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TIDAL RADIATION*

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ABSTRACT

The general theory of tides is developed within the framework of Einstein's theory of gravitation. It is based on the concept of Fermi frame and the associated notion of tidal frame along an open curve in spacetime. Following the previous work of the author an approximate scheme for the evaluation of tidal gravitational radiation is presented which is valid for weak gravitational fields. The emission of gravitational radiation from a body in the field of a black hole is discussed, and for some cases of astrophysical interest estimates are given for the contributions of radiation due to center-of-mass motion, purely tidal deformation, and the interference between the center of mass and tidal motions.

Subject headings: black holes — gravitation — relativity

I. INTRODUCTION

The nature of quasi-stellar objects (QSOs) and the origin of the violent activity in galactic nuclei are fundamental problems of astrophysics today. If the main component of the redshift of QSOs is assumed to be cosmological in origin, the source of the enormous amount of energy that is emitted is a mystery at present. The apparent concentration of the mass of a quasar in a very small region, together with the observation of high-speed astrophysical phenomena associated with QSOs and galactic nuclei, has led to the hypothesis that relativistic gravitational phenomena, such as complete gravitational collapse, play a dominant role in the activity of QSOs and galactic nuclei. The discovery of variable X-ray sources in binary star systems and in globular clusters in the Galaxy has strengthened the *premise* that the complete gravitational collapse of massive bodies can occur in nature. To put the theory on a firm foundation, however, many theoretical predictions have to be corroborated with extensive observations.

It is therefore of interest to develop the theory of interaction of matter with a strong gravitational field. In a previous paper (Mashhoon 1975, hereinafter Paper I) a simple theory has been given for the tidal interaction of a black hole with a perfect fluid model star in the harmonic approximation. It has been shown that within the framework of Einstein's theory of gravitation there is a significant new consequence of the tidal interaction, namely, the emission of tidal gravitational radiation which is expected to have important dynamical effects in the evolution of a dense stellar system. The present paper develops a general theory of tides. The gravitational radiation from an object moving in a gravitational field is discussed, and order-of-magnitude estimates are presented for the radiation due to the motion of the center of mass, tidal deformations, and the interference between the center-of-mass and tidal motions. The theory of tides is developed in the following section, and a highly approximate treatment of the gravitational radiation from a body is given in § III. The motion of a body in the field of a black hole is discussed in §§ IV and V. Appendix A presents a general discussion of the deviation equation, and Appendix B provides some general results on the tidal deformation equations in the harmonic approximation.

II. THEORY OF TIDES

Consider the motion of a body in an external gravitational field. It will be assumed that this motion causes a small perturbation on the background field. Let[1] $\bar{g}_{\mu\nu}(x)$ be the metric tensor of the external gravitational field in a given coordinate frame and $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ be the metric tensor of the background field together with the perturbation. The assumption that $h_{\mu\nu}$ is "small" compared with $g_{\mu\nu}$ in essence fixes the coordinate frame except

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[1] Greek indices run from 0 to 3. Latin indices run from 1 to 3. Units are chosen such that $G = c = 1$, unless otherwise specified. The quantity $\eta_{\mu\nu}$ denotes the Minkowski metric. The signature of the metric is +2. A semicolon denotes covariant differentiation with respect to $g_{\mu\nu}$, whereas a vertical bar denotes covariant differentiation with respect to $\bar{g}_{\mu\nu}$. The Riemann tensor is determined by $A_{\nu;\rho\sigma} - A_{\nu;\sigma\rho} = A^{\mu}R_{\mu\nu\rho\sigma}$, and the Ricci tensor is defined by $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$. Quantities with an asterisk refer to the Fermi coordinate system. Greek indices $\mu, \nu, \rho, \ldots$, refer to an arbitrary coordinate frame, while the indices $\alpha, \beta, \gamma, \ldots$, refer to the hypothetical locally Lorentzian region called the tidal frame in this paper. These latter indices may be raised or lowered with the Minkowski metric $\eta_{\alpha\beta}$ except when they appear on quantities referring to the Fermi coordinate system. Parentheses around indices denote symmetrization. A comma denotes partial differentiation.

for infinitesimal coordinate transformations. Let $x'^{\mu} = x^{\mu} - \epsilon^{\mu}(x)$ be such a transformation; then the tensor character of the total field together with the assumption of a fixed background metric results in the transformation

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) + \epsilon_{\mu|\nu} + \epsilon_{\nu|\mu}\,, \tag{1}$$

where all covariant differentiations, raising and lowering of indices, etc., are performed with the background metric. Thus the quantities $h_{\mu\nu}(x)$ are determined on the fixed background spacetime up to the gauge transformation (1). Evidently, the physical results obtained should be independent of the choice of gauge. The quantities $\psi_{\mu\nu}$ may be introduced,

$$\psi_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}h_{\rho\sigma}\,, \tag{2}$$

which under the gauge transformation (1) transform as

$$\psi'_{\mu\nu} = \psi_{\mu\nu} + \epsilon_{\mu|\nu} + \epsilon_{\nu|\mu} - \bar{g}_{\mu\nu}\epsilon^{\sigma}{}_{|\sigma}\,. \tag{3}$$

It proves convenient to impose the gauge condition

$$\psi^{\mu\nu}{}_{|\nu} = 0 \tag{4}$$

which does not completely specify the gauge, however. Any solution of

$$\epsilon^{\mu}{}_{|\nu}{}^{\nu} + \bar{R}^{\mu}{}_{\sigma}\epsilon^{\sigma} = 0 \tag{5}$$

leads to $\psi'^{\mu\nu}{}_{|\nu} = 0$ in equation (3) if equation (4) is satisfied.

In the region of spacetime under consideration the background metric $\bar{g}_{\mu\nu}(x)$ satisfies the source-free gravitational field equations. Thus the field quantities $\psi_{\mu\nu}$ satisfy (cf. Eisenhart 1926)

$$\psi_{\mu\nu|\sigma}{}^{\sigma} + 2\bar{R}_{\mu\rho\nu\sigma}\psi^{\rho\sigma} = -16\pi T_{\mu\nu}\,, \tag{6}$$

where the gauge condition (4) has been imposed. It is crucial to recognize that although equation (6) has been written in a formally covariant form, its validity is restricted to the background coordinate frame except for gauge transformations (3) that satisfy $\epsilon^{\mu}{}_{|\nu}{}^{\nu} = 0$. $T_{\mu\nu}$ is the energy-momentum tensor of the body and satisfies the conservation laws

$$T^{\mu\nu}{}_{;\nu} = 0\,. \tag{7}$$

The determination of the equation of motion using equation (7) permits the evaluation of the first-order perturbation field $\psi_{\mu\nu}$ from the wave equation (6). For the motion of a compact extended body in a gravitational field, the dynamical equations (7) are not, however, sufficient to determine the motion completely. The problem can be naturally divided into that of the motion of a characteristic point inside the body and of the internal motion of the body relative to the world line of this point. This is similar to the situation in Newtonian physics where the motion of a planet in the field of a star is described by the motion of the center of mass of the planet relative to the star together with the tidal deformations of the planet. Thus it is necessary to develop equations for the motion of the center of mass and for the internal motion relative to its world line. The moments of the energy-momentum tensor can be used to give the equations of motion for the center of mass. A detailed account of this may be found in the papers of Dixon (1970*a*, *b*, 1973, 1974). Let $\lambda^{\mu}(\tau)$, $\lambda^{\mu}\lambda_{\mu} = -1$, be the vector tangent to C, the world line of a characteristic point of the body. Here τ is the proper time along C such that $-d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$. It is possible to give a suitable definition of the moments of the energy-momentum tensor along $\lambda^{\mu}(\tau)$. It follows from the dynamical equations (7) that there exist differential relations among these moments which can be written in the form

$$\frac{DP^{\mu}}{D\tau} = \tfrac{1}{2}R^{\mu}{}_{\nu\rho\sigma}\lambda^{\nu}S^{\sigma\rho} + \mathbf{F}^{\mu}\,, \tag{8}$$

$$\frac{DS^{\mu\nu}}{D\tau} = P^{\mu}\lambda^{\nu} - P^{\nu}\lambda^{\mu} + \mathbf{T}^{\mu\nu}\,, \tag{9}$$

where $DP^{\mu}/D\tau = P^{\mu}{}_{;\nu}\lambda^{\nu}$, etc., the momentum vector P^{μ} and the spin tensor $S^{\mu\nu}$ are the first two moments of $T^{\mu\nu}$, respectively, and $\mathbf{F}^{\mu}$ and $\mathbf{T}^{\mu\nu}$ can be expressed in terms of its quadrupole and higher moments (Dixon 1973, 1974). When $\mathbf{F}^{\mu}$ and $\mathbf{T}^{\mu\nu}$ are neglected, equations (8) and (9) reduce to the set of equations first derived by Mathisson (1937) and later by Papapetrou (1951). It is possible to choose C so that it corresponds to a suitably defined center of mass.[2]

[2] The definition of a center of mass corresponds to a constraint imposed on the spin tensor. It follows from the results of Madore (1969), Beiglböck (1967), and Dixon (1970*a*, and the references cited therein) that when this constraint is taken to be $S^{\mu\nu}P_{\nu} = 0$ for a suitable definition of P^{μ} and $S^{\mu\nu}$, then the center of mass so defined is unique when certain restrictions are placed on the strength of the gravitational field.

The energy-momentum tensor can be determined by the complete set of its moments; therefore, equations (8) and (9) need to be supplemented by additional relations describing the time evolution of the quadrupole and higher moments. To find these extra relations, which describe the tidal deformations of the body, more information about the structure of $T^{\mu\nu}$ is required. The subsequent analysis is concerned with an iterative procedure for dealing with this problem. Let $\lambda^{\mu}{}_{(\alpha)}$ be a tetrad system along C such that $\lambda^{\mu}{}_{(0)} = \lambda^{\mu}$. It is important for the discussion of tidal radiation to specify how the linearly independent spacelike vectors $\lambda^{\mu}{}_{(i)}$ are transported along C. Let $\lambda^{\mu}{}_{\perp(i)} = \lambda^{\mu}{}_{(i)} + \lambda^{\nu}{}_{(i)}\lambda_{\nu}\lambda^{\mu}$ be the orthogonal component of $\lambda^{\mu}{}_{(i)}$ with respect to λ^{μ}. A manifestly nonrotating tetrad frame is determined from the requirements that the component of $\lambda^{\mu}{}_{(i)}$ along λ^{μ} be a constant and that the component of $(D/D\tau)\lambda^{\mu}{}_{\perp(i)}$ orthogonal to λ^{μ} vanish, namely,

$$\lambda^{\mu}{}_{(i)}\lambda_{\mu} = \text{constant}\,, \tag{10}$$

$$(g^{\mu}{}_{\nu} + \lambda^{\mu}\lambda_{\nu})\frac{D}{D\tau}\lambda^{\nu}{}_{\perp(i)} = 0\,. \tag{11}$$

These relations are equivalent to the assumption that the tetrad is Fermi-Walker transported along C. It will be further assumed that the tetrad system is orthonormal. At any point O along C the set of all geodesic paths starting from O and orthogonal to λ^{μ} produce a hypersurface which intersects the congruence of curves representing the trajectories of the different points in the body. It is necessary to assume that the dimensions of the body are sufficiently small compared with a characteristic length scale of the background gravitational field. Thus if P is the intersection of the path of a point in the body with the hypersurface, then it follows from the construction of Riemannian coordinates in a small neighborhood of O that there exists a unique geodesic joining O to P which lies in the hypersurface. Let ξ^{μ} be the unit vector tangent to this geodesic at O and σ be the proper distance along the geodesic such that $\sigma = 0$ at O. Consider a transformation of coordinates $x^{*\mu} = x^{*\mu}(x)$ from the background coordinate system to a new frame in the neighborhood of C where

$$x^{*0} = \tau\,, \tag{12}$$

$$x^{*i} = \sigma\lambda_{\mu}{}^{(i)}\xi^{\mu}\,, \tag{13}$$

are the Fermi coordinates of the point P (Fermi 1922; Levi-Civita 1926; Synge 1960). In terms of these coordinates the conservation laws (7) may be written explicitly as

$$T^{*\mu\nu}{}_{,\nu} + \Gamma^{*\mu}{}_{\rho\sigma}T^{*\rho\sigma} + \Gamma^{*\rho}{}_{\sigma\rho}T^{*\mu\sigma} = 0\,, \tag{14}$$

where the Christoffel symbols may be written as power series in x^{*i}. In the absence of tidal gravitational forces, equation (14) reduces to the equation of motion of matter in a Minkowski spacetime with the origin as the center of mass. Thus for the purposes of the present discussion, it is possible to regard the tetrad frame as representing a locally Minkowskian region (the "tidal frame") where the laws of the Lorentz-invariant theory hold in the neighborhood of C except for the presence of tidal forces. Let $X^{\alpha} = x^{*\alpha}$ be the coordinates of the points of the body and $\mathscr{T}^{\alpha\beta}$ be the tensor of energy-momentum of matter with respect to this tidal frame. In the tidal frame equation (14) can be expressed as

$$\mathscr{T}^{\alpha\beta}{}_{,\beta} = \mathscr{F}^{\alpha}\,, \tag{15}$$

where all tidal gravitational effects are included in the forces $\mathscr{F}^{\alpha}$ which may be written as power series in X^{i}. Equation (15) may then be solved iteratively by suitable approximation procedures based on the assumption that the size of the body is small compared to the length scale of the gravitational field.

The explicit form of the tidal forces depends on the nature of the body under consideration. As an illustration of the general procedure and for the subsequent applications, the case of a perfect fluid will now be considered. Let

$$T^{\mu\nu}(x) = (\mu + p)u^{\mu}u^{\nu} + pg^{\mu\nu} \tag{16}$$

be the energy-momentum tensor of the body, where $\mu(x)$, $p(x)$, and $u^{\mu}(x)$ are the rest energy per unit proper volume, the pressure, and the velocity unit vector of matter, respectively. One may define

$$\mathscr{T}^{\alpha\beta} = (\mu + p)U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}\,, \tag{17}$$

where $U^{\alpha} = dX^{\alpha}/ds$ is the velocity of a point in the body relative to the center of mass in the tidal frame and s, the proper time in the tidal frame, is determined by

$$\eta_{\alpha\beta}U^{\alpha}U^{\beta} = -1\,. \tag{18}$$

It follows from equations (14) and (15) that

$$\mathscr{F}^{\alpha} = -(\mu + p)(W^{\alpha} + QU^{\alpha})\,, \tag{19}$$

where W^α and Q are given by

$$(1-\chi)W^\alpha = \Gamma^{*\alpha}{}_{\beta\gamma}U^\beta U^\gamma + \chi\frac{dU^\alpha}{ds} + (1-\chi)(g^{*\alpha\beta} - \eta^{\alpha\beta})\frac{p_{,\beta}}{\mu+p}, \tag{20}$$

$$(1-\chi)Q = \frac{1}{2}\frac{d}{ds}\ln(-g^*) + (1-\chi)^{-1}\frac{d\chi}{ds} + \chi U^\alpha{}_{,\alpha} + \chi\frac{d}{ds}\ln(\mu+p), \tag{21}$$

and χ may be obtained from

$$\chi = \chi_{\alpha\beta}U^\alpha U^\beta \tag{22}$$

with $\chi_{\alpha\beta}$ defined by $g^*_{\alpha\beta} = \eta_{\alpha\beta} + \chi_{\alpha\beta}$. For the discussion of the tidal forces in the nonrelativistic approximation it is convenient to write equations (15), (17), and (19) as

$$\frac{\partial}{\partial\tau}\tilde{\rho} + \nabla_i(\tilde{\rho}\dot{X}^i) = \frac{\partial p}{\partial\tau} - \gamma^{-2}\tilde{\rho}(W^0 + \gamma Q), \tag{23}$$

$$\tilde{\rho}\ddot{X}^i = -\left(\nabla_i p + \frac{\partial p}{\partial\tau}\dot{X}^i\right) - \gamma^{-2}\tilde{\rho}(W^i - \dot{X}^i W^0), \tag{24}$$

where $\dot{X}^i = dX^i/d\tau$, $\nabla_i = \partial/\partial X^i$, $\tilde{\rho} = \gamma^2(\mu+p)$, and $\gamma = (1-\dot{X}^i\dot{X}_i)^{-1/2}$. To arrive at the nonrelativistic approximation it is important to note that in general C does not follow a geodesic and hence along C there is a uniform acceleration field. It follows that the leading power of c, the speed of light in vacuum, in $\chi_{\alpha\beta}$ is -2. Hence equations (20)–(24) reduce to

$$\frac{\partial\rho}{\partial\tau} + \nabla_i(\rho\dot{X}^i) = 0, \tag{25}$$

$$\rho\ddot{X}^i = -\nabla_i p - \rho F^i, \tag{26}$$

in the nonrelativistic approximation where $F^i = \Gamma^{*i}{}_{00}$. Here ρ is the mass density of the fluid in motion; i.e., it is the nonrelativistic limit of $\gamma\mu$.

The explicit evaluation of the tidal forces requires an examination of the metric tensor in Fermi coordinates. This is considered in detail in Appendix A, and some of the principal results obtained there are summarized below. In the harmonic approximation where the tidal forces are given to first order in the expansion in X^i, we have $W^\alpha = W_R{}^\alpha + W_A{}^\alpha$ and $Q = Q_R + Q_A$. The curvature and acceleration parts of W^α and Q are given by

$$W_R{}^0 = 2\gamma^2(R^*{}_{0i0j} - \tfrac{1}{3}R^*{}_{0ijk}\dot{X}^k)\dot{X}^i X^j, \tag{27}$$

$$W_A{}^0 = 2\gamma^2 A_i^*\dot{X}^i[1-(1+2\gamma^2)A_j^*X^j] + \gamma^2\dot{A}_i^*X^i + 2\left[(\mu+p)^{-1}\frac{\partial p}{\partial\tau} - \gamma^3\frac{d\gamma}{d\tau}\right]A_i^*X^i, \tag{27'}$$

$$W_R{}^i = \gamma^2(R^*{}_{i0k0} + 2R^*{}_{ijk0}\dot{X}^j + \tfrac{2}{3}R^*{}_{ijkl}\dot{X}^j\dot{X}^l)X^k, \tag{28}$$

$$W_A{}^i = \gamma^2 A^{*i} + \gamma^2\left[(1-2\gamma^2)A^{*i} - 2\gamma\frac{d}{d\tau}(\gamma\dot{X}^i)\right]A_j^*X^j, \tag{28'}$$

$$Q_R = [\tfrac{1}{3}\gamma(2R^*{}_{0i0j} - R^*{}_{ij}) - 2\gamma^3(R^*{}_{0i0j} + \tfrac{2}{3}R^*{}_{0ikj}\dot{X}^k)]\dot{X}^i X^j, \tag{29}$$

$$Q_A = \gamma(1-2\gamma^2)\frac{d}{d\tau}(A^*{}_iX^i) + \gamma(8\gamma^4 - 4\gamma^2 - 1)A^*{}_iA^*{}_j\dot{X}^iX^j - 2\gamma^3A^*{}_iX^i\left[3\gamma^{-1}\frac{d\gamma}{d\tau} + \nabla_j\dot{X}^j + \frac{d}{d\tau}\ln(\mu+p)\right], \tag{29'}$$

where

$$A^{*i}(\tau) = \lambda^{(i)}{}_\mu\frac{D\lambda^\mu}{D\tau} \tag{30}$$

is the acceleration of C relative to the tetrad frame and $R^*{}_{0i0j}(\tau) = R_{\mu\nu\rho\sigma}\lambda^\mu\lambda^\nu{}_{(i)}\lambda^\rho\lambda^\sigma{}_{(j)}$, etc. In the nonrelativistic approximation equations (28) and (28′) reduce to

$$F^i = A^{*i}(\tau) + R^*{}_{i0j0}X^j. \tag{31}$$

It is also of interest to consider the nonrelativistic approximation in general when C is a geodesic. Then the tidal force in equation (26) can be written as

$$F^i = K^i{}_j X^j + \frac{1}{2!} K^i{}_{jk} X^j X^k + \cdots , \tag{32}$$

where $K^i{}_{jk} = K^i{}_{kj}$, etc. A general method for the determination of $K^i{}_{jk}$, etc., is given in Appendix A. Let

$$\mathscr{R}_{\mu\nu\rho\sigma\omega} = \tfrac{1}{2}(R_{\mu\nu\rho\sigma;\omega} + R_{\mu\rho\omega\nu;\sigma}) , \tag{33}$$

$$\mathscr{S}_{\mu\nu\rho\sigma\omega} = \tfrac{2}{3}(\mathscr{R}_{\mu\nu\rho\sigma\omega} + R_{\mu\sigma\omega\nu;\rho}) , \tag{34}$$

and

$$\mathscr{T}_{\mu\nu\rho\sigma\omega\pi} = \tfrac{1}{3}\mathscr{R}_{\mu\nu\rho\sigma\omega;\pi} + \tfrac{2}{3}R^\xi{}_{\rho\omega\nu}R_{\mu\xi\pi\sigma} \tag{35}$$

be new tensors constructed out of the Riemann curvature tensor. We also need to define the quantities $\mathscr{P}_{\mu\nu\rho}$ and $\mathscr{Q}_{\mu\nu}$,

$$\mathscr{P}_{\mu\nu\rho} = \tfrac{1}{2}(\lambda_{\mu;\nu\rho} + R_{\mu\nu\sigma\rho}\lambda^\sigma) , \tag{36}$$

$$\mathscr{Q}_{\mu\nu} = \lambda_{\mu;\nu} . \tag{37}$$

Then $K_{ij} = R^*{}_{i0j0}(\tau)$ is a symmetric matrix, and the higher-order terms in equation (32) are

$$K_{ijk} = 2!K'{}_{i(jk)} , \tag{38}$$

$$K_{ijkl} = 3!K'{}_{i(jkl)} , \tag{39}$$

and so on, where $K'{}_{ijk}$ and $K'{}_{ijkl}$ are given by

$$K'{}_{ijk} = \mathscr{R}^*{}_{i0j0k} + 2R^*{}_{ilk0}\mathscr{Q}^*{}_{lj} , \tag{40}$$

$$K'{}_{ijkl} = \mathscr{T}^*{}_{i0j0kl} + 2R^*{}_{imk0}\mathscr{P}^*{}_{mjl} + \mathscr{R}^*{}_{00j0k}\mathscr{Q}^*{}_{il} + \mathscr{S}^*{}_{i0jmk}\mathscr{Q}^*{}_{ml} + 2R^*{}_{0mk0}\mathscr{Q}^*{}_{il}\mathscr{Q}^*{}_{mj} + \tfrac{2}{3}R^*{}_{imkn}\mathscr{Q}^*{}_{mj}\mathscr{Q}^*{}_{nl} . \tag{41}$$

In the Newtonian limit one finds that

$$K_{ijk\ldots} = \phi_{,ijk\ldots} , \tag{42}$$

where ϕ is the gravitational potential. In the harmonic approximation, where only the linear term in equation (32) is kept, one is naturally led to the concept of tidal potential introduced by Synge (1935) and used in the theory of tides in Paper I. In the general case, however, it is useful to introduce the tidal stress tensor Ψ_{ij}, $\Psi_{ij} = \Psi_{ji}$, which is defined by

$$F_i = \nabla_j \Psi_{ij} . \tag{43}$$

Let $\Pi_{ijk\ldots} = X_i X_j X_k \ldots$; then it is possible to write Ψ_{ij} as

$$\Psi_{ij} = \sum_{n=1,\infty} [n!(n+4)]^{-1}\Psi_{ij}{}^{(n)} , \tag{44a}$$

$$\Psi_{ij}{}^{(n)} = K_{ikl\ldots}\Pi_{kl\ldots j} + K_{jkl\ldots}\Pi_{kl\ldots i} - \frac{n}{n+3} K_{ppl\ldots}\Pi_{l\ldots ij} , \tag{44b}$$

where $K_{ikl\ldots}$ and $\Pi_{kl\ldots j}$ each have $n + 1$ indices. Thus equation (26) may be written in the general form

$$\rho \ddot{X}^i = \nabla_j P^{ij} - \rho \nabla_j \Psi^{ij} , \tag{45}$$

where $P_{ij} = -p\delta_{ij}$ is the Newtonian stress in our case. Let

$$\pi^{ij} = \rho \dot{X}^i \dot{X}^j - P^{ij} + \rho \Psi^{ij} ; \tag{46}$$

then it follows that when $\rho = \rho(\tau)$ one has

$$\frac{\partial}{\partial \tau}(\rho \dot{X}^i) = -\nabla_j \pi^{ij} , \tag{47}$$

and

$$\frac{\partial}{\partial \tau}\rho(X^i \dot{X}^j - \dot{X}^i X^j) = -\nabla_k(X^i \pi^{jk} - X^j \pi^{ik}) . \tag{48}$$

Hence π^{ij} has the interpretation of momentum flux density tensor in this case. Equation (45) has to be supplemented with boundary conditions in general, and it can be shown from equation (47) that the normal component of P^{ij} has to vanish on a free boundary of the fluid.

Appendix A develops general methods for the explicit evaluation of the tidal forces when the center-of-mass line C is given. Then it is possible to determine the rate of change of the quadrupole and higher moments of the body from the equations of internal motion (15) within the limits of the approximation schemes considered. These in turn determine the path of the center of mass according to equations (8) and (9). Thus the problem of motion of the body in the external field is reduced to the solution of a set of coupled differential equations for the internal and center-of-mass motions. The solution of these equations together with equation (6) results in the determination of the perturbation due to the motion of matter if the necessary boundary conditions are specified. The flux of gravitational radiation generated by this motion can be thought of as consisting of parts due to the center-of-mass motion, the tides, and the interference between the center of mass and internal motions. In a consistent solution of the problem of motion the reaction force due to the emission of gravitational radiation should be taken into consideration. In the present paper, however, only approximate estimates for the rate of emission of radiation is provided, and no attempt is made to take account of the radiation reaction force.

III. APPROXIMATE TREATMENT OF THE DEFORMATION EQUATIONS

In the iteration process for the determination of tidal deformation and tidal radiation, the center of mass follows a geodesic of the combined field in the lowest order of approximation. The general considerations of the previous section can be applied to the simplest case of the motion of a body in a gravitational field such that the center of mass follows a geodesic of the *external* field and the internal motion can be adequately described in the *non-relativistic* approximation. It will be further assumed that (*a*) the tidal forces dominate over the internal stresses in the spacetime region considered and (*b*) the harmonic approximation (31) is adequate for the description of the tidal forces. Then the equations of motion for a perfect fluid reduce to

$$\ddot{X}^i + K^i{}_j(\tau)X^j = 0 \tag{49}$$

and the continuity equation (25). The velocity of the fluid $\dot{X}^i$ can, in general, be expanded in a power series about the center of mass $X^i = 0$,

$$\dot{X}^i = V^i{}_j(\tau)X^j + \frac{1}{2!}V^i{}_{jk}(\tau)X^jX^k + \cdots\,, \tag{50}$$

and consistency with the harmonic approximation requires that only the linear term in this series be kept. It follows from the continuity equation that in this approximation the density is only a function of τ. The general problem of the motion of a homogeneous fluid body such that the velocity is a linear function of the position was posed by Dirichlet (1860) and solved by Riemann (1860). Let the body always keep an ellipsoidal shape with $a_i(\tau)$ the semiaxes of the ellipsoid. It is convenient to work in the principal axis frame of the ellipsoid. Thus, let Ω_i be the angular velocity of the body frame relative to the tetrad frame and $x_i = M_{ij}(\tau)X^j$ the corresponding transformation law; then

$$\dot{M}_{ij} = e_{ikl}\Omega_l M_{kj}\,, \tag{51}$$

where e_{ijk} is the alternating symbol with $e_{123} = 1$. Thus in the principal axis frame equation (49) can be written as

$$\ddot{x}_i + 2e_{ijk}\Omega_j\dot{x}_k + (\Omega_i\Omega_j - \Omega^2\delta_{ij} + e_{ilj}\dot{\Omega}_l + k_{ij})x_j = 0\,, \tag{52}$$

where $k_{ij} = M_{im}M_{jn}K_{mn}$. The velocity in the body frame can be written as

$$\dot{x}_i = Q_{ij}(\tau)x_j\,, \tag{53}$$

where Q_{ij} is restricted by the assumption that the body is ellipsoidal. Let $r_i = x_i/a_i$; then it follows from equation (53) that

$$r_i(\tau) = H_{ij}(\tau)r_j(0)\,. \tag{54}$$

If x_i are the coordinates of a point on the ellipsoid, then r_i are its coordinates on a unit sphere. In the motion of the ellipsoid this unit sphere is mapped onto itself; hence (H_{ij}) is an orthogonal matrix. It follows from equations (53) and (54) that

$$Q_{ij} = a_i^{-1}\dot{a}_i\delta_{ij} + e_{ijl}a_ia_j^{-1}\Lambda_l\,, \tag{55}$$

where $\Lambda_i = \frac{1}{2}e_{ijk}\dot{H}_{jl}H_{kl}$ is connected with the vorticity of the fluid. The continuity equation then reduces to the requirement that the mass of the ellipsoid $m = \frac{4}{3}\pi\rho(\tau)a_1a_2a_3$ be a constant of the motion. Equation (52) can now be written as nine differential equations:

$$\frac{d^2a_1}{d\tau^2} - (\Omega_2{}^2 + \Omega_3{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2)a_1 + 2(\Omega_3\Lambda_3a_2 + \Omega_2\Lambda_2a_3) + k_{11}a_1 = 0\,, \tag{56}$$

$$\frac{d\Omega_1}{d\tau}a_2 - \frac{d\Lambda_1}{d\tau}a_3 + 2\left(\Omega_1\frac{da_2}{d\tau} - \Lambda_1\frac{da_3}{d\tau}\right) + \Omega_2\Omega_3a_2 + \Lambda_2\Lambda_3a_3 - 2\Omega_2\Lambda_3a_1 + k_{32}a_2 = 0\,, \tag{57}$$

$$\frac{d\Lambda_1}{d\tau}a_2 - \frac{d\Omega_1}{d\tau}a_3 + 2\left(\Lambda_1\frac{da_2}{d\tau} - \Omega_1\frac{da_3}{d\tau}\right) + \Lambda_2\Lambda_3a_2 + \Omega_2\Omega_3a_3 - 2\Lambda_2\Omega_3a_1 + k_{23}a_3 = 0\,, \tag{58}$$

and the other six may be obtained from equations (56)–(58) by cyclic permutation of the indices. For general k_{ij}, these equations determine a_i, Ω_i, and Λ_i provided appropriate initial conditions are given. Under the assumptions of the present discussion the tidal forces may be written as the sum of the external and internal field contributions; hence

$$k_{ij} = \bar{k}_{ij} + 2\pi\rho(\tau)\mathscr{A}_i\delta_{ij}\,, \tag{59}$$

where

$$\mathscr{A}_i = a_1a_2a_3\int_0^\infty \Delta^{-1}(a_i{}^2 + u)^{-1}du\,, \tag{60}$$

$$\Delta^2 = (a_1{}^2 + u)(a_2{}^2 + u)(a_3{}^2 + u)\,, \tag{61}$$

and $\bar{k}_{ij}$ is determined from the curvature and the tetrad frame of the background field. Properties of the functions $\mathscr{A}_i$ together with a detailed discussion of Dirichlet's problem and Riemann's solution may be found in the book of Chandrasekhar (1969). In Paper I, expressions have been derived for the rate of change of energy, angular momentum, and circulation of the body as it moves in the gravitational field. The tidal matrix k_{ij} is symmetric; therefore, three of the nine deformation equations reduce to the fact that the circulation vector precesses instantaneously around the vector (Λ_i).

An approximate expression for the amount of gravitational energy radiated can be obtained if it can be assumed that the background field is weak and the motion of the body is nonrelativistic. Then the energy flux is proportional to the square of the third time derivative of the total quadrupole moment of the system. The time derivatives of the quadrupole moment can be evaluated using either the coordinate time of the background spacetime or, alternatively, the proper time τ along C since the results are approximately equal in the weak field limit. Thus, the energy radiated per unit time is given by

$$P = \frac{1}{45}\,\mathrm{Tr}\left(\frac{d^3}{d\tau^3}D\right)^2, \tag{62}$$

where $D = (D_{ij})$, the matrix of the total quadrupole moment, can be written as the sum of the tidal part referred to the background spacetime plus the center-of-mass contribution

$$D = D^T + D^{\mathrm{CM}}\,. \tag{63}$$

If the gravitational field is sufficiently weak, D^T may be evaluated with respect to the tetrad frame so that

$$D^T{}_{ij} \approx d_{kl}M_{ki}M_{lj}\,, \tag{64}$$

where $d = (d_{kl})$ is the quadrupole moment in the principal axis frame. Let $d_{(0)} = d$ and

$$d_{(n+1)} = \dot{d}_{(n)} + [d_{(n)}, \Omega^*]\,, \tag{65}$$

where $\Omega^*{}_{ij} = e_{ijk}\Omega_k$. P may be written as

$$P = \tfrac{1}{45}[\mathrm{Tr}\,(d_{(3)})^2 + 2\mathrm{Tr}\,(M\dagger d_{(3)}M\dddot{D}^{\mathrm{CM}}) + \mathrm{Tr}\,(\dddot{D}^{\mathrm{CM}})^2]\,, \tag{66}$$

where the terms in the brackets represent the purely tidal contribution, the flux due to the interference between the tidal and center-of-mass motions, and the center of mass contributions, respectively. The quantity $d_{(3)}$ can be written as

$$d_{(3)} = \delta_0 + \delta_1 + \delta_2 + \dddot{d}\,, \tag{67}$$

where

$$\delta_0 = [[[d, \Omega^*], \Omega^*], \Omega^*]\,, \tag{68}$$

$$\delta_1 = [[d, \Omega^*], \dot{\Omega}^*] + 2[[d, \dot{\Omega}^*], \Omega^*] + 3[[\dot{d}, \Omega^*], \Omega^*]\,, \tag{69}$$

$$\delta_2 = [d, \ddot{\Omega}^*] + 3[\dot{d}, \dot{\Omega}^*] + 3[\ddot{d}, \Omega^*]\,, \tag{70}$$

and $d_{ij} = \frac{1}{5}m(3a_i^2 - a_1^2 - a_2^2 - a_3^2)\delta_{ij}$. It follows from equations (56)–(58) and (66)–(70) that P is a polynomial of second order in $K_{ij} = R^*{}_{i0j0}$ and $(d/d\tau)K_{ij}$,

$$\frac{1}{2}\frac{d}{d\tau}K_{ij} = \mathscr{R}^*{}_{i0j00} - R^*{}_{k(ij)0}A^{*k}\,, \tag{71}$$

where the terms in equation (71) involving the acceleration of C can be neglected in the nonrelativistic approximation considered here. The presence of the δ_0 term in equation (67) points to the fact that the motion of a rigidly deformed body with the body frame rotating uniformly with respect to the tetrad frame contributes to the tidal and interference radiations with $d_{(3)} = \delta_0$.

Characteristic features of tidal gravitational radiation have been investigated by detailed but rather approximate calculations that are presented in subsequent sections of this paper for some cases of astrophysical interest. In these studies it is convenient to consider an initially nonrotating spherical body of radius R_0 and density ρ_0 which is subsequently deformed by the tidal forces. Let $T_0 = (2\pi\rho_0)^{-1/2}$ be the characteristic time for hydrodynamic processes in the body. Then if the internal quadrupole moment is given in units of $\frac{1}{5}mR_0^2$ and all distances (and times) connected with the center-of-mass motion are given in units of T_0, equation (66) can be written in the form

$$P = \tfrac{1}{30}\xi^3(F^{\text{CM}} + 0.6\xi F^I + 0.09\xi^2 F^T)\,, \tag{72}$$

where

$$\xi = m/R_0 \tag{73}$$

is half of the gravitational radius of the body and the power P is given in units of $L_0 = c^5/G = 3.63 \times 10^{59}$ ergs s^{-1}. Equation (72) defines the quantities F^{CM}, F^I, and F^T which represent the contributions of the center-of-mass motion, the interference between the center-of-mass and tidal motions, and the tidal deformation, respectively, to the energy radiated by gravitational waves per unit proper time.

Consider, for instance, the radial infall of a body toward a Schwarzschild black hole of mass M, $2M \gg R_0 \gg m$. In Schwarzschild coordinates the equations of the geodesic path C are given by

$$\frac{dt}{d\tau} = \gamma_0(1 - 2M/r)^{-1}\,, \tag{74}$$

$$\frac{dr}{d\tau} = -(\gamma_0^2 - 1 + 2M/r)^{1/2}\,, \tag{75}$$

where $\gamma_0 = (1 - \beta^2)^{-1/2}$ and β is the speed of the body for $r \to \infty$. If the body has no initial rotation or internal motion, one can assume $\Lambda_i = \Omega_i = 0$. In a tetrad frame so chosen that it coincides with the Schwarzschild coordinate frame at spatial infinity for $\beta = 0$ the tidal matrix is diagonal with $K_{ii} = \alpha_i M/r^3$ where $\alpha_1 = -2$, $\alpha_2 = \alpha_3 = 1$, and it has been assumed that in the spacetime region under consideration the internal gravitational binding forces are negligible compared with the external tidal forces. For $\beta \neq 0$, equations (56)–(58) reduce to

$$2z^2(1 - z)\frac{d^2a_i}{dz^2} - z\frac{da_i}{dz} + \alpha_i a_i = 0\,, \tag{76}$$

where $z = -\frac{1}{2}(\gamma_0^2 - 1)r/M$. Let ζ_i be a solution of

$$2\zeta_i^2 - 3\zeta_i + \alpha_i = 0\,; \tag{77}$$

then $(-z)^{-\zeta_i}a_i(z)$ satisfies the hypergeometric equation

$$z(1 - z)\frac{d^2v}{dz^2} + [p_3 - (p_1 + p_2 + 1)]\frac{dv}{dz} - p_1p_2v = 0 \tag{78}$$

with $p_1 = \zeta_i - 1$, $p_2 = \zeta_i$, and $p_3 = 2\zeta_i - \frac{1}{2}$. The nondegenerate case occurs when $2\zeta_i$ is not an integer (cf. Erdélyi *et al.* 1953 for a detailed discussion of the hypergeometric equation). If $2\zeta_i - \frac{1}{2}$ is not an integer and $|z| < 1$, then a_i is a constant linear combination of the two independent solutions of equation (76):

$$\psi(\zeta_i; z) = (-z)^{\zeta_i}F(\zeta_i, \zeta_i - 1; 2\zeta_i - \tfrac{1}{2}; z)$$

and

$$\psi(\tfrac{3}{2} - \zeta_i; z)\,,$$

where $3/2 - \zeta_i$ is the other solution of equation (77). The solutions for a_i are absolutely convergent for $|z| = 1$. Any hypergeometric series valid for $|z| < 1$ can be continued analytically to the domain $|\arg(-z)| < \pi$; therefore, solutions for a_i may be obtained for all z in this domain. Alternatively, for $|z| > 1$ let $w = z^{-1}$; then equation

(76) takes the form of equation (78)[3] with $v = a_i(w)$, p_1 and p_2 solutions of equation (77), and $p_3 = 2$. If $2\zeta_i$ is not an integer, then the solutions of equation (76) are (Erdélyi *et al.* 1953) $F(\zeta_i, \frac{3}{2} - \zeta_i; 2; z^{-1})$ and $F(\zeta_i, \frac{3}{2} - \zeta_i; \frac{1}{2}; 1 - z^{-1})$. In the degenerate case under discussion, i.e., $\alpha_1 = -2$, $\alpha_2 = \alpha_3 = 1$, a_1 is a constant linear combination of $(1 - z^{-1})^{1/2}$ and[4] $z^2F(1, 2; 7/2; z)$; a_2 and a_3 can be similarly expressed as linear combinations of $-z$ and $(-z)^{1/2}(1 - z)^{1/2}$. When $\beta = 0$, a_1 can be expressed as a linear combination of $r^{-1/2}$ and r^2, while a_2 and a_3 are linear combinations of r and $r^{1/2}$. Let $x \equiv r/T_0$, $a_1 = R_0(x_0/x)^{1/2}$, and $a_2 = a_3 = R_0(x/x_0)^{1/2}$ so that $\rho = \rho_0(x_0/x)^{1/2}$. Here x_0 is a constant such that $x_0 \gg 2\lambda$, where $\lambda \equiv M/T_0 = (\rho_0/\rho_{\mathrm{B.H.}})^{1/2}$ and $\rho_{\mathrm{B.H.}} = (2\pi M^2)^{-1}$ is a characteristic "density" for the black hole. It is then possible to write

$$F^{\mathrm{CM}} = 48\lambda^3/x^5\,, \tag{79}$$

$$F^I = (x_0x)^{-1}(1 + 10{x_0}^2/x^2)F^{\mathrm{CM}}\,, \tag{80}$$

$$F^T = (F^I)^2/F^{\mathrm{CM}}\,. \tag{81}$$

It follows from equations (79)–(81) that for given values of r/M and r_0/M, i.e., for the same orbit $F^{\mathrm{CM}} \propto \lambda^{-2}$, $F^I \propto \lambda^{-4}$, and $F^T \propto \lambda^{-6}$. This variation of the radiation functions, and hence of the power P, with λ, together with equation (81), holds generally for the free fall of a spheroid (i.e., $\beta = 0$, $a_2 = a_3$) into a Schwarzschild black hole when the external tidal stresses dominate over the internal binding forces.

IV. MOTION IN THE FIELD OF A BLACK HOLE

The motion of a highly simplified model star in the gravitational field of a massive black hole is discussed in this and the following section, and rough estimates are presented for the amount of gravitational radiation emitted. Far away from the black hole the model star is assumed to be a perfect fluid with the internal gravitational forces balanced by the hydrostatic pressure so that the star is initially spherical and nonrotating. As it moves close to the black hole, it is deformed but always keeps an ellipsoidal shape. The density of the fluid is constant, $\rho = \rho_0$, and its pressure is given by (cf. Paper I, § III)

$$p = p_c(\tau)\left[1 - \sum_i ({x_i}^2/{a_i}^2)\right]\,, \tag{82}$$

so that the pressure vanishes on the surface of the model star as required by the boundary conditions. Following the considerations of the previous section, the deformation equations are (56)–(58) with k_{ij} formally replaced by

$$k_{ij} - 2\frac{p_c}{\rho_0{a_i}^2}\delta_{ij}$$

and k_{ij} is given by equation (59). Let $\tilde{F}^T$, $\tilde{F}^I$, and $\tilde{F}^{\mathrm{CM}}$ be the tidal, interference, and the center-of-mass contributions to the total power emitted, respectively, computed with respect to the coordinate time such that $\tilde{P}$, given by

$$\tilde{P} = \tfrac{1}{30}\xi^3(\tilde{F}^{\mathrm{CM}} + 0.6\xi\tilde{F}^I + 0.09\xi^2\tilde{F}^T)\,, \tag{83}$$

is the amount of gravitational radiation emitted per unit coordinate time t. If the gravitational field is weak and the motion of the body is nonrelativistic, then $\tilde{P} \approx P$. Let E and $\tilde{E}$ be the amount of energy emitted according to equations (72) and (83), respectively. In the motion of the body in a strong gravitational field, solutions of equation (6) are required to give a close estimate of the energy emitted; E and $\tilde{E}$ in general differ considerably from each other and from the more exact result. However, E and $\tilde{E}$ may be used as a rough guide for the expected amount of energy radiated. If, for instance, they fall within an order of magnitude of each other, then it is expected that their average gives an estimate of energy radiated to within an order of magnitude.

Consider the radial fall of a model star into a Schwarzschild black hole of mass M, $2M \gg R_0 \gg m$. Following the discussion of the previous section, we let $\Lambda_i = \Omega_i = 0$; hence $\bar{K}_{ij} = (M/r^3)\alpha_i\delta_{ij}$. Equations (56)–(58), modified

[3] The hypergeometric equation (78) for $v(w)$ is transformed into

$$z^2(1 - z)\frac{d^2v}{dz^2} + [(p_3 - 2)z^2 - (p_1 + p_2 - 1)z]\frac{dv}{dz} + p_1p_2v = 0$$

with $w = z^{-1}$. When $p_3 = 2$ this equation has the same general form as equation (76). Moreover, if p_1 and p_2 are solutions of equation (77), then equation (76) is obtained and $v_i(z) = (-z)^{1/2}L_{\nu i}{}^{\mu}(z')$, where $z' = (1 - z^{-1})^{1/2}$ and $L_{\nu i}{}^{\mu}$ is a Legendre function with $\mu^2 = 1$ and $\nu_i(\nu_i + 1) = 2(1 - \alpha_i)$.

[4] A detailed analysis of the correspondence between the solutions of equation (76) and Legendre functions (cf. n. 2) reveals that

$$z^2F(1, 2; 7/2; z) = -\frac{15}{4}\left(z'\ln\frac{z' + 1}{z_1 - 1} + \frac{2}{3}z - 2\right)$$

for the range of variable z under consideration ($z \leq 0$).

TABLE 1

RADIAL FALL OF A MODEL STAR INTO A SCHWARZSCHILD BLACK HOLE WITH $\rho_{\text{B.H.}} = \rho_0$*

COORDINATE r (in units M)	DEFORMATION $a_1/R_0 : a_2/R_0$	RADIATION FUNCTIONS $F_1{}^T : F_1{}^I : F_1{}^{\text{CM}}$	RADIATION FUNCTIONS $\tilde{F}_1{}^T : \tilde{F}_1{}^I : \tilde{F}_1{}^{\text{CM}}$
15.0	1.01:1	$1.0 \times 10^{-7}:2.6 \times 10^{-6}:6.3 \times 10^{-5}$	$2.4 \times 10^{-7}:3.0 \times 10^{-6}:4.3 \times 10^{-5}$
5.5	1.01:0.99	$2.0 \times 10^{-4}:1.4 \times 10^{-3}:9.5 \times 10^{-3}$	$1.8 \times 10^{-6}:4.3 \times 10^{-5}:1.4 \times 10^{-3}$
5.0	1.016:0.988	$1.4 \times 10^{-4}:1.0 \times 10^{-3}:1.5 \times 10^{-2}$	$1.2 \times 10^{-5}:-6.7 \times 10^{-5}:1.5 \times 10^{-3}$
4.5	1.026:0.985	$4.45 \times 10^{-4}:2.3 \times 10^{-3}:2.6 \times 10^{-2}$	$1.7 \times 10^{-5}:-8.2 \times 10^{-5}:1.3 \times 10^{-3}$
4.0	1.041:0.981	$2.3 \times 10^{-3}:9.7 \times 10^{-3}:4.7 \times 10^{-2}$	$6.5 \times 10^{-6}:-2.9 \times 10^{-5}:7.3 \times 10^{-4}$
3.5	1.060:0.974	$1.7 \times 10^{-2}:3.9 \times 10^{-2}:9.1 \times 10^{-2}$	$8.6 \times 10^{-7}:-2.2 \times 10^{-6}:7.0 \times 10^{-6}$
3.0	1.086:0.964	0.12:0.16:0.20	$2.6 \times 10^{-5}:2.5 \times 10^{-4}:2.4 \times 10^{-3}$
2.5	1.125:0.948	1.03:0.71:0.49	$1.0 \times 10^{-4}:1.3 \times 10^{-3}:1.7 \times 10^{-2}$
2.0	1.185:0.923	12.37:4.31:1.5	0.0:0.0:0.0

* The initial conditions are specified at $r = 15M$. The initial rate of deformation is assumed to be zero. The quantity a_3 may be evaluated from $a_1a_2a_3 = R_0{}^3$. The central pressure is almost a constant except very near the horizon where it drops by about 1.4%. The increase in the internal energy of the ellipsoid at the horizon is $E - E_\infty = 0.83 \times 10^{-2}$ m^2/R_0, where $E_\infty = -\frac{3}{5}m^2/R_0$. The proper time τ and the coordinate time t may be evaluated from (cf. eq. [74] and eq. [75] with $\beta = 0$) $3\tau + 4M\cosh^3\omega = c_1$ and $3t + 4M\cosh^3\omega + 12M(\cosh\omega + \ln\tanh\frac{1}{2}\omega) = c_2$, where ω is given by $r = 2M\cosh^2\omega$, c_1 and c_2 are constants.

to take account of the pressure, may be integrated numerically to determine the tidal deformation of the body, P and $\tilde{P}$. For the free fall of a point particle from rest ($\beta = 0$), we have $E^{\text{CM}} = (2/105)m^2/M$, about twice the value obtained from the more exact computations that take account of the curvature of the background spacetime, and $E^{\text{CM}}/\tilde{E}^{\text{CM}} \approx 7.75$. Table 1 presents the results of numerical integration of the deformation equations for a_1, a_2, and the contributions of the tidal, interference, and center-of-mass parts to the power radiated by the model star with $\beta = 0$ and $\rho_0 = \rho_{\text{B.H.}}$. When λ increases, the deformation of the star generally decreases and so do $F_\lambda{}^T$, $F_\lambda{}^I$, and $F_\lambda{}^{\text{CM}}$. For $\lambda = 4$, e.g., at the horizon, $a_1/R_0 = 1.03$ and $a_2/R_0 = 0.99$; and $F_4{}^T$, $F_4{}^I$, and $F_4{}^{\text{CM}}$ are approximately 10^{-3}, 10^{-2}, and 0.9×10^{-1}, respectively. On the other hand for $\lambda = \frac{1}{4}$, $a_1/R_0 = 1.69$, $a_2/R_0 = 0.77$, $F_{1/4}{}^T = 3.1 \times 10^5$, $F_{1/4}{}^I = 2.7 \times 10^3$, and $F_{1/4}{}^{\text{CM}} = 24$ at the horizon. Taking the approximate schemes of this section to their limit of applicability, consider a neutron star of mass $m = \frac{1}{2}M_\odot$ and radius $R_0 = 3.7 \times 10^6$ cm falling radially into a black hole of mass $M = 36\,M_\odot$. Then $\lambda \approx \frac{1}{4}$, $\xi \approx 2 \times 10^{-2}$, and the energy radiated per unit proper time at the horizon is $\sim 6.5 \times 10^{54}$ ergs s^{-1} of which $\sim 35.5\%$ is due to the center of mass motion, $\sim 16.5\%$ due to the internal tidal deformation, and $\sim 48\%$ due to the interference between the tidal and center-of-mass contributions.

V. EQUATORIAL MOTION IN THE EXTREME KERR FIELD

A class of geodesic orbits in the extreme Kerr spacetime will be considered in this section and the characteristics of the radiation from a body following one such orbit will be described in detail within the approximation scheme of this paper. The path of the center of mass of the body is given by $dx^\mu/d\tau = \lambda^\mu$. In Boyer-Lindquist coordinates (Boyer and Lindquist 1967) where $x^0 = t$ and $x^i = (r, \theta, \phi)$ we have $d\theta/d\tau = 0$ and

$$\frac{dt}{d\tau} = \frac{\gamma_0 r(r^2 + M^2) + 2M^2(\gamma_0 M - P_\phi)}{r(r - M)^2}, \tag{84}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \gamma_0{}^2 - \left(1 - \frac{M}{r}\right)^2 + \frac{\gamma_0{}^2M^2 - P_\phi{}^2}{r^2} + \frac{2M(\gamma_0 M - P_\phi)^2}{r^3}, \tag{85}$$

$$\frac{d\phi}{d\tau} = \frac{rp_\phi + 2M(\gamma_0 M - P_\phi)}{r(r - M)^2}. \tag{86}$$

Here γ_0 and p_ϕ are the energy per unit mass and the orbital angular momentum per unit mass of the body, respectively. Consider the class of orbits with $\gamma_0 > 1$ and $p_\phi \neq 0$ such that the body approaches the black hole from infinity for $\tau < 0$, has a turning point at $\tau = 0$, $r = r_0 > M$, and travels to infinity for $\tau > 0$. To calculate the tidal interference radiation within the approximation scheme of this paper, it is necessary that the parallel-propagated tetrad frame carried along by the comoving observer coincide with the Minkowski frame at infinity for $\beta \to 0$. It follows from the parallel transport equation for $\lambda^2{}_{(\alpha)}$ and from the confinement of the orbit to the equatorial plane that $\lambda^\mu{}_{(2)} = \delta_{\mu 2}/r$ and $\lambda^2{}_{(i)} = \delta_{i2}/r$. From $\lambda_\mu\lambda^\mu{}_{(1)} = 0$ and $\lambda^\mu{}_{(1)}\lambda_{\mu(1)} = 1$ one finds that

$$\lambda^0{}_{(1)\pm} = q_1\lambda^1{}_{(1)} \pm \{(q_1{}^2 - q_2{}^2)[\lambda^1{}_{(1)}]^2 + q_3{}^2\}^{1/2}, \tag{87}$$

where q_1, q_2, and q_3 are given by

$$q_1(r) = \lambda_1\lambda^0(\gamma_0\lambda^0 - p_\phi\lambda^3)^{-1}\,, \tag{88}$$

$$q_2(r) = (r - M)^{-1}(\gamma_0\lambda^0 - p_\phi\lambda^3)^{-1/2}[g_{11}p_\phi{}^2 + g_{33}(\lambda_1)^2]^{1/2}\,, \tag{89}$$

$$q_3(r) = p_\phi(r - M)^{-1}(\gamma_0\lambda^0 - p_\phi\lambda^3)^{-1/2}\,, \tag{90}$$

with $g_{11}(r) = r^2(r - M)^{-2}$ and $g_{33}(r) = r^2 + M^2 + 2M^3/r$. Equations (87)–(90) together with $D\lambda^1{}_{(1)}/D\tau = 0$ and $\lambda_\mu\lambda^\mu{}_{(1)} = 0$ determine $\lambda^1{}_{(1)}$ when appropriate boundary conditions are specified. A detailed examination reveals that it is possible to choose $\lambda^0{}_{(1)}$ and $\lambda^3{}_{(1)}$ such that they are odd functions of τ. Thus in the determination of $\lambda^1{}_{(1)}$, $\lambda^0{}_{(1)-}$ can be used for $\tau \leq 0$ and $\lambda^0{}_{(1)+}$ for $\tau \geq 0$. The quantity $\lambda^1{}_{(1)}$ is then an even function of τ and its value at $\tau = 0$ is given by $\lambda^0{}_{(1)}(\tau = 0) = 0$, hence $\lambda^1{}_{(1)}(\tau = 0) = \pm(1 - M/r_0)$. If $\lambda^1{}_{(1)}(\tau = 0) = -1 + M/r_0$ is chosen, then the value of $\lambda^1{}_{(1)}$ is γ_0 for $r \to \infty$. In this way $\lambda^\mu{}_{(1)}$ is completely determined, and the choices made here have been used in the numerical work reported in this section. If instead $\lambda^0{}_{(1)+}$ is used for $\tau \leq 0$, $\lambda^0{}_{(1)-}$ for $\tau \geq 0$, and $\lambda^1{}_{(1)}(\tau = 0) = 1 - M/r_0$ (i.e., $\lambda^1{}_{(1)} \to -\gamma_0$ as $r \to \infty$), then $\lambda^\mu{}_{(1)}$ so obtained is simply the negative of the tetrad vector chosen above (cf. Appendix B). In the numerical work $\lambda^1{}_{(1)}$ is required at an initial value of r, $r \gg M$, for the integration of the deformation equations. Thus $D\lambda^1{}_{(1)}/D\tau = 0$ is integrated using $\lambda^1{}_{(1)}(\tau = 0) = -1 + M/r_0$ to find $\lambda^1{}_{(1)}$ at some initial value of r. It is next necessary to determine $\lambda^\mu{}_{(3)}$ from $\lambda^\mu\lambda_{\mu(3)} = 0$, $\lambda^\mu{}_{(1)}\lambda_{\mu(3)} = 0$, and $\lambda^\mu{}_{(3)}\lambda_{\mu(3)} = 1$. To this end, $\lambda^0{}_{(3)}$ and $\lambda^3{}_{(3)}$ can be found from $\lambda^\mu\lambda_{\mu(3)} = 0$ and $\lambda^\mu{}_{(1)}\lambda_{\mu(3)} = 0$ in terms of $\lambda^1{}_{(3)}$ and the results substituted in $\lambda^\mu{}_{(3)}\lambda_{\mu(3)} = 1$ to give an expression for $[\lambda^1{}_{(3)}]^2$. Up to a change in sign, $\lambda^\mu{}_{(3)} \to -\lambda^\mu{}_{(3)}$, two possibilities present themselves. If $\lambda^1{}_{(3)}$ is taken to be an odd function of τ with $\lambda^1{}_{(3)} \geq 0$ for $\tau \leq 0$ and $\lambda^1{}_{(3)} \leq 0$ for $\tau \geq 0$, then $\lambda^0{}_{(3)}$ and $\lambda^3{}_{(3)}$ are even functions of τ and $\lambda^3{}_{(3)} \to 1/r$ as $r \to \infty$. Alternatively, if $\lambda^1{}_{(3)}$ is assumed to be even with $\lambda^1{}_{(3)} \geq 0$ for $-\infty < \tau < \infty$, then $\lambda^0{}_{(3)}$ and $\lambda^3{}_{(3)}$ are odd functions of τ with a jump discontinuity at $\tau = 0$. Thus in the latter case $\lambda^\mu{}_{(3)}$ coincides with the result of the former case for $\tau < 0$ but becomes its negative for $\tau > 0$. For the purposes of the present discussion we assume that $\lambda^1{}_{(3)}$ is an odd function of τ with $\lambda^3{}_{(3)} \to 1/r$ as $r \to \infty$.

Once the tetrad frame is completely specified, the tidal matrix may be evaluated. The only off-diagonal element of the tidal matrix is K_{13} as in Paper I; therefore, it may be assumed that the body is initially spherical and nonrotating, $\Omega_1 = \Omega_3 = 0$, $\Lambda_1 = \Lambda_3 = 0$, and the circulation vanishes. A general discussion of the resulting deformation equations is contained in Appendix B. The orbits under consideration fall into two classes. For $p_\phi < 0$, a test body approaching the black hole from infinity can first encounter a turning point in ϕ, i.e., it can reach $r = r_\phi \equiv 2M(1 - \gamma_0 M/p_\phi)$ before reaching r_0. For $p_\phi > 0$, however, a test body simply reaches the turning point r_0 and then goes to infinity. The remainder of this section is devoted to the study of gravitational radiation from bodies following the latter type of orbits with $\beta \ll 1$. A separate study is necessary for the orbits with $p_\phi < 0$. The $p_\phi = 0$ case has been considered by Hiscock (1977). Following the method outlined in the previous section, expressions (72) and (83) are evaluated for each $p_\phi > 0$ and compared. It appears from the numerical work that P is always greater than $\tilde{P}$ by at most a certain factor which is approximately $[\lambda^0(r_0)]^6$. The same statement also applies to $F_\lambda{}^{\mathrm{CM}}$, $F_\lambda{}^T$, and $|F_\lambda{}^I|$. Thus when $r_0 \gg M$ this factor is $\gamma_0{}^6$, and for the orbit chosen in Paper I ($\beta = 0.1$, $p_\phi = 2.5\,M$, and $r_0 \approx 1.961\,M$) it is $\sim 2 \times 10^3$. In the present analysis the orbit is so chosen that P is greater than $\tilde{P}$ by no more than an order of magnitude. Thus we let $\beta = 0.01$, $p_\phi = 3.75\,M$, and hence $r_0 \approx 5.704\,M$. This orbit is depicted in Figure 1. The initial conditions for the deformation equations are specified at $r = 10.002\,M$ such $a_1 = 1.01\,R_0$,

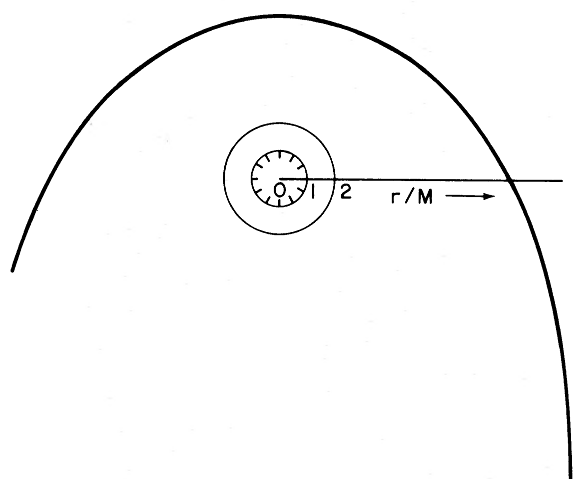

FIG. 1.—Plot of the trajectory of a test body with orbital angular momentum per unit mass $p_\phi = 3.75\,M$ and initial speed at infinity $\beta = 0.01$ following a geodesic orbit in the equatorial plane of an extreme Kerr black hole of mass M. The turning point occurs at $r_0 \equiv 5.704\,M$.

TABLE 2

MOTION OF A MODEL STAR IN THE EQUATORIAL PLANE OF AN EXTREME KERR BLACK HOLE WITH $\lambda = 0.35$*

Proper Time (in units M)	Coordinate r (in units M)	Deformation $a_1/R_0 : a_2/R_0$	Frequency (in units T_0^{-1})	Central Pressure	Excess Internal Energy $(E - E_\infty)/(0.3\ m^2/R_0)$
−23.1922......	10.002	1.01:1	0	1	0.53×10^{-4}
−20.3.........	9.2	1.00:0.99	−6.95	1.00	0.46×10^{-3}
−15.6.........	8.0	1.07:0.97	0.05	1.00	0.33×10^{-2}
−5.1.........	6.0	1.24:0.88	0.25	0.97	0.29×10^{-1}
0.0.........	5.704	1.42:0.84	0.23	0.92	0.78×10^{-1}
5.1.........	6.0	1.69:0.79	0.22	0.83	0.16
11.2.........	7.0	1.96:0.74	0.23	0.74	0.25
17.2.........	8.4	2.06:0.75	0.25	0.71	0.30
21.5.........	9.5	2.01:0.77	0.28	0.73	0.31
35.1.........	13.3	1.76:0.90	0.37	0.80	0.28
41.0.........	14.9	1.78:0.85	0.36	0.81	0.27
49.1.........	17.1	1.87:0.82	0.32	0.77	0.28
37.8.........	14.1	1.77:0.89	0.36	0.80	0.27

* The model star follows the orbit of Fig. 1. The deformation of the body depends on the initial conditions which are specified at $r = 10.002\ M$. The frequency given is $-\Omega_2 = d\Theta/d\tau$, so that it is positive when the body corotates with the black hole. The central pressure function given here is defined to be unity at the starting point. The sum of vibrational, rotational, and gravitational (potential) energies of the ellipsoid due to the tides is given in the last column where $E_\infty = -0.6\ m^2/R_0$.

$a_2 = R_0$, $a_1 a_2 a_3 = R_0^3$ with the initial rate of deformation assumed to be zero, $\Theta = 0$, $\Omega_2 = 0$, and $\lambda^1{}_{(1)}$ is obtained from the integration of the parallel transport equation from $r = r_0$, $\lambda^1{}_{(1)} = -1 + M/r_0$, to $r = 10.002\ M$ which results in $\lambda^1{}_{(1)} \approx 0.02$. In Paper I similar initial conditions were specified at $r = 10\ M$ except that $\lambda^1{}_{(1)} = 0.945$ was used as an estimate, which is to be compared with $\lambda^1{}_{(1)} = 0.935$ obtained from the integration of the parallel transport equation from $r_0 \approx 1.961\ M$. The deformation equations were then integrated from $r = 10\ M$ to $r_0 \approx 1.961\ M$ and again from r_0 to $r \geq M$ using the same tetrads as in the present work except that $\lambda^1{}_{(3)} \geq 0$ was assumed for $-\infty < \tau < \infty$. The crudeness of our numerical estimates for the amount of gravitational radiation emitted should be emphasized again. The radiation due to the tides becomes significant when the body is very close to the black hole, and this is precisely where the weak field approximation that is used here breaks down. The relationship between P and $\tilde{P}$ is thus a rough guide as to how different the actual results may be from our estimates. The orbit used in the present analysis is such that r_0/M is not too large to make the tidal radiation insignificant and yet not too small to make our approximation scheme totally inadequate.

For a given orbit and given initial conditions, the deformation equations contain only $\lambda > 0$ as a free parameter. The deformation of the body near the turning point generally increases as λ decreases. The numerical work for the orbit of Figure 1 shows that for $\lambda \geq 0.35$ the tidal force is not great enough to cause a permanent deformation of the model star and the body oscillates as it moves away from the turning point and goes to infinity. For $\lambda \leq 0.3$, however, the body continues to be elongated as it moves to infinity. Tables 2 and 3 illustrate the former case for

TABLE 3

MOTION OF A MODEL STAR IN THE EQUATORIAL PLANE OF AN EXTREME KERR BLACK HOLE WITH $\lambda = 0.5$*

Proper Time (in units M)	Coordinate r (in units M)	Deformation $a_1/R_0 : a_2/R_0$	Frequency (in units T_0^{-1})	Central Pressure	Excess Internal Energy $(E - E_\infty)/(0.3\ m^2/R_0)$
−23.1922......	10.002	1.01:1	0	1	0.53×10^{-4}
−20.5.........	9.3	1.003:0.996	−6.93	1.00	0.24×10^{-3}
−15.6.........	8.0	1.06:0.98	0.06	1.00	0.14×10^{-2}
−9.6.........	6.7	1.07:0.96	0.26	1.00	0.26×10^{-2}
−3.8.........	5.9	1.13:0.94	0.11	0.99	0.73×10^{-2}
0.0.........	5.704	1.18:0.91	0.16	0.98	0.013
4.2.........	5.9	1.20:0.91	0.23	0.97	0.018
6.3.........	6.2	1.21:0.93	0.22	0.97	0.019
14.2.........	7.7	1.13:0.98	0.38	0.99	0.014
17.7.........	8.5	1.12:0.98	0.32	0.99	0.011
22.3.........	9.8	1.09:0.98	0.50	1.00	0.009
26.6.........	11.0	1.11:1.00	0.30	0.99	0.98×10^{-2}
30.2.........	11.9	1.10:0.99	0.41	0.99	0.011
35.0.........	13.3	1.13:0.98	0.29	0.99	0.011
40.3.........	14.7	1.10:0.99	0.45	1.00	0.01
43.9.........	15.7	1.11:1.01	0.31	0.99	0.01

* See note to Table 2.

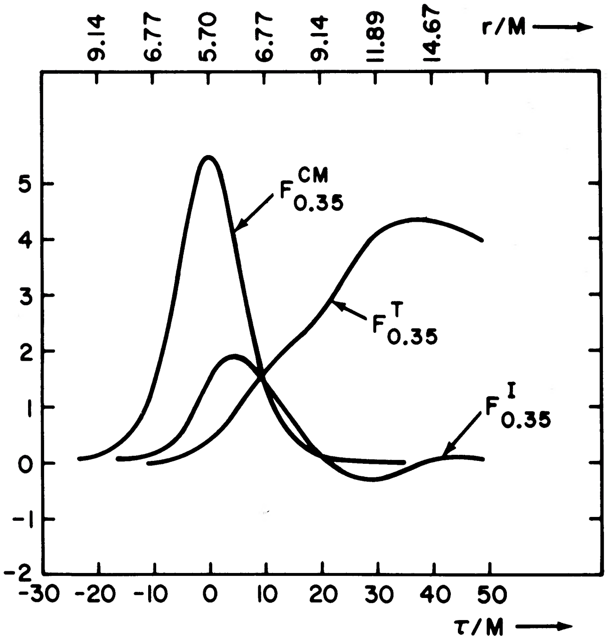

FIG. 2

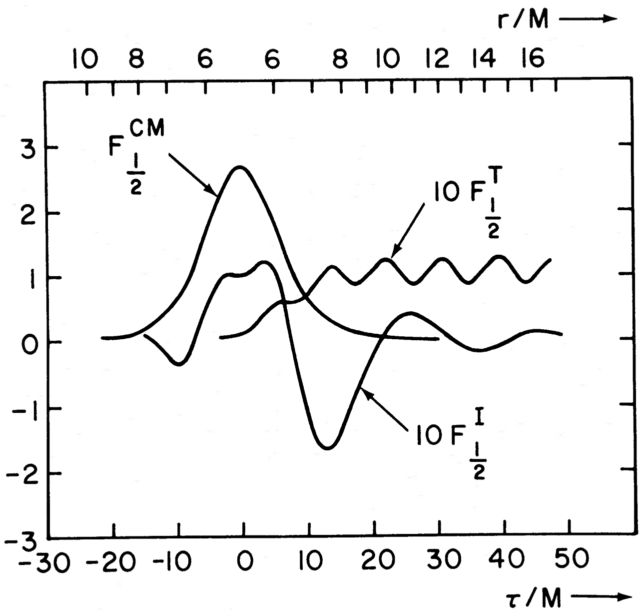

FIG. 3

FIG. 2.—The radiation functions for $\lambda = 0.35$, $\lambda = (\rho/\rho_{\rm B.H.})^{1/2}$, versus proper time (and the radial coordinate r) for an initially ($-\tau \gg M$) spherical nonrotating fluid ellipsoid whose center of mass follows the trajectory of Fig. 1.

FIG. 3.—The radiation functions for $\lambda = 0.5$, $\lambda \equiv (\rho/\rho_{\rm B.H.})^{1/2}$, versus proper time (and the radial coordinate r) for an initially ($-\tau \gg M$) spherical nonrotating fluid ellipsoid whose center of mass follows the trajectory of Fig. 1.

$\lambda = 0.35$ and $\lambda = 0.5$, respectively. The corresponding results for the radiation functions are given in Figures 2 and 3, respectively. The period of oscillations of the body and of Ω_2 appears to be $T_2 \approx 8.6\,T_0$, the natural frequency for quadrupole oscillations of a sphere. The oscillations are damped by frictional forces, including the gravitational radiation reaction force, which have been ignored in our simple analysis. Tables 4 and 5 illustrate the $\lambda \leq 0.3$ case for $\lambda = 0.25$ and $\lambda = 0.05$, respectively, with the corresponding results for the radiation functions given in Figures 4 and 5, respectively. The radiation function $F_\lambda^{\rm CM}$ peaks at the turning point and has a width at half-maximum of $\sim 13\,M$ independently of λ. For example for $\lambda = 1$, the body has a deformation of $a_1/R_0 = 1.025$, $a_2/R_0 = 0.98$ at the turning point, and $F_1^{\rm CM}(\tau = 0) = 0.67$; F_1^{T} has an oscillatory character and for $\tau > 0$ becomes almost periodic with a period of $\frac{1}{2}T_2 \approx 4.3\,T_0$ and oscillates between 0 and 2×10^{-3}. $F_1^{I}(\tau)$ also has an oscillatory character, but for $r > 10\,M$ these oscillations are damped since $F_1^{\rm CM}$ decreases rapidly.

Some of the astrophysical phenomena associated with strong gravitational fields are expected to result in the emission of substantial fluxes of gravitational radiation. The experimental discovery of this radiation will mark an

TABLE 4

MOTION OF A MODEL STAR IN THE EQUATORIAL PLANE OF AN EXTREME KERR BLACK HOLE WITH $\lambda = 0.25$*

Proper Time (in units M)	Coordinate r (in units M)	Deformation a_1/R_0:a_2/R_0	Frequency (in units T_0^{-1})	Central Pressure	Excess Internal Energy $(E - E_\infty)/(0.3\,m^2/R_0)$
−23.1922......	10.002	1.01:1	0	1	0.53×10^{-4}
−20.1.........	9.2	1.00:0.99	−8.00	1.00	0.89×10^{-3}
−14.3.........	7.7	1.10:0.95	0.78	1.00	0.01
−3.7.........	5.9	1.50:0.78	0.26	0.90	0.12
0.0.........	5.704	1.75:0.71	0.29	0.83	0.25
1.1.........	5.72	1.85:0.70	0.30	0.80	0.30
8.2.........	6.4	2.66:0.64	0.27	0.60	0.72
15.6.........	8.0	3.83:0.56	0.21	0.41	1.15
23.2.........	10.0	5.30:0.41	0.16	0.29	1.50
30.5.........	12.0	6.91:0.37	0.12	0.22	1.75
37.6.........	14.0	8.67:0.39	0.09	0.17	1.95
49.2.........	17.2	11.83:0.26	0.06	0.12	2.20

* See note to Table 2.

TABLE 5

MOTION OF A MODEL STAR IN THE EQUATORIAL PLANE OF AN EXTREME KERR BLACK HOLE WITH $\lambda = 0.05$*

Proper Time (in units M)	Coordinate r (in units M)	Deformation $a_1/R_0 : a_2/R_0$	Frequency (in units T_0^{-1})	Central Pressure	Excess Internal Energy $(E - E_\infty)/(0.3\ m^2/R_0)$
−23.1922......	10.002	1.01:1	0	1	0.53×10^{-4}
−19.9.........	9.1	1.01:0.99	−32.07	1.02	0.22×10^{-1}
−15.6.........	8.0	1.07:0.96	0.20	1.16	0.18
−2.7.........	5.8	2.07:0.60	0.89	2.24	6.21
0.0.........	5.704	2.54:0.49	1.02	2.09	10.39
4.1.........	5.9	3.43:0.35	1.11	1.37	18.10
15.5.........	8.0	6.49:0.15	0.79	0.17	28.98
23.2.........	10.0	8.64:0.10	0.56	0.05	27.79
30.4.........	12.0	10.62:0.07	0.42	0.03	25.49
37.6.........	14.0	12.55:0.06	0.34	0.01	23.29
49.2.........	17.2	15.52:0.05	0.23	0.76×10^{-2}	20.49

* See note to Table 2.

advance in the theory of gravitation. It is therefore of interest to have estimates of the amount of gravitational energy radiated in such processes. Two illustrative examples for a body following the orbit of Figure 1 will now be examined. Consider first the $\lambda = 0.3$ case. The deformation at $\tau = 0$ is $a_1/R_0 = 1.56$, $a_2/R_0 = 0.786$, and $F_{0.3}{}^{\mathrm{CM}}$ reaches a maximum value of 7.47. The maximum of $F_{0.3}{}^{I}$ occurs at $\tau = 3.1\ M$ where $F_{0.3}{}^{\mathrm{CM}} = 6.35$, $F_{0.3}{}^{I} = 3.40$, and $F_{0.3}{}^{T} = 2.59$. It follows from $M/R_0 = (2\lambda^2/3\xi)^{1/2}$ that if we let $\xi = 0.06$, then $M \approx R_0$ and thus we are at the limit of applicability of our approximation scheme. The amount of energy radiated per unit proper time at $\tau = 3.1\ M$ is then $\sim 1.7 \times 10^{55}$ ergs s^{-1} of which $\sim 1.9\%$ is due to the tidal interference radiation. On the other hand, when $\lambda = 0.5$, $F_\lambda{}^{T}$ has an oscillatory character for $\tau \geq 10\ M$ with a period of $\frac{1}{2}T_2 \approx 4.3\ T_0$ and an average value of $F_{0.5}{}^{T} \approx 0.11$. Thus, with $R_0 \approx 0.8\ M$ and $\xi = 0.1$ we get an average flux of $\sim 1.2 \times 10^{51}$ ergs s^{-1} from the purely tidal contribution. This flux lasts until all the excess energy of the model star, $\sim 3 \times 10^{-3}\xi(mc^2)$, is radiated away. For a neutron star of $m \approx 2$–$3\ M_\odot$ this time is of the order of 1 second.

In the motion of a body in a gravitational field the presence of the tides and the emission of gravitational radiation due to the center-of-mass motion continually alter the path of the center of mass, while the tidal radiation damps the tidal deformation. Thus when a star approaches a black hole these processes tend to

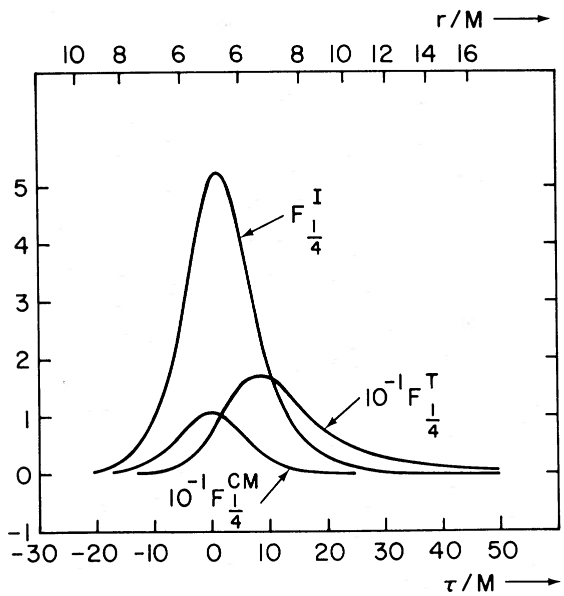

FIG. 4

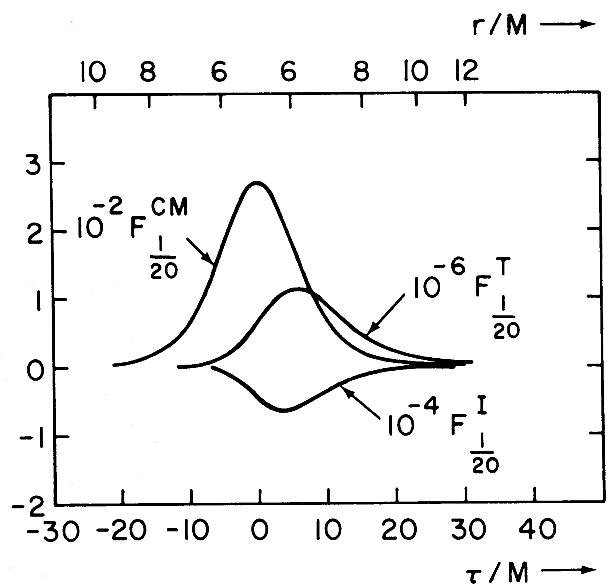

FIG. 5

FIG. 4.—The radiation functions for $\lambda = 0.25$, $\lambda \equiv (\rho/\rho_{\mathrm{B.H.}})^{1/2}$, versus proper time (and the radial coordinate r) for an initially $(-\tau \gg M)$ spherical nonrotating fluid ellipsoid whose center of mass follows the trajectory of Fig. 1.

FIG. 5.—The radiation functions for $\lambda = 0.05$, $\lambda \equiv (\rho/\rho_{\mathrm{B.H.}})^{1/2}$, versus proper time (and the radial coordinate r) for an initially $(-\tau \gg M)$ spherical nonrotating fluid ellipsoid whose center of mass follows the trajectory of Fig. 1.

increase the probability of its capture by the collapsed body. Therefore, gravitational radiation can have significant dynamical effects in the evolution of a dense stellar system. Moreover, if the star is endowed with a strong magnetic field, then the tidal deformations of the body are followed by variations in the flux of the radiation emitted by charged particles in its magnetic field. If the tidal deformations have an oscillatory character, this tidal electromagnetic radiation results in a similar modulation of the electromagnetic flux.

APPENDIX A

GENERAL DEVIATION EQUATION

In this Appendix general methods are developed for the derivation of the deviation equation. In part (*a*) the general deviation equation is given to first order in the Fermi frame, and in part (*b*) the nonrelativistic geodesic deviation equation is found to third order. The geodesic deviation equation was first discussed by Levi-Civita (1926) and Synge (1926) following the earlier work of Jacobi. The paper of Levi-Civita contains a detailed exposition of the original theorem of Fermi (1922) that it is possible to choose a coordinate system in the neighborhood of any open curve such that all the connection coefficients vanish and the metric tensor is Minkowskian along the curve. This coordinate system, which coincides with the Fermi frame when the curve is a geodesic, was then used by Levi-Civita (1926) in the discussion of the geodesic deviation equation.

a) Relativistic Case

Consider a congruence of initially neighboring curves that have an arbitrary rate of separation. Let C be a characteristic curve in the congruence along which a Fermi coordinate frame is constructed as in § II. In the derivation of the deviation equation to a given order it is useful to have the Taylor expansion of the metric tensor near the base curve C to one order higher. All connection coefficients vanish along C except for $\Gamma^{*i}{}_{00} = \Gamma^{*0}{}_{0i} = A^*{}_i(\tau)$, hence

$$g^*{}_{00} = -1 - 2A^*{}_iX^i + \tfrac{1}{2}(g^*{}_{00,ij})_0X^iX^j + \cdots, \tag{A1}$$

$$g^*{}_{0i} = \tfrac{1}{2}(g^*{}_{0i,jk})_0X^jX^k + \cdots, \tag{A2}$$

$$g^*{}_{ij} = \delta_{ij} + \tfrac{1}{2}(g^*{}_{ij,kl})_0X^kX^l + \cdots. \tag{A3}$$

To arrive at expressions for the partial derivatives of $g^*{}_{\mu\nu}$ along C in equations (A1)–(A3) it is necessary to use the fact that a curve of constant τ is a geodesic. Then it follows from equations (12) and (13) that in the Fermi frame

$$\Gamma^{*\mu}{}_{ijk\ldots} = 0 \quad \text{along } C, \tag{A4}$$

where $\Gamma^\mu{}_{\nu\rho\sigma}$, etc., are expressions involving the Christoffel symbols and their derivatives (see Eisenhart 1926 for their definition).

The general method may be illustrated by evaluating $g^*{}_{\mu\nu}$ to second order. The definition of the Riemann tensor implies that

$$\tfrac{1}{2}g^*{}_{00,ij} = -(R^*{}_{0i0j} + A^*{}_iA^*{}_j) \quad \text{along } C. \tag{A5}$$

For the case under consideration it is sufficient to consider $(\Gamma^{*\mu}{}_{ijk})_0 = 0$, or

$$\Gamma^{*\mu}{}_{(ij,k)} - 2\Gamma^{*\mu}{}_{\nu(i}\Gamma^{*\nu}{}_{jk)} = 0 \quad \text{along } C. \tag{A6}$$

To evaluate $(g^*{}_{0i,jk})_0X^jX^k$ consider (A6) for $\mu = 0$, which implies that

$$g^*{}_{0(i,jk)} = 0 \quad \text{along } C. \tag{A7}$$

This relation together with $2R^*{}_{0kij} = g^*{}_{0j,ik} - g^*{}_{0i,jk}$ along C results in

$$\tfrac{1}{2}g^*{}_{0i,jk}X^jX^k = -\tfrac{2}{3}R^*{}_{0jik}X^jX^k \quad \text{along } C. \tag{A8}$$

In a similar way, $g^*{}_{ij}$ can be evaluated using equation (A6) for $\mu = l$, hence

$$G^*{}_{ijkl} \equiv g^*{}_{i(j,kl)} - \tfrac{1}{2}g^*{}_{(jk,l)i} = 0 \quad \text{along } C. \tag{A9}$$

When this equation is combined with $G^*{}_{jikl} = 0$ they result in

$$2g^*{}_{ij,kl} + g^*{}_{l(i,j)k} + g^*{}_{k(i,j)l} = g^*{}_{kl,ij} \quad \text{along } C. \tag{A10}$$

On the other hand, $G^*_{kjil} = 0$ can be written as

$$\tfrac{1}{2}g^*_{ij,kl} + g^*_{l(i,j)k} - 2g^*_{k(i,j)l} = g^*_{kl,ij} \quad \text{along } C . \tag{A11}$$

Equations (A10) and (A11) imply that along C

$$g^*_{ij,kl} + g^*_{ik,jl} + g^*_{jk,il} = 0 \tag{A12}$$

and

$$g^*_{ij,kl} = g^*_{kl,ij} . \tag{A13}$$

It then follows from the definition of the Riemann tensor that along C

$$R^*_{ikjl} = g^*_{jk,il} - g^*_{ij,kl} , \tag{A14}$$

which together with (A12) and (A13) imply

$$\tfrac{1}{2}g^*_{ij,kl}X^kX^l = -\tfrac{1}{3}R^*_{ikjl}X^kX^l \quad \text{along } C . \tag{A15}$$

The Taylor expansion of the inverse matrix $g^{*\mu\nu}$ and the connection coefficients $\Gamma^{*\mu}{}_{\nu\rho}$ can be simply obtained from (A5), (A8), and (A15). The results are

$$g^{*00} = -1 + 2A^*{}_iX^i + (R^*{}_{0i0j} - 3A^*{}_iA^*{}_j)X^iX^j + \cdots , \tag{A16}$$

$$g^{*0i} = -\tfrac{2}{3}R^*{}_{0jik}X^jX^k + \cdots , \tag{A17}$$

$$g^{*ij} = \delta_{ij} + \tfrac{1}{3}R^*{}_{ikjl}X^kX^l + \cdots , \tag{A18}$$

$$\Gamma^{*0}{}_{00} = \frac{dA^*{}_i}{d\tau} X^i , \tag{A19}$$

$$\Gamma^{*0}{}_{0i} = A^*{}_i + (R^*{}_{0i0j} - A^*{}_iA^*{}_j)X^j , \tag{A20}$$

$$\Gamma^{*0}{}_{ij} = \tfrac{2}{3}R^*{}_{0(ij)k}X^k , \tag{A21}$$

$$\Gamma^{*i}{}_{00} = A^{*i} + (R^*{}_{0i0j} + A^*{}_iA^*{}_j)X^j , \tag{A22}$$

$$\Gamma^{*i}{}_{0j} = -R^*{}_{0kij}X^k , \tag{A23}$$

$$\Gamma^{*i}{}_{jk} = -\tfrac{2}{3}R^*{}_{i(jk)l}X^l . \tag{A24}$$

The expansion of $g^*_{\mu\nu}$ to second order in the special case that C is a geodesic has been obtained by Manasse and Misner (1963) using a different method.

The equation for an arbitrary curve in the congruence is

$$\frac{d^2x^{*\mu}}{ds^{*2}} + \Gamma^{*\mu}{}_{\nu\rho}\frac{dx^{*\nu}}{ds^*}\frac{dx^{*\rho}}{ds^*} = A^{*\mu}(\tau, x^{*i}) , \tag{A25}$$

where $-ds^{*2} = g^*_{\mu\nu}dx^{*\mu}dx^{*\nu}$. From the definition $\Gamma \equiv d\tau/ds^*$ it follows that

$$\Gamma^{-2} = 1 - \dot{X}^i\dot{X}_i + 2A^*{}_iX^i + (R^*{}_{0i0j} + \tfrac{4}{3}R^*{}_{0ijk}\dot{X}^k - \tfrac{1}{3}R^*{}_{ikjl}\dot{X}^k\dot{X}^l + A^*{}_iA^*{}_j)X^iX^j + \cdots , \tag{A26}$$

and equation (A25) can be written as

$$\frac{d^2x^{*\mu}}{d\tau^2} + \left(\frac{1}{\Gamma}\frac{d\Gamma}{d\tau}\right)\frac{dx^{*\mu}}{d\tau} + \Gamma^{*\mu}{}_{\nu\rho}\frac{dx^{*\nu}}{d\tau}\frac{dx^{*\rho}}{d\tau} = \Gamma^{-2}A^{*\mu}(\tau, X^i) . \tag{A27}$$

The general deviation equation (in the Fermi frame) can be obtained from (A27), and it can be written to first order as

$$\begin{aligned}\ddot{X}^i &+ (\delta_{il}\delta_{jk} - 2\delta_{ik}\delta_{jl})A^*{}_l\dot{X}^j\dot{X}^k + (R^*{}_{0j0k} - A^*{}_jA^*{}_k)(\delta_{ik} - 2\dot{X}^i\dot{X}^k)X^j - \frac{dA^*{}_j}{d\tau}\dot{X}^iX^j \\ &- \gamma^{-2}(A^*{}_{0,j}\dot{X}^i + A^{*i}{}_{,j})X^j - \tfrac{2}{3}(3R^*{}_{0jik} + R^*{}_{0lkj}\dot{X}^i\dot{X}^l - R^*{}_{iljk}\dot{X}^l)\dot{X}^kX^j = 0\end{aligned} \tag{A28}$$

where $\gamma^{-2} \equiv 1 - \dot{X}^i\dot{X}_i$ and $A^*{}_{\alpha,j} = A_{\mu;\nu}\lambda^\mu{}_{(\alpha)}\lambda^\nu{}_{(j)}$. In the absence of the acceleration field, equation (A28) can also be obtained (Paper I, Appendix) from a result of Hodgkinson (1972).

b) *Nonrelativistic Case*

The nonrelativistic limit of the deviation equation can, of course, be obtained from the general one in the approximation that $|\dot{X}^i| \ll 1$. However, it is simpler to develop a general procedure for dealing with this special case. Let the congruence of curves under consideration be neighboring and their rate of separation be small compared with unity. In a general coordinate frame the base curve C is given by $x^\mu = x^\mu(\tau)$. Let $x'^\mu(s^*)$ represent a point P on a neighboring curve connected to point 0 (with coordinates x^μ) on C as in § II. Let $\eta^\mu \equiv \sigma\xi^\mu$; then

$$x'^\mu = x^\mu + \eta^\mu - \frac{1}{2!}\Gamma^\mu{}_{\nu\rho}(\tau)\eta^\nu\eta^\rho - \frac{1}{3!}\Gamma^\mu{}_{\nu\rho\omega}(\tau)\eta^\nu\eta^\rho\eta^\omega - \cdots, \tag{A29}$$

so that $\lambda'^\mu(s^*) \equiv dx'^\mu/ds^*$, the tangent vector to the neighboring curve, is given by

$$\lambda'^\mu(s^*) = \Gamma\frac{dx'^\mu}{d\tau} = \Gamma\left(\lambda^\mu + \frac{d\eta^\mu}{d\tau} - \frac{1}{2}\frac{d\Gamma^\mu{}_{\nu\rho}}{d\tau}\eta^\nu\eta^\rho - \Gamma^\mu{}_{\nu\rho}\frac{d\eta^\nu}{d\tau}\eta^\rho - \cdots\right), \tag{A30}$$

where Γ may be determined from

$$-\Gamma^{-2} = g_{\mu\nu}(x')\frac{dx'^\mu}{d\tau}\frac{dx'^\nu}{d\tau}.$$

It follows from equation (A29) that

$$\Gamma^{-2} = 1 - 2\lambda_\mu\frac{D\eta^\mu}{D\tau} - \left(\frac{D\eta^\mu}{D\tau}\right)^2 + R_{\mu\nu\rho\sigma}(\tau)\lambda^\mu\eta^\nu\lambda^\rho\eta^\sigma + \cdots, \tag{A31}$$

which corresponds to equation (A26) with

$$\frac{D\eta^\mu}{d\tau} = \dot{X}^i\lambda^\mu{}_{(i)} + A^*{}_iX^i\lambda^\mu. \tag{A32}$$

The requirement that the rate of separation of the curves be small compared with unity implies that

$$\lambda'^\mu = \lambda^\mu + (\lambda'^\mu{}_{,\nu})_0(x'^\nu - x^\nu) + \frac{1}{2!}(\lambda'^\mu{}_{,\nu\rho})_0(x'^\nu - x^\nu)(x'^\rho - x^\rho) + \cdots, \tag{A33}$$

which to second order may be written as

$$\lambda'^\mu = \lambda^\mu + \lambda^\mu{}_{,\nu}(\eta^\nu - \tfrac{1}{2}\Gamma^\nu{}_{\rho\sigma}\eta^\rho\eta^\sigma) + \tfrac{1}{2}\lambda^\mu{}_{,\nu\sigma}\eta^\nu\eta^\sigma + \cdots. \tag{A34}$$

It follows from equations (A30), (A31), and (A34) that to first order (cf. Paper I, Appendix)

$$(g^\mu{}_\nu + \lambda^\mu\lambda_\nu)\frac{D\eta^\nu}{D\tau} = \lambda^\mu{}_{;\rho}\eta^\rho. \tag{A35}$$

The derivation of the deviation equation to orders higher than the first is much simplified if it is assumed that the congruence is geodesic. The rate of change of η^μ along C is found, after some algebra, to be

$$\frac{D\eta^\mu}{D\tau} = \lambda^\mu{}_{;\nu}\eta^\nu + (g^{\mu\nu} + \lambda^\mu\lambda^\nu)\mathscr{P}_{\nu\rho\sigma}\eta^\rho\eta^\sigma + \cdots, \tag{A36}$$

where $\mathscr{P}_{\nu\rho\sigma}$ is defined by equation (36). The covariant differentiation of equation (A36) results in

$$\frac{D^2\eta^\mu}{D\tau^2} + R^\mu{}_{\rho\nu\sigma}\lambda^\rho\eta^\nu\lambda^\sigma - \tfrac{1}{2}(g^{\mu\nu} + \lambda^\mu\lambda^\nu)(\Lambda_{\nu\rho\sigma} + R_{\nu\rho\omega\sigma;\xi}\lambda^\omega\lambda^\xi$$
$$+ R^\omega{}_{\rho\xi\sigma}\lambda_{\nu;\omega}\lambda^\xi + R_{\nu\xi\omega\sigma}\lambda^\xi{}_{;\rho}\lambda^\omega + R_{\nu\rho\omega\xi}\lambda^\xi{}_{;\sigma}\lambda^\omega)\eta^\rho\eta^\sigma + \cdots = 0, \tag{A37}$$

where $\Lambda_{\nu\rho\sigma}$ is defined by

$$\Lambda_{\nu\rho\sigma} = \lambda_{\nu;\rho\sigma\omega}\lambda^\omega + \lambda_{\nu;\omega}\lambda^\omega{}_{;\rho\sigma} + \lambda_{\nu;\omega\rho}\lambda^\omega{}_{;\rho} + \lambda_{\nu;\rho\omega}\lambda^\omega{}_{;\sigma}. \tag{A38}$$

The identities

$$(\lambda_{\mu;\omega}\lambda^\omega)_{;\rho\sigma} = \lambda_{\nu;\omega\rho\sigma}\lambda^\omega + \lambda_{\nu;\omega\rho}\lambda^\omega{}_{;\sigma} + \lambda_{\nu;\omega\sigma}\lambda^\omega{}_{;\rho} + \lambda_{\nu;\omega}\lambda^\omega{}_{;\rho\sigma} = 0, \tag{A39}$$

and

$$\lambda_{\nu;\omega\rho\sigma} - \lambda_{\nu;\rho\sigma\omega} = \lambda_{\xi;\sigma}R^\xi{}_{\nu\omega\rho} + \lambda_{\xi;\rho}R^\xi{}_{\nu\omega\sigma} + \lambda_{\nu;\xi}R^\xi{}_{\rho\omega\sigma} + \lambda_\xi R^\xi{}_{\nu\omega\rho;\sigma}, \tag{A40}$$

may be combined to show that

$$\Lambda_{\nu\rho\sigma} + R_{\xi\nu\omega\rho;\sigma}\lambda^{\xi}\lambda^{\omega} + R_{\xi\nu\omega\rho}\lambda^{\xi}{}_{;\sigma}\lambda^{\omega} + R_{\xi\nu\omega\sigma}\lambda^{\xi}{}_{;\rho}\lambda^{\omega} + R^{\xi}{}_{\rho\omega\sigma}\lambda_{\nu;\xi}\lambda^{\omega} + R_{\xi\nu\omega\rho}\lambda^{\xi}\lambda^{\omega}{}_{;\sigma} = 0\,. \tag{A41}$$

When the expression for $\Lambda_{\nu\rho\sigma}$ given in equation (A41) is substituted in (A37), the geodesic deviation equation to second order takes the form

$$\frac{D^2\eta^{\mu}}{D\tau^2} + R^{\mu}{}_{\rho\nu\sigma}\lambda^{\rho}\eta^{\nu}\lambda^{\sigma} + (g^{\mu\nu} + \lambda^{\mu}\lambda^{\nu})(\mathscr{R}_{\nu\xi\rho\omega\sigma}\lambda^{\xi} + 2R_{\nu\xi\sigma\omega}\lambda^{\xi}{}_{;\rho})\lambda^{\omega}\eta^{\rho}\eta^{\sigma} + \cdots = 0\,. \tag{A42}$$

In a similar manner all the higher-order terms in the deviation equation may be derived. However, to obtain the third-order term it is simpler to combine equation (A36) with a third-order equation derived by Hodgkinson (1972, eq. [2.51]). The result as it would appear on the left side of equation (A42) is of the form $\Phi^{\mu}{}_{\rho\sigma\zeta}\eta^{\rho}\eta^{\sigma}\eta^{\zeta}$, where $\Phi^{\mu}{}_{\rho\sigma\zeta}$ is given by

$$\begin{aligned}\Phi^{\mu}{}_{\rho\sigma\zeta} = {} & (2R_{\pi\xi\sigma\omega}\lambda^{\xi}{}_{;\rho} + \mathscr{R}_{\pi\xi\rho\omega\sigma}\lambda^{\xi})\lambda^{\pi}\lambda^{\omega}\lambda^{\mu}{}_{;\zeta} + 2(g^{\mu\nu} + \lambda^{\mu}\lambda^{\nu})(g^{\xi\pi} + \lambda^{\xi}\lambda^{\pi})\mathscr{P}_{\pi\rho\zeta}R_{\nu\xi\sigma\omega}\lambda^{\omega} + (g^{\mu\nu} + \lambda^{\mu}\lambda^{\nu}) \\ & \times (\mathscr{S}_{\nu\pi\rho\omega\sigma}\lambda^{\pi}\lambda^{\omega}{}_{;\zeta} + \tfrac{2}{3}R_{\nu\pi\rho\omega}\lambda^{\omega}{}_{;\zeta}\lambda^{\pi}{}_{;\sigma} + \mathscr{T}_{\nu\pi\rho\xi\sigma\zeta}\lambda^{\pi}\lambda^{\xi})\,. \end{aligned} \tag{A43}$$

The Newtonian approximation to the deviation equation may be discussed using the fact that in this limit the only nonvanishing connection coefficient and Riemann tensor component (except for the symmetries of the Riemann tensor) are $\Gamma^{i}{}_{00} = \phi_{,i}$ and $R^{i}{}_{0j0} = \phi_{,ij}$, where ϕ is the gravitational potential with $|\phi| \ll 1$. The geodesic deviation equation (to third order) in the Fermi frame together with the Newtonian limit is given in § II.

APPENDIX B

TIDAL DEFORMATION EQUATIONS

The purpose of this Appendix is to discuss some general properties of the system of deformation equations (56)–(58). For the sake of simplicity let the only nonzero off-diagonal element of (K_{ij}) be K_{13} and set $\Omega_1 = \Omega_3 = \Lambda_1 = \Lambda_3 = 0$. In terms of the quantities $a_{\pm} = \frac{1}{2}(a_1 \pm a_3)$, $\omega_{\pm} = \Omega_2 \pm \Lambda_2$, $k_{\pm} = \frac{1}{2}(k_{11} \pm k_{33})$, and $k_0 = k_{13}$, the deformation equations can be written as

$$\frac{d^2}{d\tau^2}a_{\pm} + (k_{+} - \omega^2{}_{\mp})a_{\pm} + k_{-}a_{\mp} = 0\,, \tag{B1}$$

$$\frac{d^2}{d\tau^2}a_2 + k_{22}a_2 = 0\,, \tag{B2}$$

$$\frac{d}{d\tau}(a_{+}{}^{2}\omega_{-}) = \frac{d}{d\tau}(a_{-}{}^{2}\omega_{+}) = k_0 a_{+}a_{-}\,. \tag{B3}$$

The circulation of the fluid $C_2 = 2\pi(a_{+}{}^{2}\omega_{-} - a^{2}{}_{-}\omega_{+})$ is a constant of the motion since the viscosity is assumed to be absent and the tidal matrix is symmetric. Equation (B3) expresses the rate of change of the angular momentum L_2, $L_2 = (2m/5)(a_{+}{}^{2}\omega_{-} + a_{-}{}^{2}\omega_{+})$, due to the presence of tidal forces. In equations (B1)–(B3) we have $k_{+} = K_{+}$, $k_{22} = K_{22}$, and

$$k_0 = K_0 \cos 2\Theta - K_{-} \sin 2\Theta\,, \tag{B4}$$

$$k_{-} = K_0 \sin 2\Theta + K_{-} \cos 2\Theta\,, \tag{B5}$$

with $\Omega_2 = -d\Theta/d\tau$, where Θ is the angle of rotation of the body frame with respect to the tetrad frame in the negative X^2-direction.

The deformation equations depend on a given initial tetrad frame, or any other frame related to it by a proper spatial rotation. The tidal matrix in the body frame (k_{ij}) is invariant under all proper rotations of the spacelike tetrads. Improper transformations, however, produce changes in the deformation equations. Consider, for instance, an inversion which changes $\lambda^{\mu}{}_{(3)}$ into $-\lambda^{\mu}{}_{(3)}$. It follows from equations (53), (55), (B4), and (B5) that this transformation causes a change in the sign of Θ, $\omega_{\pm}$, and k_0. The same changes occur for the inversion $\lambda^{\mu}{}_{(1)} \to -\lambda^{\mu}{}_{(1)}$, whereas the deformation equations (B1)–(B5) remain invariant under $\lambda^{\mu}{}_{(2)} \to -\lambda^{\mu}{}_{(2)}$.

Inspection of equations (57)–(58) reveals the possible existence of a singularity when two of the semiaxes of the ellipsoid become equal. To investigate this in the context of equations (B1)–(B3), let $L_2 = \frac{4}{5}ml$ and

$$l(\tau) = \int_{\tau_0}^{\tau} k_0 a_{+}a_{-}d\tau'\,, \tag{B6}$$

where the initial angular momentum of the ellipsoid is assumed to be zero. Equations (B1) and (B3) can then be written as

$$\frac{d^2}{d\tau^2} a_\pm = l^2 a_\pm{}^{-3} - k_+ a_\pm - k_- a_\mp , \tag{B7}$$

$$\Omega_2 = \tfrac{1}{2} l (a_-{}^{-2} + a_+{}^{-2}) , \tag{B8}$$

$$\Lambda_2 = \tfrac{1}{2} l (a_-{}^{-2} - a_+{}^{-2}) , \tag{B9}$$

if the initial circulation is also assumed to be zero. It follows from equation (B7) that if $l(\tau) \neq 0$ as $|a_-| \to 0$, there is a "force" approaching infinity that repels $|a_-|$ from zero. Thus no singularity occurs in Ω_2 and Λ_2. This argument assumes the absence of viscous forces; otherwise, it is quite general. Evidence for this type of behavior can be found in the present work (cf. Tables 2–5) and also in some previous work of the author (Mashhoon 1972). It has also been noted in the context of further numerical work by Lattimer and Schramm (1976).

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