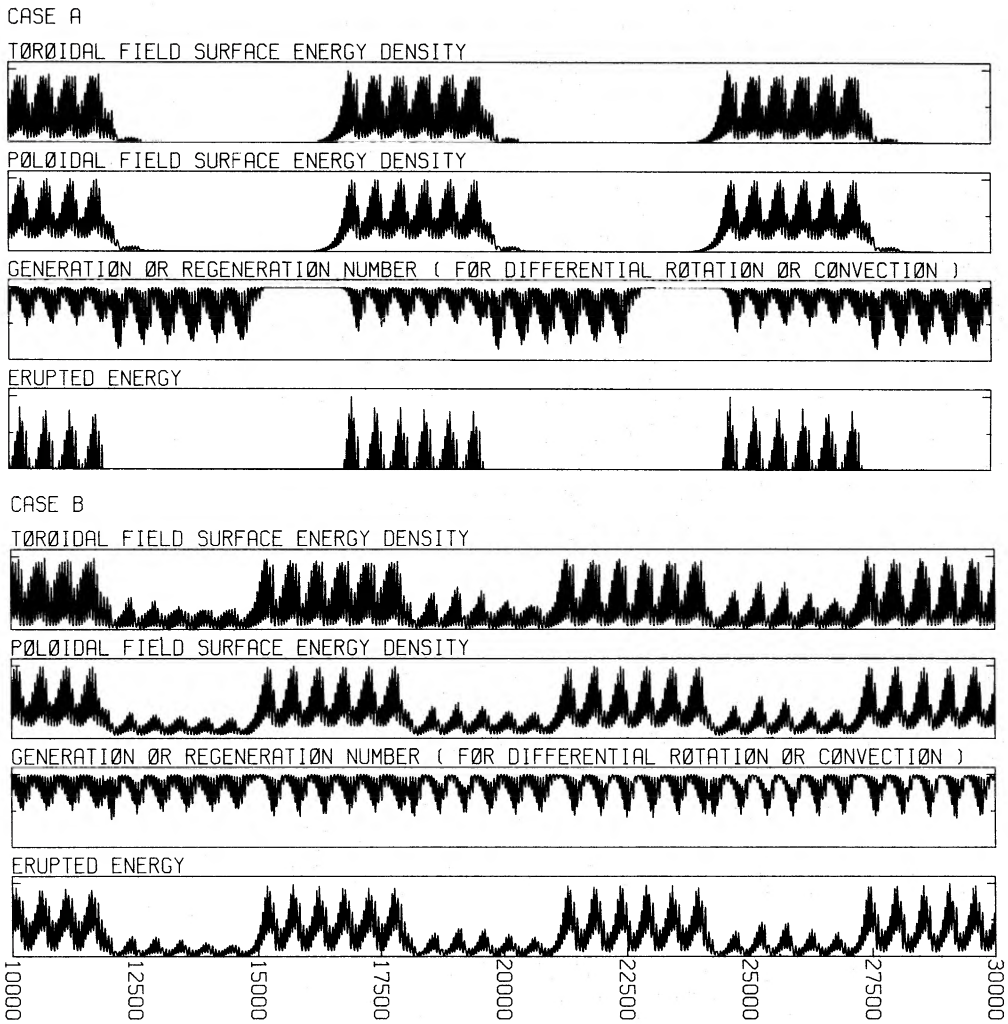

FIG. 9.—Examples of the magnetic oscillation with multiple (three) periods when two delay-time parameters represent the feedback process. A spike in the diagrams corresponds to an 11 year cycle, a mountain to an 80 year modulation, and a range of mountains to a higher-order (1000 year) modulation. Notice in case A that there are extended eras of a total absence of surface activity. The duration of a nonactive era, however, is comparable to that of an active era in the two delay-time parameter case. The values of parameters are: $a_{N1} = 0.2$, $a_{N2} = 0.1$, $N_{f1} = 5$, $N_{f2} = 5$, $t_{d1} = 3000$ steps, $t_{d2} = 150$ steps, and $\Delta t = 0.0002$. Other parameters are the same as in Fig. 8. A larger value of t_{d1} can make the period of the higher-order modulation longer. Case B is similar to case A, but with $a_{N1} = 0.1$, $B_{\rm crit} = 0.3$, and $E_x = 0.0001$. Three kinds of oscillation are visible even in a great-grand-minimum of a higher-order modulation. The rugged appearance of the curves is due to the coarse resolution of the graphic display. Internally, the evolutionary curves are more smooth.

feedback process with two and three time parameters were studied following the arguments of § II. This is an attempt to simulate the actually observed long-term modulations of the solar cycle and their two basic properties (§ I). Figure 9 shows an example of multiple-period oscillation described by two time parameters, where equations (III-3) and (III-4) are replaced by the following:

$$N_G = N_{G0} \exp\left[-\sum_{i=1}^{N} a_{N_i} |\Phi(t - t_{di})|_{\max}^{N_{fi}}\right], \quad \text{(V-1)}$$

$$N_R = N_{R0} \exp\left[-\sum_{i=1}^{N} a_{N_i} |\Phi(t - t_{di})|_{\max}^{N_{fi}}\right], \quad \text{(V-2)}$$

where $N = 2$, each spike represents an 11 year cycle, each mountain represents second-period modulation, and a succession of mountains represents third-order modulation (approximately 1200–1500 years).

One striking feature of Figure 9*a* is not only that the third-order modulation appears but that it does not show a simple periodic behavior. During a great-grand-minimum (meaning a minimum phase of the third-order modulation), the level of oscillation is suppressed to such a degree that no conspicuous oscillation is visible in the evolution of the internal fields or of the erupted fields at the surface. This is typical of a nonlinear system. Even for the basic-period oscillation, the profile of one cycle can be deformed to such a degree that no activity can be seen at the surface at its minimum phase when the nonlinear process is strong. The nonlinear process related to the third-order modulation is rather strong in this case of Figure 9*a*. When it is not so strong, as in Figure 9*b*, however, three kinds of oscillation (11 year or 22 year oscillation, 80 year modulation, 1000 year modulation) become visible. Thus, if we assume two discrete time parameters, we can reproduce nonlinear oscillations with three periods.

When we try to simulate the observed long-term modulation of solar activity (especially the occasional occurrence of an era without sunspots, like the Maunder minimum) by a nonlinear dynamo model with two delay-time parameters, we have the basic difficulty that the length of a maximum phase is comparable to that of a minimum phase. As the work of Eddy (1976*a*, 1977) suggests, the length of an anomalous era is somewhat shorter than that of a phase of high activity. In order to describe this phenomenon, we need at least three delay-time parameters. This means there must be at least three channels of time-delayed feedback process (§ II).

Case A of Figure 10 represents a category of the general behavior of the nonlinear dynamo system

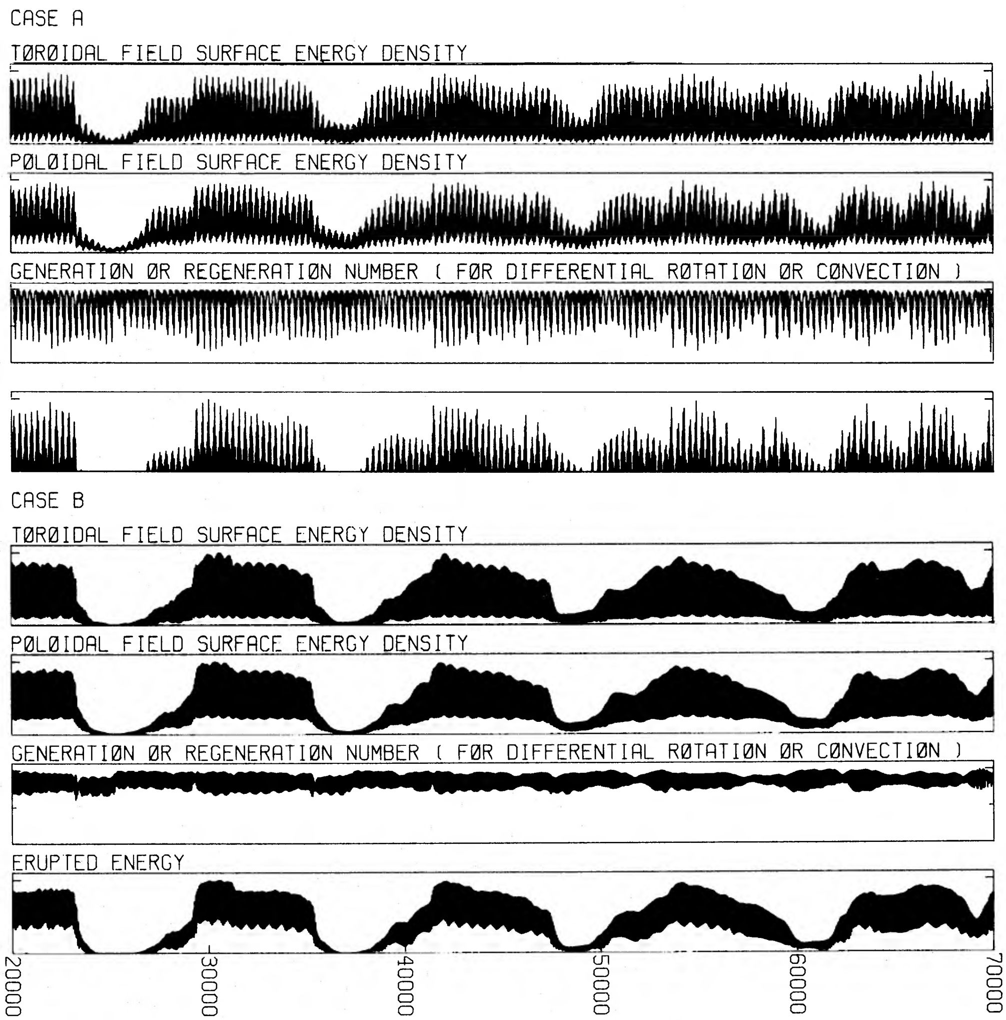

FIG. 10.—Multiple-period magnetic oscillations when three delay-time parameters represent the feedback process. These are examples of numerical simulations of the actually observed long-term modulations of the solar cycle. In order to describe the phenomenon that the nonactive phase is shorter than the active phase and an especially anomalous era like the 17th century (Maunder minimum), we need at least three delay-time parameters corresponding to three channels in the time-delayed feedback process. The three delay-time parameters are: $t_{d1} = 6000$ steps (1200 years), $t_{d2} = 2000$ steps (300 years), and $t_{d3} = 100$ steps (19 years). For case A, $a_{N1} = 0.22$, $a_{N2} = 0.1$, $a_{N3} = 0.1$, $N_{f1} = 6.0$, $N_{f2} = 4.1$, $N_{f3} = 8.5$, $B_{\rm crit} = 1.0$, and $E_x = 0.0001$. For case B, $a_{N1} = 0.1$, $a_{N2} = 0.05$, $a_{N3} = 0.1$, $N_{f1} = 5.0$, $N_{f2} = 4.0$, $N_{f3} = 5$, $B_{\rm crit} = 0.3$, and $E_x = 0.0001$. Integration was done over 13,440 years (70,000 steps) and the time span displayed in the diagram is 9600 years (50,000 steps). Case A is a typical example of the cases in which the first (55 year) and third (2300 year) kinds of modulations dominate the oscillation. Each spike consists of approximately five basic-period (11 year) cycles. Indicators of the surface energy densities in the upper two diagrams are calculated at one level below the surface. Internal fields are thus shown to be in an oscillating state even in such an era without sunspots. Case B is a typical example of the cases in which the second (1000 year) and the third (2300 year) kinds of modulations dominate the oscillations beside the basic-period (11 year) cycle.

with a time-delayed feedback process described by three time parameters. The oscillation is characterized by three kinds of modulation beside the basic-period (11 year) cycle, i.e., (i) second-period modulation, on the order of 55 years; (ii) third-period modulation, on the order of 1000 years; and (iii) fourth-period modulation, on the order of 2300 years. Each spike in this diagram represents the first kind of modulation. The 11 year cycle cannot be resolved in this diagram though it is the basic component in the solution. In case A, the first (55 year) and the third (2300 year) kinds of modulation dominate the evolution of the envelope of solar activity, and the second kind (1000 year) of modulation is only slightly visible. However, the interplay of the second and third kinds of modulation determines the evolution of the envelope. When a minimum of the second kind coincides with a minimum of the third kind, the level of activity becomes so low that no magnetic activity can erupt up to the surface. Notice the minima between 20,000 steps and 40,000 steps. The anomalous era of the 17th century (Maunder minimum, Eddy 1976*a*) could well be such a case. As the cycle progresses, the depth of a minimum becomes shallow and activity can appear even in a great-grand-minimum. Another no-activity minimum era would soon appear again after 70,000 steps, as the minima of the second and third kinds of modulation coincide with each other like a beat phenomenon.

Case B of Figure 10 represents another category of the three-parameter nonlinear oscillation. Here only the second and third kinds of modulations are evident. (The width of the broad black band corresponds to the width between the maxima and minima of the basic 11 year cycle oscillation.) An important property of the nonlinear dynamo system which this diagram demonstrates is that the long-term (1000 year and 2300 year) modulations do not need any large degree of modification of the dynamo-driving differential rotation and convection (second diagram from the bottom). This is not the case for the 55 year modulation (case A). This suggests that only a slight (time-delayed) modification of thermal structure which drives the global convection (and eventually the differential rotation) in association with the oscillating magnetic field can significantly cause long-term modulations. The longer the period of the modulation, the smaller the degree of modification of the feedback process that can cause a sufficiently large modulation. In case B of Figure 10, the second kind of modulation (1000 year) is still large. Hence, the degree of modification of the dynamo numbers is still fairly large. However, other cases, in which only the third kind of modulation is evident, show a much smaller degree of modification of the feedback process.

VI. DISCUSSION

The present study has attempted to understand the long-term modulations of the solar cycle (especially the second-period modulation) as a regular and intrinsic property of the nonlinear oscillation, caused by the time-delayed feedback action of the oscillating magnetic field. If this understanding is correct, we will be able to use the present model as a diagnostic tool to study the dynamics of the interior of the Sun by examining the mechanisms responsible for this process and their time scales. The extended era of low solar activity in the 17th–18th centuries may not necessarily be a truly anomalous period, but could well be a part of the second-period modulation (Yoshimura 1975*a*; Link 1977 and Wittman 1978 recently studied it from the observational point of view). Notice especially that the length of the period is similar to that of the modulation. If the extended period is a truly anomalous era and there is no maximum or minimum of magnetic activity (or field), then a drastic change should have taken place in the dynamics of the dynamo system driving the magnetic oscillation of the solar cycle (Yoshimura 1978*a*). Although this possibility cannot, at present, be denied totally, it may be rather reasonable to regard the era as a part of the second-period modulation suppressed below the critical level by higher order modulations (Fig. 10) or by a slight modification in the dynamics of the dynamo. Since fluctuations in the dynamics of the convection system responsible for the dynamo are also common phenomena in large-scale fluid systems, it is difficult to separate the two possible causes. Both could at least partially contribute to the mechanism of the phenomenon.

With regard to the thermal effects of the long-term modulation, the suppressive effects of the magnetic field on convection should be more prominent in longer-time-scale phenomena than in 22 year (or 11 year) phenomena. In a previous study (Yoshimura 1978*a*), it was suggested that the modulation of global convection by the magnetic field, oscillating with a period of 22 years, can well result in a 22 year cyclic modulation of the solar energy output (solar constant) because the convection zone can work as a heat reservoir. At the same time, however, the convection zone can smear out or lessen the short-time-scale modulation of energy flow in the interior. Even in this case, long-time-scale modulation should remain more conspicuous (see the locus of the envelope of the 80 year modulation in Fig. 8). In this regard, it is interesting to note that modulation with a time scale longer than 11 years has been found to be more likely to have taken place (Öpik 1968). It should be stressed here, however, that the 11 year cycle modulation is not ruled out and may well actually take place, as the first part of the solar constant measurements of solar cycle No. 20 from space suggest (Fröhlich 1977; Fig. 5), though some ambiguity remains in the last part of cycle No. 20 (Willson and Hickey 1977).

The present model of the nonlinear dynamo describes the nonlinear feedback process by the three parameters a_N, N_f, and t_d (§ III). The parameter a_N is related mainly to the relative strength of the thermal driving force of the fluid motions and the Lorentz force of the magnetic field. The parameters N_f and t_d are mainly related to the configuration of the magnetic and velocity fields and to the behavior of the fields in three-dimensional space and in time. In order to deduce the values of a_N, N_f, and t_d in the interiors of the Sun and other stars and planets from the basic equations of physics, it is necessary to know the actual structure of the magnetic field. It is especially important to know whether the magnetic field lines are bundled even in the deep interior of the convection zone. Since simple integration of the equation of motion under the influence of the (global) magnetic field cannot answer this question and an understanding of other local physics concerning the state of the local magnetic field is necessary, the present approach from the opposite direction (assuming a_N, N_f, and t_d, thus assuming a nonlinear mechanism, and examining the resulting solutions to compare them with observed facts) can be an important tool for understanding the physics of the interior of the Sun, the other stars, and the planets.

One particular aspect of the nonlinear oscillation with long-term modulations which we should discuss here is the fact that the time-delayed feedback process is not a function of the magnetic field of a discrete time lag but rather a function of the magnetic field over the wide range of time in the past. In other words, the delay time t_d should be regarded as representing a contribution function of time prior to the epoch under consideration. However, in order to reproduce the 80 year modulation, for example, the function t_d must have a peak at around 29 years. (Various cases in which the function t_d has a broad profile were tested. The results, however, were not much different from the case in which t_d has a single discrete value as far as the function has a peak around 29 years.) Although it is assumed in this study that the nonlinear interaction is a function of the overall state of the magnetic field and does not depend on the spatial distribution of the field, it is, in reality, a function of the spatial distribution, so the time-delayed feedback could be an interaction of the magnetic fields of different parts (different layers) which originate from the fields generated at different periods. (See the evolution of internal structure of the field in Yoshimura 1975*a*, 1978*a*, and in Fig. 4*a*.) Thus the mechanism represented by the delay time t_d has more profound meaning in studying the magnetohydrodynamics of the deep interior of the Sun. It represents not only delayed interactions between different parts of the same wave trains (timelike delay) but also delayed interactions between different parts of the system (spacelike delay). The concept of limit cycle has been developed in theories of ordinary (nonlinear) differential equations. If a physical system, extended in space and described by a partial differential equation, behaves in unison like one entity, then the concept of limit cycle with one period should apply to it (Yoshimura 1978*a*). However, if different parts of the system interact with each other with a time delay, the concept of limit cycle with one cycle is not necessarily valid and should be modified according to the intrinsic properties of the system. Thus, the trajectory of the solution does not cross itself as it appeared to do when the trajectory was drawn in the actual multidimensional phase space which completely describes the system. In this multidimensional phase space, a different limit cycle with a much longer time scale might exist. In that limit cycle, short-time-scale oscillations may be contained as pseudoperiodic components.

When some regular oscillatory phenomena are found in astrophysical bodies such as pulsars, only the rotation and pulsation of the bodies have usually been taken into account to explain them. However, if field-generating material motions are constantly available, the magnetic oscillations can be remarkably regular and can display remarkably various behavior, as this study has demonstrated. Thus magnetic oscillations should be considered a possible candidate for a mechanism responsible for regular pulsating phenomena even in such objects as white dwarfs, pulsars, and X-ray bursters (see Figs. 8 and 9). If only some material motions are available, the magnetic oscillations can be driven easily in such rapidly rotating objects and various magnetic activities in their atmospheres can provide a regular, periodic radiating mechanism. Since the period of magnetic oscillations of small-scale, rapidly rotating objects should be much shorter than our life span, it may be more appropriate to study these objects in order to test the present theory and to understand the long-term evolution of the nonlinear wave system in our lifetime.

To summarize, the present simple parametrization of the nonlinear interaction between the magnetic field and fluid motions seems to describe some important intrinsic properties of the nonlinear oscillation resulting from the interaction. It also presents useful abstract concepts, the limit cyclic nature of the oscillation and the time-delayed feedback process, which help us to understand physical processes in nature and in more complicated models, which can describe the Sun and other rotating astrophysical bodies more closely.

This study is the answer to the question "Is it possible to understand long-term modulations of the solar cycle as a natural behavior of the nonlinear oscillation of the general magnetic field?" which was raised when the author discovered a book entitled *Studies in Nonlinear Vibration Theory*, edited by R. Courant (1946, Institute for Mathematics and Mechanics, New York University) in the library of the Department of Astronomy, University of Tokyo, in the summer of 1976. Drs. Bernard R. Durney, Robert F. Howard, John A. Eddy, Peter A. Gilman, and Edward R. Benton read the original manuscript before publication and made helpful comments on the presentation of the paper. The author wishes to thank Dr. Gordon A. Newkirk for giving him the opportunity to visit the High Altitude Observatory, where most of the computational work was done. The computational part of this work was performed on the HITAC 8800/8700 of the Computer Center of the University of Tokyo and on the CDC 7600/CRAY-1 of the National Center for Atmospheric Research.

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