

Black hole electrodynamics and the Carter tetrad

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Summary. The orthonormal frame first found by Carter (1968) is the natural frame to use when interpreting Newman–Penrose formalism outside the event horizon of a Kerr black hole. Observers at rest in this frame rotate round the hole at constant r and θ with angular velocity $d\phi/dt = a/(r^2 + a^2)$. A simple set of Maxwell equations satisfied by field components in the Carter frame is obtained. Boundary conditions for a force-free black hole magnetosphere are found. Expressions for the rate of loss of energy and angular momentum by a black hole to an electromagnetic field are derived in terms of the toroidal components of the field at the event horizon. They are evaluated for a field that is slowly rotating with angular frequency ω and is approximately uniform at radii satisfying $M \ll r \ll 1/\omega$. If $B_\perp$ is the magnitude of the component of the field perpendicular to the axis of rotation of the hole, the rates of loss of energy and angular momentum are $(2/3)\omega B_\perp^2 a M^2$ and $(2/3)B_\perp^2 a M^2$ respectively in geometrical units.

1 Introduction

Many of the most luminous objects in the Universe appear to derive their power from very compact massive bodies. Matter accreting on to these bodies goes down a deep potential well, and the energy that powers the sources is presumed to originate as gravitational energy. If the compact body is a neutron star or white dwarf, there is no difficulty in explaining how the gravitational energy gets converted into photons: these bodies possess solid surfaces at which the accreting matter has to come to rest. Kinetic energy acquired by matter during infall is converted into thermal energy and then radiated away.

It is, however, difficult to avoid the conclusion that a very massive compact body is going to end up as a black hole. Unlike neutron stars and white dwarfs, black holes do not possess solid surfaces. If matter accretes radially, only a small fraction of the potential energy goes into compressing and heating the gas, and of that only another small fraction can be radiated away before the matter disappears behind the event horizon (Shapiro 1973).

This picture changes radically if one introduces any of the complexities that are liable to be present in reality. If the accreting matter possesses angular momentum with respect to the

hole, an accretion disk (Pringle & Rees 1972; Shakura & Sunyaev 1973) is likely to be formed, converting between 6 and 42 per cent of the rest-mass energy of the accreting matter into heat (Bardeen 1970a) and enabling the heat to be radiated away before the matter plunges into the hole. If a large-scale magnetic field is present in accreting plasma, it gets amplified and slows down the accretion process. The field grows until some resistive process becomes important, and then gravitational energy gets converted into heat via electromagnetic energy (Bisnovatyi-Kogan & Ruzmaikin 1974, 1976). If both magnetic field and angular momentum are present, energy is removed from the vicinity of the hole in the form of an electromagnetic Poynting flux, which can be converted into relativistic electrons over a region of sufficient size for inverse Compton scattering to be unimportant, and this energy can come from the rotational energy of the hole as well as from the accretion disk (Blandford 1976; Blandford & Znajek 1977).

It is therefore important to study electromagnetic processes around black holes. Black holes are likely to rotate (Bardeen 1970a), but are unlikely to have enough electric charge when plasma is present to significantly affect the gravitational field (Wald 1974). We can therefore assume that the gravitational field of a hole is given by the Kerr metric. Over the last 15 yr Newman–Penrose formalism and its expanded version, Geroch–Held–Penrose formalism, have been developed to study zero rest-mass fields. These formalisms are ideally suited to metrics (like the Kerr) which are ‘algebraically special’. There is an underexploited physical interpretation of GHP in terms of orthonormal tetrads. In Section 2 we outline GHP formalism and find that the Carter orthonormal tetrad (first found by Carter in 1968) is the natural physical frame to use outside the event horizon of a Kerr hole when interpreting results obtained from GHP. We obtain a set of relatively simple Maxwell equations satisfied by Carter components of the electromagnetic field.

In Section 3 we study the behaviour of the Carter field components in the neighbourhood of the event horizon. This is used to derive boundary conditions satisfied by a black hole magnetosphere (Blandford & Znajek 1977). It is shown that a black hole behaves like an imperfectly conducting rotating body. In Section 4 we use the Carter tetrad to find the rate of loss of energy and angular momentum by a black hole to a vacuum electromagnetic fields in terms of the toroidal field components at the event horizon. The rates are explicitly calculated for a slowly rotating field that is uniform and oblique to the hole at large (but not necessarily infinite) distances from the hole.

2 Orthonormal and null tetrads around Kerr holes

NP formalism (Newman & Penrose 1962) uses null tetrads $\{l, n, m, \bar{m}\}$ satisfying

$$l \cdot n = 1 = -m \cdot \bar{m}, \quad (2.1)$$

all other contractions being zero. There is a natural correspondence between any null tetrad and the orthonormal tetrad $\{e_0, e_1, e_2, e_3\}$ given by

$$\begin{aligned} e_0 &= (l + n)/\sqrt{2}, & e_1 &= (l - n)/\sqrt{2}, \\ e_2 &= (m + \bar{m})/\sqrt{2}, & e_3 &= (m - \bar{m})/i\sqrt{2}. \end{aligned} \quad (2.2)$$

Vectors and symmetric tensors can be described by null tetrad components like $J_l = J^\alpha l_\alpha$ and $T_{lm} = T^{\alpha\beta} l_\alpha m_\beta$. Physical components in the corresponding orthonormal frame (2.2) can be immediately deduced from the null tetrad components, e.g. $J^0 = J_0 = (J_l + J_n)/\sqrt{2}$ and

$J^1 = -J_1 = -(J_l - J_n)/\sqrt{2}$. Antisymmetric tensors like the Maxwell tensor have null tetrad components

$$\begin{aligned}\phi_0 &= F_{\mu\nu} l^\mu m^\nu \\ \phi_1 &= \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu) \\ \phi_2 &= F_{\mu\nu} \bar{m}^\mu n^\nu.\end{aligned}\quad (2.3)$$

The electric and magnetic fields in the orthonormal frame are then given by

$$\begin{aligned}E_1 + iB_1 &= 2\phi_1 \\ E_2 + iB_2 &= \phi_2 - \phi_0 \\ E_3 + iB_3 &= i(\phi_2 + \phi_0).\end{aligned}\quad (2.4)$$

Using GHP formalism (Geroch, Held & Penrose 1973; Breuer 1975) it is very easy to perform a Lorentz transformation of the type known as a null rotation

$$\begin{aligned}l &\rightarrow \zeta l, & n &\rightarrow \zeta^{-1} n, \\ m &\rightarrow \exp(i\psi) m, & \bar{m} &\rightarrow \exp(-i\psi) \bar{m}.\end{aligned}\quad (2.5)$$

Here $\zeta > 0$ and ψ is real. The corresponding orthonormal tetrad is boosted in the $\mathbf{e}_1$ -direction with 3-velocity $(\zeta^2 - 1)/(\zeta^2 + 1)$ and spatially rotated about $\mathbf{e}_1$ through the angle ψ . Any quantity η is said to have type (p, q) if under the null rotation (2.5) it transforms like

$$\eta \rightarrow \chi^p \bar{\chi}^q \eta$$

where $\chi = \sqrt{\zeta} \exp(i\psi/2)$. The NP differential operators $\Delta \equiv \mathbf{n} \cdot \nabla$ etc. and the spin coefficients α, β, γ and ϵ fail to have type because derivatives of ζ and ψ appear when they are transformed. Fortunately it is always possible to combine quantities without type so that their sum does have type. Specifically, the following operators (known as derivations) can be defined to operate on quantities of type (p, q)

$$\begin{aligned}\mathfrak{D} &= D - p\epsilon - q\bar{\epsilon}, \\ \mathfrak{D}' &= D' + p\bar{\epsilon}' + q\epsilon', \\ \mathfrak{d} &= \delta - p\beta + q\bar{\beta}', \\ \mathfrak{d}' &= \delta' + p\beta' - q\bar{\beta}.\end{aligned}\quad (2.6)$$

$\mathfrak{D}$ ('thorn') has type $(1, 1)$ in the sense that $\mathfrak{D}\eta$ has type $(p+1, q+1)$. Similarly, $\mathfrak{d}$ ('edth') has type $(1, -1)$. For our purposes we shall require the Maxwell equations

$$\begin{aligned}2\pi J_l &= \mathfrak{D}\phi_1 - \mathfrak{d}'\phi_0 + \tau'\phi_0 - 2\rho\phi_1 + \kappa\phi_2, \\ 2\pi J_n &= \mathfrak{d}\phi_2 - \mathfrak{D}'\phi_1 - \tau\phi_2 + 2\rho'\phi_1 - \kappa'\phi_0, \\ 2\pi J_m &= \mathfrak{d}\phi_1 - \mathfrak{D}'\phi_0 + \rho'\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \\ 2\pi J_{\bar{m}} &= \mathfrak{D}\phi_2 - \mathfrak{d}'\phi_1 - \rho\phi_2 + 2\tau'\phi_1 - \sigma'\phi_0.\end{aligned}\quad (2.7)$$

We now apply GHP formalism to Kerr black holes. In Boyer–Lindquist coordinates with geometrical units ($c = G = 1$), the metric of a hole of mass M and angular momentum aM is

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{A \sin^2 \theta}{\Sigma} d\phi^2 \quad (2.8)$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$ and $A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$. $(a/M)^2 < 1$ for a 'realistic' hole. One can write $\Delta = (r - r_+)(r - r_-)$ where $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. r_+ is the radius of the event horizon.

The null tetrad normally used in Kerr spacetime was derived by Kinnersley (1969). Its contravariant Boyer–Lindquist components are

$$\begin{aligned} \mathbf{l} &= \left(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta} \right), \\ \mathbf{n} &= \left(\frac{r^2 + a^2}{2\Sigma}, -\frac{\Delta}{2\Sigma}, 0, \frac{a}{2\Sigma} \right), \\ \mathbf{m} &= (ia \sin \theta, 0, 1, i/\sin \theta)/\sqrt{2}(r + ia \cos \theta). \end{aligned} \quad (2.9)$$

It has been chosen so that $\mathbf{l}$ and $\mathbf{n}$ lie along the two principal repeated null directions of the Weyl tensor. The practical consequence of this is that κ , σ , $\nu = -\kappa'$ and $\lambda = -\sigma'$ all vanish (Newman & Penrose 1962). Furthermore, the length of $\mathbf{l}$ (and therefore of $\mathbf{n}$, by (2.1)) are such that $\text{Re}(\epsilon) = 0$, and $\mathbf{m}$ has been rotated so that $\text{Im}(\epsilon) = 0$.

The Kinnersley null tetrad has proved very useful for separating and solving the equations governing scalar, electromagnetic and gravitational perturbations of the Kerr geometry (see, e.g. Teukolsky & Press 1974; Breuer 1975, and references therein). However, the corresponding orthonormal tetrad suffers from two disadvantages. It is singular on the event horizon, and an observer at rest in the orthonormal tetrad has non-zero radial velocity. The latter is caused by the asymmetric normalization of $\mathbf{l}$ and $\mathbf{n}$, and means that the orthonormal tetrad is unnatural in the sense that it frequently hides interesting features of the fields.

In the next Section we shall use a null tetrad that is non-singular on the horizon to obtain boundary conditions for field components in frames that are singular, but otherwise useful. Here we obtain a null tetrad that is symmetric with respect to the principal null directions by null rotating the Kinnersley tetrad with $\xi = \sqrt{(\Delta/2\Sigma)}$. The timelike and radial orthonormal vectors are then

$$\mathbf{e}_0^c = \left(\frac{r^2 + a^2}{\sqrt{(\Sigma\Delta)}}, 0, 0, \frac{a}{\sqrt{(\Sigma\Delta)}} \right), \quad (2.10a)$$

$$\mathbf{e}_r^c = (0, \sqrt{(\Delta/\Sigma)}, 0, 0)$$

in Boyer–Lindquist components. An observer at rest in this frame travels round the hole at constant r and θ with angular velocity $a/(r^2 + a^2)$ independent of θ . His 4-velocity $\mathbf{e}_0^c$ is always timelike outside the event horizon. We choose ψ in the transformation so that $\exp(i\psi) = -\rho\sqrt{\Sigma}$, where $\rho = -1/(r - ia \cos \theta)$. Then $\mathbf{e}_\theta^c$ lies along the usual θ direction

$$\begin{aligned} \mathbf{e}_\theta^c &= (0, 0, 1/\sqrt{\Sigma}, 0), \\ \mathbf{e}_\phi^c &= \left(\frac{a \sin \theta}{\sqrt{\Sigma}}, 0, 0, \frac{1}{\sin \theta \sqrt{\Sigma}} \right). \end{aligned} \quad (2.10b)$$

The tetrad of 1-forms dual to the tetrad (2.10) was found by Carter (1968)

$$\omega_c^0 = \sqrt{(\Delta/\Sigma)}(dt - a \sin^2 \theta d\phi),$$

$$\omega_c^r = \sqrt{(\Sigma/\Delta)} dr,$$

$$\omega_c^\theta = \sqrt{\Sigma} d\theta,$$

$$\omega_c^\phi = \frac{\sin \theta}{\sqrt{\Sigma}} [(r^2 + a^2) d\phi - a dt].$$

The spin coefficients κ , λ , ν and σ are still zero, but the symmetry requirement implies that ϵ no longer vanishes. Components of tensors in the symmetric null tetrad or Carter orthonormal tetrad will be labelled with a c .

The physical significance of the Carter frame is that observers at rest in it see principal null congruence photons* moving with purely radial velocities. This leads one to suppose that Maxwell's equations should be relatively simple in the Carter frame. It is shown below that this is indeed the case. Carter observers are *not* 'locally non-rotating' (LNR) observers (Bardeen 1970b) whose angular momentum with respect to infinity is zero and whose 4-velocities are orthogonal to the hypersurfaces $t = \text{const}$. However, the Boyer–Lindquist components of the Carter tetrad are simpler than the corresponding components of the LNR tetrad because the former do not contain the function A and because the angular velocity of the LNR observers depends on θ . The Riemann tensor also takes a simpler form in the Carter frame (Carter 1973) than in the LNR frame (Bardeen 1970b). This is not algebraically fortuitous, but is because the Kerr metric is of Petrov type D, i.e. it possesses the two principal repeated null directions from which we derive the Carter tetrad.

If the hole has electric charge Q and magnetic charge P , the vector potential can be expressed as the 1-form

$$A = \frac{-Qr}{\sqrt{(\Sigma\Delta)}} \omega_c^0 - \frac{P \cot \theta}{\sqrt{\Sigma}} \omega_c^\phi$$

(Carter 1973), and the only non-zero field components in the Carter frame are

$$E_r^c = Q(r^2 - a^2 \cos^2 \theta)/\Sigma^2 - 2Par \cos \theta/\Sigma^2,$$

$$B_r^c = P(r^2 - a^2 \cos^2 \theta)/\Sigma^2 + 2Qar \cos \theta/\Sigma^2.$$

An observer moving with a different toroidal velocity would also detect meridional fields. General stationary vacuum axisymmetric fields have been derived by Petterson (1975) and King, Lasota & Kundt (1975), using the Kinnersley null tetrad. Such fields were found to satisfy

$$\phi_2 = -\rho^2(\Delta/2) \phi_0. \quad (2.11)$$

ϕ_0 , ϕ_1 and ϕ_2 have type $(2, 0)$, $(0, 0)$ and $(-2, 0)$ respectively, and hence

$$\phi_0^c = -\rho\sqrt{(\Delta/2)} \phi_0, \quad \phi_2^c = (-1/\rho)\sqrt{(2/\Delta)} \phi_2. \quad (2.12)$$

Transforming (2.12) to the symmetric null tetrad we find that

$$\phi_0^c + \phi_2^c = 0$$

which from (2.4) means that an observer moving with constant r and θ sees no toroidal field components when the field is stationary and axisymmetric. Given any solution ϕ_0 , ϕ_1 and ϕ_2 , the Carter field components E_k^c and B_k^c are easily obtained by transforming to the symmetric tetrad and using (2.4).

To conclude this Section we derive the Maxwell equations satisfied by E_k^c and B_k^c . Substituting into (2.7) the symmetric null tetrad and spin coefficients given in Table 1, and then

* Photons moving along null geodesics whose tangent vectors always lie along one or other of the principal null directions; these geodesics "mold themselves into" the Kerr–Newman geometry in an especially harmonious way' (Misner, Thorne & Wheeler 1973).

Table 1. In the first two columns ρ and Δ are respectively the NP spin coefficient and derivative operator. In the final three columns giving the explicit forms of the coefficients in the Kerr metric, it is convenient to use ρ and Δ as the functions $-1/(r - ia \cos \theta)$ and $r^2 - 2Mr + a^2$.

Coefficient	Definition	Type	Kinnersley tetrad	Spin coefficients	
				Symmetric tetrad	Ingoing tetrad
ρ	$m^\alpha \bar{\delta} l_\alpha$	(1, 1)	$\rho \equiv -1/(r - ia \cos \theta)$	$\rho \sqrt{\Delta/2\Sigma}$	$\rho \Delta / \sqrt{2\Sigma}$
σ	$m^\alpha \delta l_\alpha$	(3, -1)	0	0	0
τ	$m^\alpha \Delta l_\alpha$	(1, -1)	$-ia \sin \theta / \sqrt{2\Sigma}$	$ia \rho \sin \theta / \sqrt{2\Sigma}$	$-ia \rho^2 \sin \theta / \sqrt{2}$
κ	$m^\alpha D l_\alpha$	(3, 1)	0	0	0
$\lambda = -\sigma'$	$-\bar{m}^{\alpha\bar{\delta}} n_\alpha$	(-3, 1)	0	0	0
$\mu = -\rho'$	$-\bar{m}^{\alpha\delta} n_\alpha$	(-1, -1)	$\rho \Delta / 2\Sigma$	$\rho \sqrt{\Delta/2\Sigma}$	$\rho / \sqrt{2}$
$\nu = -\kappa'$	$-\bar{m}^{\alpha\Delta} n_\alpha$	(-3, -1)	0	0	0
$\pi = -\tau'$	$-\bar{m}^{\alpha D} n_\alpha$	(-1, 1)	$ia \rho^2 \sin \theta / \sqrt{2}$	$-ia \rho \sin \theta / \sqrt{2\Sigma}$	$ia \sin \theta / \sqrt{2\Sigma}$
ϵ	$\frac{1}{2}(n^\alpha D l_\alpha - \bar{m}^\alpha D m_\alpha)$	-	0	$(\rho \Delta + r - M)/2\sqrt{2\Delta\Sigma}$	$(\rho \Delta + r - M)/\sqrt{2\Sigma}$
$\gamma = -\epsilon'$	$\frac{1}{2}(n^\alpha \Delta l_\alpha - \bar{m}^\alpha \Delta m_\alpha)$	-	$(\rho \Delta + r - M)/2\Sigma$	$(\rho \Delta + r - M)/2\sqrt{2\Delta\Sigma}$	0
β	$\frac{1}{2}(n^\alpha \delta l_\alpha - \bar{m}^{\alpha\delta} m_\alpha)$	-	$-\bar{\rho} \cot \theta / 2\sqrt{2}$	$-\frac{\rho(r \cos \theta - ia)}{2\sqrt{2\Sigma} \sin \theta}$	$-\frac{\rho}{\sqrt{2}} \left(ia \rho \sin \theta + \frac{\cot \theta}{2} \right)$
$\alpha = -\beta'$	$\frac{1}{2}(n^{\alpha\bar{\delta}} l_\alpha - \bar{m}^{\alpha\bar{\delta}} m_\alpha)$	-	$\frac{\rho}{\sqrt{2}} \left(\frac{\cot \theta}{2} + ia \rho \sin \theta \right)$	$\frac{\rho(r \cos \theta - ia)}{2\sqrt{2\Sigma} \sin \theta}$	$\bar{\rho} \cot \theta / 2\sqrt{2}$

separating the real and imaginary parts using (2.4), we obtain the homogeneous equations

$$0 = \frac{1}{\Sigma} \frac{\partial}{\partial r_c} (\Sigma B_r^c) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta_c} (\sin \theta B_\theta^c) + \frac{\partial}{\partial \phi_c} (B_\phi^c) + \frac{2\sqrt{\Delta}}{\Sigma^{3/2}} a \cos \theta E_r^c \quad (2.13a)$$

$$-\frac{\partial}{\partial t_c} B_r^c = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta_c} (\sin \theta E_\phi^c) - \frac{\partial}{\partial \phi_c} (E_\theta^c) \quad (2.13b)$$

$$-\frac{\partial}{\partial t_c} B_\theta^c = \frac{\partial}{\partial \phi_c} (E_r^c) - \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial r_c} (\sqrt{\Delta} E_\phi^c) \quad (2.13c)$$

$$-\frac{\partial}{\partial t_c} B_\phi^c = \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial r_c} (\sqrt{\Delta} E_\theta^c) - \frac{1}{\Sigma} \frac{\partial}{\partial \theta_c} (\Sigma E_r^c) + \frac{2ar \sin \theta}{\Sigma^{3/2}} B_r^c \quad (2.13d)$$

and the inhomogeneous equations

$$4\pi J_c^t = \frac{1}{\Sigma} \frac{\partial}{\partial r_c} (\Sigma E_r^c) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta_c} (\sin \theta E_\theta^c) + \frac{\partial}{\partial \phi_c} (E_\phi^c) - \frac{2\sqrt{\Delta}}{\Sigma^{3/2}} a \cos \theta B_r^c \quad (2.14a)$$

$$4\pi J_c^r = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta_c} (\sin \theta B_\phi^c) - \frac{\partial}{\partial \phi_c} (B_\theta^c) - \frac{\partial}{\partial t_c} (E_r^c) \quad (2.14b)$$

$$4\pi J_c^\theta = \frac{-1}{\sqrt{\Delta}} \frac{\partial}{\partial r_c} (\sqrt{\Delta} B_\phi^c) + \frac{\partial}{\partial \phi_c} (B_r^c) - \frac{\partial}{\partial t_c} (E_\theta^c) \quad (2.14c)$$

$$4\pi J_c^\phi = \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial r_c} (\sqrt{\Delta} B_\theta^c) - \frac{1}{\Sigma} \frac{\partial}{\partial \theta_c} (\Sigma B_r^c) - \frac{\partial}{\partial t_c} (E_\phi^c) - \frac{2ar \sin \theta}{\Sigma^{3/2}} E_r^c \quad (2.14d)$$

where $\partial/\partial t_c = \mathbf{e}_0^c \cdot \nabla$, $\partial/\partial r_c = \mathbf{e}_r^c \cdot \nabla = \sqrt{\Delta/\Sigma}(\partial/\partial r)$ etc. Apart from the radial sources at the ends of equations (2.13a), (2.13d), (2.14a) and (2.14d), all the terms correspond directly to terms in the flat space Maxwell equations. The new terms result in magnetic fields generating electric fields and vice versa: this is the origin of the field-mixing noted by Wald (1974) and Petterson (1975).

3 The event horizon and the force-free magnetosphere

All that is required of physical fields in the neighbourhood of an event horizon is that their components be finite in any well-behaved frame. The latter can be obtained from a null tetrad whose vectors are finite and non-zero on the event horizon; this is verified by using a non-singular coordinate system, e.g. Kerr coordinates. An appropriate null tetrad can be obtained by null-rotating the Kinnersley or symmetric tetrads by an amount that becomes suitably infinite on the horizon (*cf.* Teukolsky 1973). We choose to transform away from the symmetric tetrad with $\zeta = \sqrt{(\Delta/\Sigma)}$ and $\exp(i\psi) = -\rho\sqrt{\Sigma}$. This results in a null tetrad that is non-singular on the horizon, with the corresponding orthonormal tetrad stationary at infinity, and with $\epsilon_i' = 0$. (Components in this ingoing system are labelled with an i.)

Null tetrad components in the ingoing frame must be finite. From the relations

$$J_l^i = \sqrt{(\Delta/\Sigma)} J_l^c, \quad J_n^i = \sqrt{(\Sigma/\Delta)} J_n^c, \quad J_m^i = -\rho\sqrt{\Sigma} J_m^c \quad (3.1)$$

for the current and

$$\phi_0^i = -\rho\sqrt{\Delta}\phi_0^c, \quad \phi_1^i = \phi_1^c, \quad \phi_2^i = \frac{-1}{\rho\sqrt{\Delta}} \phi_2^c \quad (3.2)$$

for the electromagnetic field, we deduce that

$$\begin{aligned} J_c^\theta &= 0(1) = J_c^\phi, \\ J_c^t &= 0(1/\sqrt{\Delta}) = J_c^r, \\ J_c^t + J_c^r &= 0(\sqrt{\Delta}), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} E_r^c &= 0(1) = B_r^c, \\ E_\theta^c, E_\phi^c, B_\theta^c \text{ and } B_\phi^c &= 0(1/\sqrt{\Delta}), \\ E_\theta^c + B_\phi^c &= 0(\sqrt{\Delta}) = B_\theta^c - E_\phi^c, \end{aligned} \tag{3.4}$$

as we take the limit $r \rightarrow r_+$. These relations must be true everywhere on the event horizon. If we try to take the limit holding t and ϕ constant, we merely trace out a path spiralling round the event horizon into the infinite past. It is generally necessary to use non-singular coordinates. Suppose however that the field is axisymmetric and nearly stationary. If the field and the hole change over timescales much longer than the light-travel time, a strictly stationary solution is a good approximation. Such a solution extends back into the infinite past and it is possible to check its behaviour at the event horizon by taking $r \rightarrow r_+$ holding t and ϕ constant, because the field and metric are invariant under translation along the trajectories of the Killing vectors $(\partial/\partial t) = (\partial/\partial v)$ and $(\partial/\partial \phi) = (\partial/\partial \bar{\phi})$, so that there is no difference between this limit and the one we would have obtained holding the Kerr coordinates v and $\bar{\phi}$ constant.

How general are the boundary conditions (3.3) and (3.4)? Consider any system of observers rotating around the hole with constant r and θ and angular velocity $\omega(r, \theta)$. Their 3-velocity with respect to the Carter observers is

$$v_c^\phi = \frac{\sin \theta}{\sqrt{\Delta}} \cdot \frac{\omega(r^2 + a^2) - a}{1 - a\omega \sin^2 \theta}. \tag{3.5}$$

For physical observers $|v_c^\phi| < 1$, so

$$\lim_{r \rightarrow r_+} \omega = a/(r_+^2 + a^2) \equiv \Omega_H,$$

the angular velocity of the hole (Misner, Thorne & Wheeler 1973). The rate at which ω approaches Ω_H determines

$$\lim_{r \rightarrow r_+} v_c^\phi \equiv v_+^\phi.$$

If v_+^ϕ is non-zero, then the boundary conditions satisfied by fields measured by the new observers can be obtained by Lorentz-transforming (3.3) and (3.4), and are rather more complicated. Boundary conditions in the Carter frame are simple because it is possible to obtain a physical frame by giving the Carter frame a purely radial boost. This cannot be done with a frame for which $v_+^\phi \neq 0$, as the timelike orthonormal vector is pointing in the wrong direction. It happens that for LNR observers $v_c^\phi = -a \sin \theta \sqrt{\Delta}/(r^2 + a^2)$, and so field components in the LNR frame also satisfy (3.4). King, Lasota & Kundt (1975) obtained a well-behaved frame by giving the LNR frame just such a radial boost.

When the electromagnetic field is sufficiently strong and there are enough charged particles present to support the necessary space-charges and currents, a force-free magnetosphere is established. This is described in detail in Blandford & Znajek (1977). We now

derive constraints that a force-free magnetosphere must satisfy at the event horizon. In a force-free situation particle inertia is negligible, so that the electromagnetic force on each particle has to be zero. Summing over all particles,

$$F_{\mu\nu}J^\nu = 0. \quad (3.6)$$

It is convenient to express $F_{\mu\nu}$ in terms of a vector potential: $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. A_ν is required to be stationary and axisymmetric. A_0 and A_ϕ are then unique up to a constant. Equation (3.6) is a sufficient condition for magnetic field lines to lie in the surfaces of constant A_ϕ (Carter 1976, private communication).

Having obtained a solution of Maxwell's equations satisfying (3.6), it is necessary to check that J^ν can be produced by particles of opposite charge travelling slower than light with zero electric field in their rest frames. Clearly a necessary condition is that the Lorentz invariants satisfy

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad B^2 - E^2 > 0. \quad (3.7)$$

It turns out that this is also a sufficient condition (Znajek 1976). Equation (3.7) has to be strictly satisfied in the limit $r \rightarrow r_+$; the force-free solution is then consistent with particles never leaving the hole.

Equation (3.6) automatically implies that $\mathbf{E} \cdot \mathbf{B} = 0$ because

$$A_{0,r} = -\omega A_{\phi,r}, \quad A_{0,\theta} = -\omega A_{\phi,\theta}, \quad (3.8)$$

with A_0 and ω functions of A_ϕ only. Furthermore, $B_T \equiv (\Delta/\Sigma) \sin \theta \cdot F_{r\theta}$ is also a function of A_ϕ . The Carter field components are

$$\begin{aligned} E_r^c &= [(a - \omega(r^2 + a^2)) A_{\phi,r}]/\Sigma, & E_\theta^c &= [(a - \omega(r^2 + a^2)) A_{\phi,\theta}]/\sqrt{(\Delta\Sigma)}, \\ B_r^c &= [(1 - a\omega \sin^2 \theta) A_{\phi,\theta}]/(\Sigma \sin \theta), & B_\theta^c &= [-\sqrt{\Delta}(1 - a\omega \sin^2 \theta) A_{\phi,r}]/(\Sigma \sin \theta), \\ E_\phi^c &= 0, & B_\phi^c &= \frac{\sqrt{\Delta}}{\Sigma} F_{r\theta} = \frac{B_T}{\sqrt{\Delta} \sin \theta}. \end{aligned} \quad (3.9)$$

Conditions (3.4) are satisfied if and only if A_ϕ is finite at the horizon and

$$B_T(A_\phi(r_+, \theta)) = \frac{2Mr_+\{\omega - \Omega_H\}}{r_+^2 + a^2 \cos^2 \theta} \frac{\partial}{\partial \theta} A_\phi(r_+, \theta). \quad (3.10)$$

Conditions (3.3) then follow from the Maxwell equations. Given $B_T(A_\phi)$ and A_ϕ , this fixes ω on field lines crossing the horizon. Energy is extracted from the hole if $0 < \omega < \Omega_H$.

Neutron stars and accretion disks are very good conductors, so that in any locally co-moving frame there is no electric field. As tangential electric fields are continuous across surfaces, the boundary condition that replaces (3.10) for a neutron star magnetosphere (Goldreich & Julian 1969) or accretion disk magnetosphere (Blandford 1976) is that $\omega(A_\phi)$ is the material angular velocity of the conductor at the foot of the field line A_ϕ . Equation (3.10) shows that a black hole does not behave like a perfect conductor. When $\omega \neq \Omega_H$, dissipation occurs in the form of the irreducible mass of the hole increasing (Blandford & Znajek 1977).

It remains to show that $B^2 - E^2 > 0$. Because

$$(E_r^c)^2 + (E_\theta^c)^2 = (v_\phi^c)^2 [(B_r^c)^2 + (B_\theta^c)^2],$$

where v_c^ϕ is given by (3.5), it is only necessary to do this outside the outer light surface $v_c^\phi = 1$ and inside the inner light surface $v_c^\phi = -1$. In those two regions there has to be enough toroidal magnetic field for the particles to be able to 'slip back' along the field lines and avoid travelling faster than light. As the direction of the toroidal field is the same in each hemisphere, this means that particles have to travel outwards through the outer light surface and inwards through the inner light surface, so that the particle-production mechanism described in Blandford & Znajek (1977) must operate between the two light surfaces.

4 Energy extraction by a slowly rotating uniform vacuum field

Suppose we have a rotating hole in a vacuum electromagnetic field whose dependence on t and ϕ is given by $\exp(im\phi - i\omega t)$. Then the angular velocity of the field pattern around the hole is ω/m . If $0 < \omega/m < \Omega_H$, the hole loses energy to the field (Press & Teukolsky 1972). As in the case of the magnetospheric field of Section 3, there are then two light surfaces, which can be found by setting $|v_c^\phi| = 1$ in equation (3.5) and replacing ω by ω/m . When $|v_c^\phi| > 1$, the field behaviour can be described in terms of wave propagation (*cf.* the field outside the light surface of non-axisymmetric rotating dipole). Within the inner light surface ($v_c^\phi = -1$) the field behaves like an inwardly propagating wave from the point of view of a local physical observer, but like an outwardly propagating wave in Boyer–Lindquist coordinates (Press & Teukolsky 1972). Outside the outer light surface ($v_c^\phi = 1$) the field consists of two waves, an inwardly propagating wave carrying energy towards the hole and an outwardly propagating wave carrying energy at a faster rate away from the hole. Although the two waves are independent at infinity, they have to be present together in a certain ratio and phase in order for the field to be physical at the event horizon. The ratio can be interpreted as an amplification factor. It was calculated analytically by Starobinsky & Churilov (1973, hereafter SC) in the limits $\omega M \ll 1$ and $1 - \omega/m\Omega_H \ll 1$, and numerically by Teukolsky & Press (1974, hereafter TP) for general ω and m .

It is difficult to conceive of a likely situation in which radiation is incident from infinity in a manner that results in the hole losing energy. In fact a reasonable 'radiation condition' at infinity is that there is no incident radiation. It is more plausible to postulate a source of electromagnetic field at a finite distance R from the hole, with angular velocity ω in the same direction as that of the hole. If ω is the Keplerian angular velocity, and $R \gg M$, $\omega R = \sqrt{M/R} \ll 1$. We assume that there are no significant charges and current in $r_+ < r \leq R$, so that the results of SC and TP can be used in the vicinity of the hole. (This is not the case if the magnetic field is too strong as the vacuum near the hole would break down due to pair production and a magnetosphere would be established (Blandford & Znajek 1977).) As $0 < \omega/m < \Omega_H$, the hole will lose energy which will either be absorbed by matter at the source of the field or will be radiated away to infinity.

In this case there is no point in calculating an amplification factor as there is no wave incident from infinity. It is more useful to relate the rate of energy extraction to the strength of the field at its source. Below, we use the formalism of Section 2 to derive generally valid expressions for the rate of loss of energy and angular momentum by a black hole to a field varying like $\exp(im\phi - i\omega t)$, in terms of the toroidal fields at the event horizon. We then perform an explicit calculation for the case of a field that is an essentially uniform magnetic field in the region $M \ll r \leq R$. As $\omega R \ll 1$, the field is approximately stationary, and we use the results of SC to relate the field at the horizon to the field at the source.

We proceed in a manner common in flat space electrodynamics and assume that all quantities depend on t and ϕ as $\exp(im\phi - i\omega t)$. It must be emphasized that this is a

different complexification from that of NP formalism. In equations (2.4), E_k and B_k are real. We note that

$$\frac{\partial}{\partial t_c} = \frac{-iK}{\sqrt{(\Sigma\Delta)}}, \quad \frac{\partial}{\partial \phi_c} = \frac{iN}{\sin \theta \cdot \sqrt{\Sigma}},$$

where

$$K = (r^2 + a^2) \omega - am, \quad N = m - a\omega \sin^2 \theta.$$

The 'real' fields will be given by $E_k^c = \text{Re}(\tilde{E}_k \exp[im\phi - i\omega t])$, etc. $\tilde{E}_k$ and $\tilde{B}_k$ depend only on r and θ . We define $\tilde{B}_T = \sin \theta \sqrt{\Delta} \tilde{B}_\phi$, $\tilde{E}_T = \sin \theta \sqrt{\Delta} \tilde{E}_\phi$, and solve equations (2.13b), (2.13c), (2.14b) and (2.14c) assuming no sources to be present, obtaining

$$\begin{aligned} \tilde{B}_r &= \frac{i}{\mathcal{K}} \left\{ N\Delta \frac{\partial \tilde{B}_T}{\partial r} - K \sin \theta \frac{\partial \tilde{E}_T}{\partial \theta} \right\} \\ \tilde{B}_\theta &= \frac{i\sqrt{\Delta}}{\mathcal{K}} \left\{ K \sin \theta \frac{\partial \tilde{E}_T}{\partial r} + N \frac{\partial \tilde{B}_T}{\partial \theta} \right\} \\ \tilde{E}_r &= \frac{i}{\mathcal{K}} \left\{ N\Delta \frac{\partial \tilde{E}_T}{\partial r} + K \sin \theta \frac{\partial \tilde{B}_T}{\partial \theta} \right\} \\ \tilde{E}_\theta &= \frac{i\sqrt{\Delta}}{\mathcal{K}} \left\{ -K \sin \theta \frac{\partial \tilde{B}_T}{\partial r} + N \frac{\partial \tilde{E}_T}{\partial \theta} \right\} \end{aligned} \quad (4.1)$$

where $\mathcal{K} = K^2 \sin^2 \theta - N^2 \Delta$. $\mathcal{K} = 0$ on the light surfaces, and $K = 0$ at the radius at which the Carter observers corotate with the field. It is interesting that at $K = 0$ the effective potential derived by Chandrasekhar (1976) is singular and the radial coordinate he uses to remove the explicit dependence of the equations on m takes its minimum value. If we substitute (4.1) into (2.13a), (2.13d), (2.14a) and (2.14d) we obtain two independent equations for $\tilde{E}_T$ and $\tilde{B}_T$. Putting

$$X = \tilde{B}_T + i\tilde{E}_T, \quad Y = \tilde{B}_T - i\tilde{E}_T, \quad (4.2)$$

we find two separate complex conjugate equations for X and Y . As it does not appear to be possible to separate the independent variables we have to rely on known solutions, using

$$2\sqrt{\Delta} \sin \theta (\phi_0^c + \phi_2^c) = \bar{X} \exp(-im\phi + i\omega t) + Y \exp(im\phi - i\omega t) \quad (4.3)$$

obtained from (2.4) by substituting the real part of $\tilde{B}_\phi \exp(im\phi - i\omega t)$ for B_ϕ^c , etc. TP found the Kinnersley null tetrad components ϕ_0 and ϕ_2 by separation of variables

$$\begin{aligned} \phi_0 &= \exp(im\phi - i\omega t) {}_1S_{\omega lm}(\theta) {}_1R_{\omega lm}(r) \\ \rho^{-2} \phi_2 &= B \exp(im\phi - i\omega t) {}_{-1}S_{\omega lm}(\theta) {}_{-1}R_{\omega lm}(r) \end{aligned} \quad (4.4)$$

where B is a constant. The incoming and outgoing waves at infinity for such modes will both either have positive or negative helicity. In our context the field is linearly rather than circularly polarized, and symmetric between $\theta < \pi/2$ and $\theta > \pi/2$, so we have adopted a notation which can deal with (ω, m) and $(-\omega, -m)$ modes simultaneously, as is demonstrated by (4.2) and (4.3).

The conserved radial energy and angular momentum fluxes (Blandford & Znajek 1977) are respectively

$$\mathcal{E}^r = T_t^r = \frac{a \sin \theta \sqrt{\Delta}}{\Sigma} T_c^{r\phi} + \frac{\Delta}{\Sigma} T_c^{rt},$$

$$\mathcal{L}^r = -T_\phi^r = \frac{(r^2 + a^2) \sin \theta \sqrt{\Delta}}{\Sigma} T_c^{r\phi} + \frac{a \sin^2 \theta \Delta}{\Sigma} T_c^{rt},$$

where T_t^r , T_ϕ^r are Boyer–Lindquist components of the energy-momentum tensor and the Carter components are given by

$$4\pi T_c^{r\phi} = -E_r^c E_\phi^c - B_r^c B_\phi^c$$

$$4\pi T_c^{rt} = E_\theta^c B_\phi^c - E_\phi^c B_\theta^c.$$

Averaging over ϕ and t ,

$$4\pi \langle \mathcal{E}^r \rangle = \frac{1}{2\Sigma\mathcal{K}} \left\{ \omega \Sigma \Delta \mathcal{A} + \left(Ka \sin \theta + \frac{N\Delta}{\sin \theta} \right) \mathcal{B} \right\}$$

$$4\pi \langle \mathcal{L}^r \rangle = \frac{1}{2\Sigma\mathcal{K}} \{ m \Sigma \Delta \mathcal{A} + \sin \theta (K(r^2 + a^2) + aN\Delta) \mathcal{B} \}$$

where

$$\mathcal{A} = \text{Im} \left(\frac{\partial \tilde{B}_T}{\partial r} \right) \text{Re} (\tilde{B}_T) - \text{Re} \left(\frac{\partial \tilde{B}_T}{\partial r} \right) \text{Im} (\tilde{B}_T) + \text{Im} \left(\frac{\partial \tilde{E}_T}{\partial r} \right) \text{Re} (\tilde{E}_T) - \text{Re} \left(\frac{\partial \tilde{E}_T}{\partial r} \right) \text{Im} (\tilde{E}_T)$$

and

$$\mathcal{B} = \frac{\partial}{\partial \theta} \{ \text{Re} (\tilde{E}_T) \text{Im} (\tilde{B}_T) - \text{Im} (\tilde{E}_T) \text{Re} (\tilde{B}_T) \}.$$

The energy flow across a surface of constant radius is

$$2\pi \int_0^\pi d\theta \langle \mathcal{E}^r \rangle \Sigma \sin \theta$$

and similarly for the angular momentum.

From Teukolsky (1973) we find, using (4.3), that near the horizon $\tilde{B}_T$ and $\tilde{E}_T$ both vary radially as $\exp(-ikr_*)$, where $k = K/2Mr_+$ and $dr_*/dr = (r^2 + a^2)/\Delta$. Hence at $r = r_+$, $\mathcal{A} = -(K/\Delta)(\tilde{B}_T \tilde{B}_T + \tilde{E}_T \tilde{E}_T)$. Integrating over the horizon, the terms in $\mathcal{B}$ disappear as $\tilde{B}_T = \tilde{E}_T = 0$ when $\theta = 0$ or π . The rates of change of the hole's energy and angular momentum are therefore

$$\begin{aligned} \dot{L} &= \frac{m}{4K} \int_0^\pi d\theta \frac{\tilde{B}_T \tilde{B}_T + E_T \tilde{E}_T}{\sin \theta} \Sigma \\ &= \frac{m}{8K} \int_0^\pi d\theta \frac{X\bar{X} + Y\bar{Y}}{\sin \theta} \Sigma, \end{aligned} \tag{4.5}$$

$$\dot{M} = \frac{\omega}{m} \dot{L},$$

with $r = r_+$.

As we shall only require $\dot{L}$ and $\dot{M}$ to lowest order in ωM , we can assume that X and Y are static solutions ($\omega = 0$). If $0 < \omega M \ll 1$, the torque L is essentially the same as when $\omega = 0$, and the rate at which the field gains energy is just the product of its angular velocity and the torque. X and Y would only be significantly affected by ω in the outer radiation zone, which is of no interest to us.

The radial functions in (4.4) satisfying correct boundary conditions at the horizon were found by SC to be of the form

$${}_s R_{lm}(r) = {}_s C_{lm} \left(\frac{x}{x+1} \right)^{iQ} (x(x+1))^{-s} {}_2F_1(-l-s, l-s+1, 1-s-2iQ; x+1) \quad (4.6)$$

for $\omega r \ll 1$, independent of ω , where

$$x = (r - r_+)/ (r_+ - r_-), \quad Q = am/2\sqrt{(M^2 - a^2)},$$

the ${}_s C_{lm}$ are normalization constants and ${}_2F_1$ is the hypergeometric function. As $-l-s$ is a non-positive integer, the series expression for ${}_2F_1$ truncates. For a uniform field at R we only require the $l = 1, m = 0, \pm 1$ modes. Adopting the normalization of TP,

$$\int_0^\pi ({}_s S_{\omega lm})^2 \sin \theta d\theta = 1.$$

Neglecting $a\omega$, the angular functions ${}_s S_{lm}$ are the spin-weighted spherical harmonics (Goldberg *et al.* 1967). We require

$$\begin{aligned} {}_1 S_{11} &= \sqrt{(3/8)}(\cos \theta - 1) = -{}_1 S_{1-1}, \\ {}_1 S_{1-1} &= -\sqrt{(3/8)}(\cos \theta + 1) = -{}_1 S_{11}. \end{aligned} \quad (4.7)$$

${}_s R_{lm}$ and ${}_{-s} R_{lm}$ describe different aspects of the same field and are therefore connected by 'ladder relations' (TP). For $r\omega \ll 1, r \gg M$, these relations give

$${}_{-1} R_{lm} = \frac{r^2}{2B} {}_1 R_{lm} \quad (4.8)$$

where $B = l(l+1) + 0(a\omega)$ is the constant first encountered in (4.4). Here $B = 2$. We choose the constants ${}_s C_{lm}$ to be

$$\begin{aligned} {}_1 C_{1m} &= \frac{2Q^2 + iQ}{\sqrt{3}} B_\perp, \\ {}_{-1} C_{1,m} &= \frac{-(M^2 - a^2)}{\sqrt{3}} B_\perp. \end{aligned}$$

Then the ladder relations are satisfied for $r \gg M$ and therefore everywhere, and for $M \ll r \ll 1/\omega$ the field is a slowly rotating uniform magnetic field of strength $B_\perp$, perpendicular to the hole's axis of rotation. The radial functions are then

$$\begin{aligned} {}_1 R_{1m} &= \frac{4B_\perp(M^2 - a^2)}{\sqrt{3}\Delta} \exp(iQ \ln((r - r_+)/ (r - r_-))) \{2Q^2 + iQ + (1 - 2iQ)(x+1) - (x+1)^2\} \\ {}_{-1} R_{1m} &= \frac{-B_\perp \Delta}{4\sqrt{3}} \exp(iQ \ln((r - r_+)/ (r - r_-))). \end{aligned} \quad (4.9)$$

From (4.4) and (2.12),

$$\begin{aligned} \phi_0^c + \phi_2^c = & -\rho \left(\sqrt{\frac{\Delta}{2}} {}_1R_{11} {}_1S_{11} + 2\sqrt{\frac{2}{\Delta}} {}_{-1}R_{11} {}_{-1}S_{11} \right) \exp(i\phi - i\omega t) \\ & - \rho \left(\sqrt{\frac{\Delta}{2}} {}_1R_{1-1} {}_1S_{1-1} + 2\sqrt{\frac{2}{\Delta}} {}_{-1}R_{1-1} {}_{-1}S_{1-1} \right) \exp(-i\phi + i\omega t). \end{aligned} \quad (4.10)$$

Using (4.7) and (4.9), we find that in the limit $r \gg M$

$$B_\phi^c + iE_\phi^c = \phi_0^c + \phi_2^c = B_\perp \cos(\phi - \omega t).$$

Taking $B_\perp$ to be real, this is the required field. Using (4.3), we see that $X(r, \theta) = Y(r, \pi - \theta)$. On the horizon

$$Y(r_+, \theta) = \frac{-iaB_\perp \sin \theta}{r_+ - ia \cos \theta} (\sqrt{M^2 - a^2} + ia)(x/(x+1))^{iQ} (\cos \theta - 1).$$

Hence (4.5) gives

$$\dot{M} = \frac{-2}{3} \omega B_\perp^2 a M^2$$

and

$$\dot{L} = \frac{-2}{3} B_\perp^2 a M^2. \quad (4.11)$$

As the $l=1, m=0$ mode is orthogonal to the $l=1, m=\pm 1$ modes, the case of an oblique field can be treated by considering the perpendicular ($B_\perp$) and parallel ($B_\parallel$) modes separately. But the parallel field is axisymmetric, so $X=Y=0$, no energy or angular momentum is extracted by it, and (4.11) is still valid. This agrees with a calculation by Pollock (1976, private communication) for a strictly stationary field in the limit $r \rightarrow \infty$.

5 Summary and applications

We have used the Carter tetrad as a link between null tetrad formalisms and their physical applications around Kerr holes. The boundary conditions satisfied by electromagnetic fields at the event horizon show that a force-free magnetosphere need not necessarily rotate (in the sense of Blandford & Znajek 1977) with the same angular velocity as the hole, and so the hole does not behave like a perfect conductor of electricity. This is relevant when plasma accreting in a disk amplifies a magnetic field to a level at which pair creation can occur and establish a magnetosphere above and below the disk (Blandford & Znajek 1977).

It is possible for a black hole to interact with electromagnetic fields even when no plasma is present in the immediate vicinity of the hole. For example, a cloud of plasma left over after the gravitational collapse of a star could orbit round the hole that has been formed, and any large-scale non-axisymmetric field embedded in the plasma would affect the energy and angular momentum of the hole. Alternatively, the field could be generated by a neutron star near the hole. There is no reason why accretion should necessarily take place, and the fields are unlikely to be strong enough to break down the vacuum by pair creation. We have obtained expressions for the rates of loss of energy and angular momentum by the hole to the field in terms of the toroidal electric and magnetic fields at the horizon, for the case when the field varies like $\exp(im\phi - i\omega t)$ and no charges or currents are present very close

to the horizon. These expressions have been explicitly evaluated for a quasi-uniform field of magnitude $B_{\perp}$ perpendicular to the hole's axis of rotation.

The energy and angular momentum is transported by stresses in the field due to curvature of the field lines induced by the hole. If the field is frozen into a cloud of plasma orbiting the hole at radius R , then the cloud will absorb this energy and angular momentum, but it will also act as a source of radiation to infinity, losing energy and angular momentum at rates we estimate to be

$$\dot{M}_{\infty} \sim \frac{2}{3} \omega^4 p^2, \quad \dot{L}_{\infty} \sim \frac{2}{3} \omega^3 p^2 \quad (5.1)$$

where p is the dipole moment of the field outside the cloud and $\omega = \sqrt{M/R^3}$. An equilibrium may occur if the net loss of energy by the cloud is zero. If the field near the hole is quasi-uniform, then equating (5.1) and (4.11) we have

$$B_{\perp}^2 a M^2 \approx \omega^3 p^2. \quad (5.2)$$

p will be of the order of magnitude

$$R \times (\text{Flux through source})$$

and depends on the geometry of the source as well as R and $B_{\perp}$. Whether or not the equilibrium (5.2) is stable depends on how p varies with R ; for stability an increase in R must result in more energy being lost to infinity than is gained from the hole. If stability can occur, then the angular momentum and some of the energy of the hole will be transported to infinity by very low frequency electromagnetic waves on a timescale $L/\dot{L} \sim 1/B_{\perp}^2 M$.

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