THE INTERPRETATION OF CYCLICAL PHOTOMETRIC VARIATIONS IN CERTAIN DWARF ME-TYPE STARS

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Abstract. A method of determination of parameter sets characterizing models of starspots is described. The method makes use of a systematic integral notation in the description of the darkening due to spots and optimization procedures to evaluate appropriate parameters. The method is applied to light curves of YY Gem and CC Eri. In the latter case a comparison is made with the results of Bopp and Evans's (1973) study. The physical meaning of the derived parameter set and possibly correlated effects are considered for YY Gem.

1. Introduction

In recent years interest has developed in the notion of 'starspots', or regions of relative darkness on stellar photospheres, and whether such ideas can be used to account for the cyclical photometric variations observed in certain stars (see, for example, Krzeminski, 1969; Evans, 1971; Torres *et al.*, 1972; Chugainov, 1966, 1971; Mullan, 1974). In particular, the concept has been applied to certain flare stars where quantitative results have been presented (Bopp and Evans, 1973).

Observations of YY Gem – the late-type flaring dwarf binary component of the Castor system – were carried out by the author in the winter of 1974–75 working with the co-operation of Japanese observers at Tokyo Astronomical Observatory. The work was described separately (Budding, 1975a) where particular attention was paid to points of observational procedure.

The starspot idea, with reference to dwarf Me-type stars, came from suggestions originally made by Kron in 1952, who first drew attention to the photometric irregularities of YY Gem. Quantitative details were, however, lacking in Kron's discussion of the system.

It is the purpose of this paper to establish a method for dealing with light curves which may exhibit the effects of long-lived starspots and to supply a set of optimal numerical parameters for a suitable representation of the spot in relation to the photosphere. The treatment will be found to be quite parallel to that of the author's method for the determination of optimal parameter sets to specify the characteristics of eclipsing binary light-curves (Budding, 1973). It will be applied to the observations of YY Gem and also to Evans's observations of CC Eri for comparison purposes. Closer attention to some particular features of the YY Gem system will be paid in Section 6.

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2. The Photometric Effect of Starspots

The photometric effect produced by a dark 'spot' on the photosphere of an observed star as it is moved around in relation to the line of sight by the axial rotation of the star will depend upon various quantities. As with the light curves of other kinds of photometric variable we may expect, after analysis of the observed variations, to be able to derive a set of n parameters k_j (or 'elements' as they are sometimes called) which will enable a theoretical function $l_c(k_1, k_2, \ldots, k_n; t)$ to represent, to a certain accuracy Δl, a set of m observations ($m > n$) of light intensity l_{0_t} at given times t_i.

We should not presume that a set of parameters k_j will be necessarily uniquely determinable from the observations, nor indeed can we presume the uniqueness of the underlying hypothesis to explain the observed effects. However, we may aim at a principle of simplicity in specifying a minimal number of parameters and assumptions which are capable of providing a satisfactory curve fit.

Hence, we shall suppose that the 'spot' is of circular outline, – i.e., as though formed by the intersection of a plane with a spherical star surface. The star is assumed to rotate uniformly about an axis of rotation which is inclined at angle i to the line of sight.

The description of light losses in an eclipsing system was extensively developed by Kopal (1942, 1959) in whose treatment integrals of the form

$$\pi\alpha_n^m = \iint_{\substack{\text{Eclipsed}\\\text{area}}} x^m z^n \,\mathrm{d}x\,\mathrm{d}y$$

played a fundamental role. In our treatment we shall, in a similar way, recourse to integrals of the form

$$\pi\sigma_n^m = \iint_{\substack{\text{Spot}\\\text{area}}} x^m z^n \,\mathrm{d}x\,\mathrm{d}y,$$

where the xyz rectangular coordinate system, convenient from the observers point of view, will be related to a spherical polar system in which it may be more natural to express coordinates of features on the surface of the star.

Thus, for instance, the latitude β and longitude λ of the spot centre, which we may take to be effectively constant for a given period of time, would count as two of the unknowns required to be determined from the observations. Along with these we shall introduce another five parameters sufficient for an initial simple representation to be described in this section. In Section 5 we shall consider the extension of the parameter set to more than seven quantities. Listing these parameters we have:

λ – longitude of spot centre,
β – latitude of spot centre,

i – inclination of the rotation axis of the line of sight,

γ – angular extent of spot. It is found to be slightly more convenient to express the spot size in angular measure. The apparent semi-major axis of the spot, k, is then simply given by $k = \sin\gamma$,

U – fiducial intensity level from which the darkening will be reckoned. Normally it would be set at unity,

κ_w – the ratio of the mean flux over the starspot to the normal photospheric flux (over spectral window w). κ_w is generally a small quantity for the stars and spectral ranges likely to be encountered,

u – the coefficient of linear limb-darkening which we expect to adequately account for the effects of limb-darkening over the range of observed light variations.

2.1. Coordinate transformation

Consider the radius of the spherical star to be the unit of length. We then erect a rectangular coordinate system ξ, η, ζ in which we may denote by ξ_0, η_0 and ζ_0 the coordinates of the centre of the spot. Recalling that, with respect to axes fixed in the body, longitude is usually defined by a negative rotation about the rotation axis (ζ) we may write

$$\xi_0 = \cos\lambda\cos\beta, \qquad \eta_0 = -\sin\lambda\cos\beta, \qquad \zeta_0 = \sin\beta.$$

In this way the ξ axis lies in the equatorial plane directed to a point of zero longitude (which may be conveniently defined by reference to a suitable epoch). To transfer from ξ, η, ζ to x, y, z, first rotate about the ζ axis by negative angle ϕ; then a rotation about the new η axis by positive angle i will bring the ζ axis into coincidence with the z axis. A third rotation, of say ψ, about the z axis may then be made to make the newly redirected ξ axis coincide with the x axis which is defined to pass through the centre of the apparent elliptical outline of the spot on the disk (see Figure 1). The third rotation is done for convenience in defining the integrand and integration limits – actually ψ does not appear explicitly in any of the resulting integrals.

We obtain

$$\begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \cos i & 0 & -\sin i \\ 0 & 1 & 0 \\ \sin i & 0 & \cos i \end{vmatrix} \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \xi \\ \eta \\ \zeta \end{vmatrix}. \qquad (2.1)$$

Since the y coordinate of the spot centre is zero, we can write for the coordinates of this point in the x, y, z system (d', 0, z_0). It may be easily shown that the separation of the apparent spot centre from the centre of the disk d is related to the separation of the actual central point of the spot d' by $d' = d(1-k^2)^{-1/2}$, k having been introduced already in the definition of the parameter γ. The introduction of the terms k, d was made in order to bring out certain formal resemblances of the σ-integrals

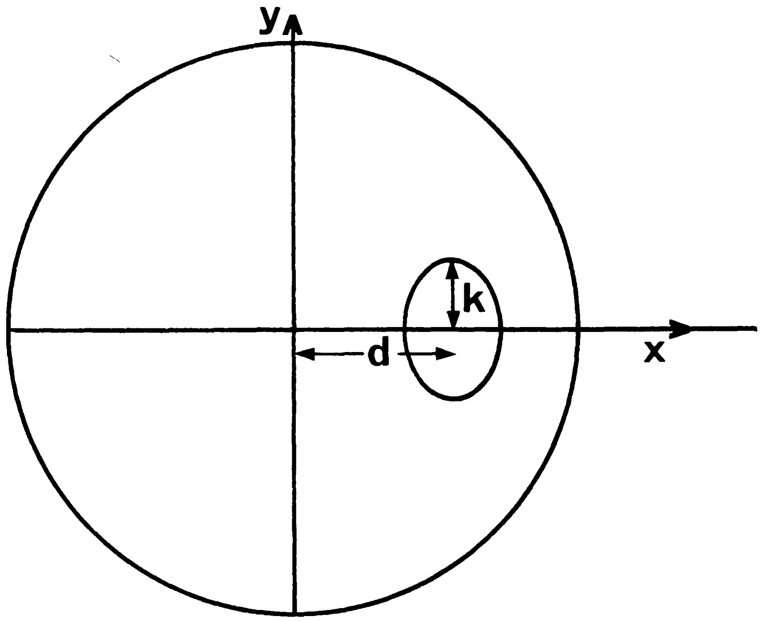

Fig. 1. Schematic representation of starspot of radius k whose apparent elliptical outline is centred about a point distant d from the centre of the star. x and y coordinate axes are chosen for convenience in the directions indicated.

to the α-integrals as previously discussed in the form $\alpha_n^m(k, d)$ (see, for example, Budding, 1973). For computation purposes, however, it became more convenient to consider σ_n^m as dependent on γ and $z_0 - z_0$ is, of course, simply related to d' by

$$z_0^2 = 1 - d'^2.$$

Multiplying out (2.1) we obtain

$$z_0 = \cos(\lambda - \phi)\cos\beta\sin i + \sin\beta\cos i. \tag{2.2}$$

2.2. σ-INTEGRALS

We may now write out the basic σ-integrals (i.e., σ_0^0 and σ_1^0) in a convenient form. In the Appendix the formulae necessary to evaluate the whole network of σ_n^m for arbitrary m and n are provided, where use is made of auxiliary quantities and recursion relations in a manner similar to that originated by Kopal (1947).

The forms of the basic σ-integrals are as follows:

(i) 'Annular case' (i.e., entire outline visible), $d \leqslant 1 - k^2$

$$\sigma_0^0 = k^2 z_0, \tag{2.3}$$

$$\sigma_1^0 = \frac{2}{3}\left[1 - \sqrt{1 - k^2}\left\{(1 - k^2) + \frac{3d^2k^2}{2(1 - k^2)}\right\}\right]; \tag{2.4}$$

(ii) 'Partial' case, $d > 1 - k^2$,

$$\sigma_0^0 = \frac{1}{\pi}\{\cos^{-1} s - s\sqrt{1-s^2} + k^2 z_0(\cos^{-1} v - v\sqrt{1-v^2})\}, \tag{2.5}$$

$$\sigma_1^0 = \frac{2}{3\pi}\left[\cos^{-1}\left(\frac{v}{s}\right) + \frac{\sqrt{1-k^2}}{2s}\{kz_0(3k^2-1)\sqrt{1-v^2} - - (2s(1-k^2) + 3dk^2)\cos^{-1} v\}\right], \tag{2.6}$$

where

k = 'spot' radius ($=\sin\gamma$);
d = apparent separation of spot centre from the star's centre;
z_0 = z-coordinate of spot centre;
$s = (1-k^2)/d$;
$v = (d-s)/kz_0$.

In the case of a 'totality' ($\gamma \geqslant \pi/2$, $d \leqslant 1-k^2$), we obtain the trivial values $\sigma_0^0 = 1$, $\sigma_1^0 = \frac{2}{3}$.

3. Application to Observations

As a test case, the foregoing formulae were first applied to Evans's (1959) observations of the star CC Eri. Evans remarked on the similarities between CC Eri and YY Gem, though it must be admitted that repetitive photometric features which can be easily and definitely linked to a starspot model seem to be much clearer in the 1956/57 series of observations of CC Eri than any which have so far been presented for YY Gem.

In fitting the observations to a theoretical curve use was made of the χ^2 statistic, defined in a similar way to that given previously by the author in dealing with eclipsing binary light curves (Budding, 1973).

If we first write

$$l_c = U\{1 - (1 - \kappa_w)\sigma_c(u, \gamma, z_0)\}, \tag{3.1}$$

then we may form χ^2 by

$$\chi^2 = \sum_{i=1}^{m} \frac{\{l_{0_i} - l_{c_i}\}^2}{\Delta l_i^2}. \tag{3.2}$$

Equation (3.1) as it stands contains four of the unknowns of the problem, i.e. U, κ_w, u and γ; the remaining three unknowns λ, β and i are involved through the definition of z_0 given in Equation (2.2). σ_c is the appropriately weighted spot-darkening function given by

$$\sigma_c = \frac{3}{(3-u)}\{(1-u)\sigma_0^0 + u\sigma_1^0\}, \tag{3.3}$$

where the explicit forms of σ_0^0 and σ_1^0 were written out in the preceding section. The suffix i on l_c in Equation (3.2) means that this quantity is to be calculated at the time of each observation of l_{0_i} measured in intensity units. The value of l_{c_i} evidently relates to the value of the phase ϕ_i which is to be substituted in Equation (2.2) for z_0. The conversion of time values into phases implies that the rotation period P and epoch of zero phase E are known quantities. In the systems which have been dealt with this can be regarded as being the case, though in principle the equations of condition could be used to determine the two additional parameters given by

$$\phi = 360 \frac{(t - NE)}{P} \quad \text{degrees},$$

where N is an integer chosen so that

$$0 \leqslant \frac{t - NE}{P} \leqslant 1.$$

An optimal curve fit is defined ('in the χ^2 sense') by the minimization of χ^2. This may be achieved by well known computational methods (see, for example, Bevington, 1969). The introduction of a large number of unknowns, particularly when the effect of variation of two or more parameters can combine to produce little resultant decrease in χ^2, decreases the accuracy of determination of any single one of them. It may be considered advantageous, therefore, if we can reasonably evaluate certain parameters independently. If the starspot variable happens also to be an eclipsing variable, as is the case with YY Gem, or a spectroscopic binary as is the case with CC Eri, the phases can be reckoned from an independently determined epoch, provided we can assume synchronism of rotation and revolution periods, although this may not hold true exactly in practice (Budding, 1975b). However, errors introduced by this assumption should not be too large provided the time-base of observational coverage is not too long.

An additional circumstance is, however, introduced if the spotted star happens to be a member of a close binary system. This is that the light level multiplying the fractional loss of light in Equation (3.1) is not the unit luminosity U, but the fractional luminosity of the star on which the spot is supposed to exist, which may be designated L_1 so that Equation (3.1) should in this case be replaced by

$$l_c = U - L_1(1 - \kappa_w)\sigma_c(u, \gamma, z_0). \tag{3.4}$$

4. Results

An optimal theoretical curve and corresponding geometric parameters appropriate to Evans's (1959) observations of CC Eri are given in Figure 2 and Table I. The results of Bopp and Evans (1973) are also presented for comparison purposes. In these results only the geometric parameters λ, β, i and γ have been optimized as these

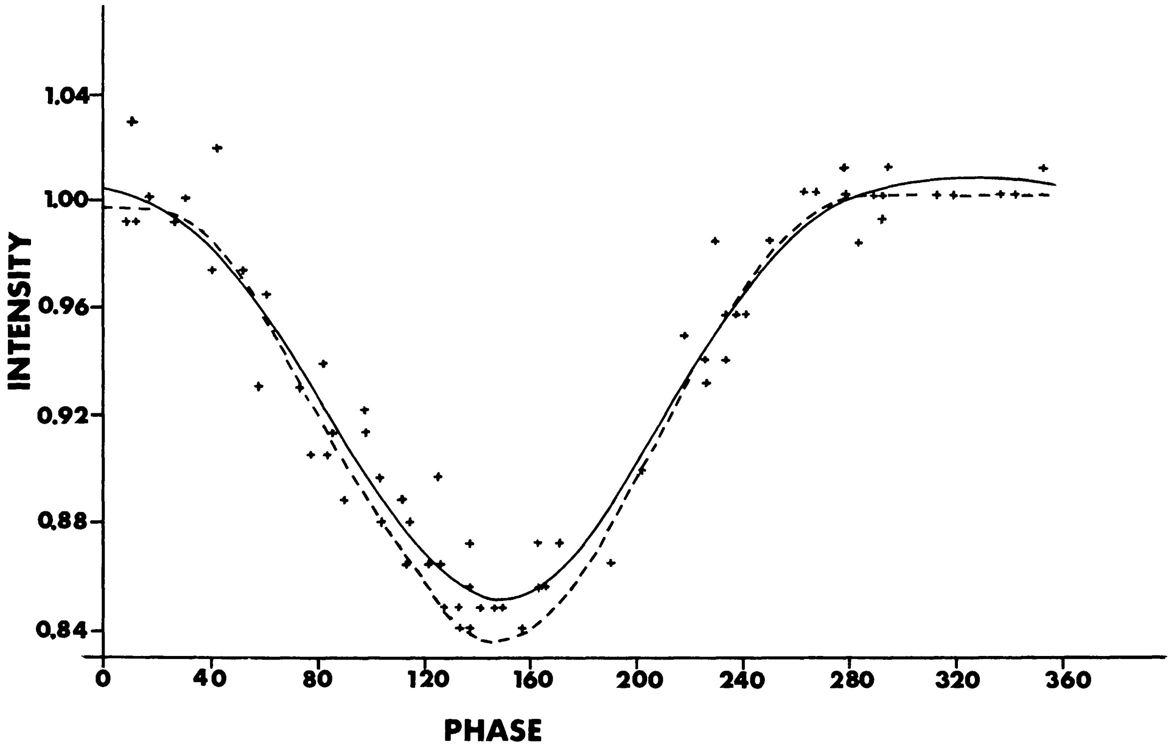

Fig. 2. The observations of Evans (1959) of CC Eri with an optimal curve fit as defined by the model of this paper shown by a continuous line. The dashed line indicates the curve constructed by Bopp and Evans (1973).

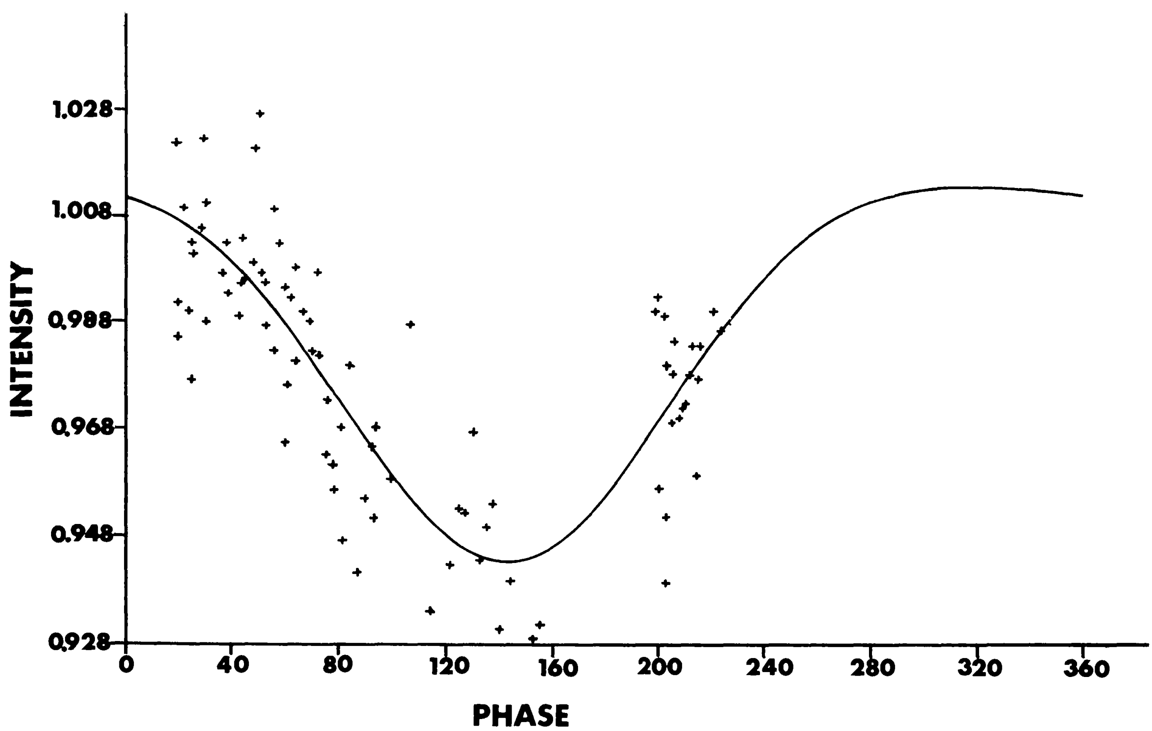

Fig. 3. An optimal curve fit, on the basis of the circular starspot model, to Budding's (1975a) observations of YY Gem. (The V-band observations have been used for this purpose.)

TABLE I

Starspot parameters for CC Eri (Evans's (1959)–1956/57 data)

Parameter	Optimal value	Bopp and Evans's (1973) value
λ	148°.6 ± 1.1	152°.5
β	38°.0 ± 4.7	10°
i	50°.1 ± 5°.0	42°
γ	21°.1 ± 0°.2	21°
m_0*	8^m.755 ± 0^m.005	8^m.76
κ_V	0.0	0.0
u	0.5	0.5
Δl	0.015	
χ^2	56.3	
ν†	62	

* In Table I and Table II the fiducial reference luminosity is given in the more directly relatable magnitude measure. This value is obtained in a simple way from the initial trial value for this reference magnitude (such as would normally be quoted in sources) and the fiducial intensity U, which would be exactly equal to unity if no further correction to the reference magnitude was required.

† ν is the number of observations minus the number of free unknowns.

TABLE II

Starspot parameters for YY Gem

Parameter	Optimal value
λ	143°.4 ± 1.7
β	43°.2 ± 6.0
i	55°.6 ± 6.5
γ	18°.4 ± 0.8
m_0	9^m.169 ± 0^m.006
κ_V	0.0
u	0.8
Δl	0.015
χ^2	68.2
ν	81

were the only elements sought by Bopp and Evans. A low value of the limb-darkening coefficient was retained in order to provide the comparison results for CC Eri, though this may be numerically inappropriate for the photosphere of a late type dwarf in the V range. Bopp and Evans argued reasonably in favour of a spot that was essentially black in comparison to the surrounding photosphere ($\kappa_V \approx 0$) and this hypothesis has been retained. Their assumption that the light contribution of the secondary star was negligible ($L_1 = U$) has also been held.

Similar assumptions were made for YY Gem. In this case, however, the value of the limb-darkening coefficient was raised to 0.8 in accordance with the value used in the analysis of the eclipses ($u=0.8$) (Budding, 1975a). Equality of primary and secondary luminosities was also assumed ($L_1=0.5U$), since this was determined to within the quoted probable errors of the eclipse solutions for the V-band light curves. The results for the out-of-eclipse light variation curve fit are presented in Table II and shown diagrammatically in Figure 3.

It will be noted that we cannot say, from the results as presented, on which the star spot is located, and this situation is likely to be generally true for close binary situations where the amplitude of light variation due to the spots is less than the fractional luminosity of either star. If, however, theoretical arguments can be advanced which favour particular spot sizes, it may be possible to deduce which of two stars is more likely to be spotted in a case like, say, BY Dra where the luminosity ratio is about $\frac{3}{2}$ (Bopp and Evans, 1973). Actually, part of the light variation due to the starspot, as assumed for the observations of YY Gem, is interrupted by the secondary eclipse. If observations were really precise, therefore, an appropriate decision could be made concerning which star is darkened by the supposed single spot. The observations under investigation, however, were not of sufficient number or quality to permit such a high-grade photometric resolution.

Concerning more detailed points of the results as presented in Tables I and II and Figures 2 and 3 the following remarks have some relevance. Bopp and Evans's (1973) numerical integration procedures for the evaluation of spot parameters seem somewhat inaccurate. Thus a 'spot' which is bounded by parallels of latitude and meridians of longitude is unrealistic. The doubtful value of their limb-darkening parameter has been mentioned already. The value of the inclination supports further consideration however. Bopp and Evans have assumed that the rotation axis of the primary star of CC Eri is parallel to that of the binary orbit whose value was obtained by independent spectroscopic means. The comparison value given, obtained by free optimization curve fitting, is not so far away from this value as could plausibly be expected – though 'forcing' the value of the orbital inclination onto the rotation axis may be reckoned a methodologically undesirable restriction. A significant difference between rotation and revolution axes also seems possible from the results on YY Gem, although it should be kept in mind that the quoted uncertainties refer to the properties of the curve fit for the particular model used, and may not be so realistic with reference to the true physical picture.

The longitude and spot size values for CC Eri compare reasonably well, though a fairly large discrepancy appears in the quoted latitudes. Actually, the duration of the darkening as a fraction of the total period imposes a strong correlation between the spot latitude and the rotation axis inclination of the form

$$\tan \beta = \cos \psi \tan i,$$

where

$$\frac{(\pi - \psi)}{\pi} = \frac{D}{P},$$

P being the period and D the duration just referred to. Indeed, the imposition of this constraint on the parameters was found not to affect the resulting optimal χ^2 value by more than a few per cent. A rough application of this constraint to Evans's 1956–57 observations yields $\beta \approx 30°$ even using the quoted value of 42° for the rotational inclination. Bopp and Evans's low value for the spot latitude in the 1956–57 observations of CC Eri is thus difficult to understand.

The quoted error estimates of these highly correlated variables have been obtained by noting their wander over a number of improvement iterations in the vicinity of the adopted χ^2 minimum at χ^2 values differing from this minimum value by unity – not, as, for instance, with the much less correlated longitude and size values, from their independent effects on the variation of χ^2. The error estimates are greater in the case of YY Gem which must be a reflection of the greater relative observational scatter in the case of this star.

5. Choice of Parameters

It was remarked in Section 2 that a minimum of seven independent parameters was required for the specification of a simple model to be used in providing a curve fit to the available data. The possibility of including a further two (or three in the case of close binaries) basic parameters as unknowns to be elucidated from the curve fit was suggested at the end of Section 3. The hypothesis of a circular spot and the use of definite integrals has, however, proved successful and advantageous in optimizing the curve fit.

We may now consider whether such a parameter set is really adequate or appropriate, and perhaps whether there could be circumstances in which the number of parameters in the set, or indeed the model should be changed.

In order to do this we can consider the objectives and circumstantial factors which concern the determination of the parameters. Since the stars under study are, in general, low intensity objects we can expect correspondingly reduced signal to noise ratios to operate in their observation, so that if magnitude determinations are correct to within $0\overset{m}{.}01$ we can consider that we are doing relatively well by typical standards. When we consider that the maximum amplitude of cyclical photometric variations of the type under consideration which have been discussed so far are of order $0\overset{m}{.}2$, we can quickly form some feeling for the probable significance to be attached to more complicated models. This point can also be assessed from the scatter of points about the theoretical curves in Figures 2 and 3.

Data on the spots observed on the much more easily observable solar photosphere are, to a very large extent, to be found used in statistical studies connected with the solar cycle. By analogy we might expect interest to develop in similar studies of starspots. For these purposes the parameters already used are probably quite adequate. Effective spot sizes as measured by γ, and positions given in terms of λ and β are representative values for statistical studies concerning, say, spot growths or declines, preferred latitudes or longitudes and migrations. Relative constancy of the determined value of i provides some additional check on model plausibility.

Of course, with better sets of observations the possibility of more detailed analysis becomes more feasible. The most likely first step in model development would be to include the provision of a description of the darkening in terms of more than one spot. A spot model which includes a 'penumbra' can be easily dealt with in the foregoing method by writing in place of Equation (3.1),

$$l_c = U[1 - \{p'\sigma_c(u, \gamma', z_0) + p''\sigma_c(u, \gamma'', z_0)\}], \tag{5.1}$$

where (say) $\gamma' < \gamma''$ so that p' and p'' would combine to represent the umbral effect while p'' alone represents the fractional flux loss in the penumbra. In these circumstances it would be realistic to write $p' = 1 - p'' = \kappa_w''$ where κ_w'' is the penumbral flux ratio. Whether the two contributions could be reliably separated is not obvious. Further refinements which can be dealt with by the present methodology are to allow a more complicated limb-darkening law than one which is merely linear in terms of the cosine of the angle of foreshortening, or else that the photosphere on which the spot is found is, to some extent, distorted from sphericity. In the former case an extension of Equation (3.3) to include higher order integrals $\sigma_i^0(i > 1)$ is all that is required. For the latter purpose more complicated expressions such as along the lines presented by Kopal (1959) are adaptable. The integrals involved in such representations will be considered in the Appendix.

6. Conclusions

We may take the results as presented to be evidence for the existence of large dark areas on stars like those of the system YY Gem. Moreover, the circular spot model and σ-integral formulation provides a useful and convenient means of curve fitting for the photometric variations associated with such dark areas. The adequacy of the given set of specification parameters for statistical studies of starspot variations seems reasonable. Mullan (1974) has argued theoretically that it is not unreasonable to expect spots covering an area of the order of 10–20% of the photospheres of late stars of the type in question. That such features last long enough for a consistent description to be derivable from the observations is supported by the work of Chugainov (1971) and others, as also, of course, in the earlier series of observations of Evans (1959) in addition to the more recent analysis of Bopp and Evans (1973). The fact that we cannot easily say, in the case of YY Gem, just which is darkened by the spot (or indeed whether or not spots of comparable size and similar longitude are present on both stars) may be taken as something of a limitation.

The existence of large spots on flare stars implies the existence of a whole range of related phenomena. To some extent this has already been noted in the correlation of flares with broad-band cyclical variations of intensity associated with spots (Torres *et al.*, 1972). Also, the possibility of a long term cycle of activity seems expectable. Bopp (1974a, b) has already drawn a distinction between 'active' and 'quiescent' phases for YY Gem. With such a categorization in mind, the interpretation could be made that in an observed sample of similar Me stars the percentage which are observed flaring represents some sort of average ratio of active to quiescent phases in these stars.

Concerning the formation of emission lines in the spectra of YY Gem, Joy and Sanford (1926) once gave rise to the interesting idea that the effect of rapid orbital motion through a resisting medium might be such as to cause the emission line forming atmospheric layer to be dragged behind, i.e., the receding star would tend to show the strengthened bright lines which they had observed as a repetitive feature. The stars do indeed move relatively rapidly – the revolution velocity at the centre of each star is about 120 km s^{-1} (Bopp, 1974b) – so that if there was some circumbinary medium at a nominal temperature of 10 000 K the stars would be immersed in a hypersonic flow regime with Mach number ~ 10, shock fronts being formed about either star. The existence of the associated shock fronts may have interesting consequences for the coronal flow regime, however, on the basis of free molecular flow notions for realistic particle densities in the hypothetical circumbinary material (a figure of 10^7 particles cm^{-3} quoted by Bopp (1974b) could be appropriate) it can be seen that the Balmer emission lines, which must predominantly originate in higher density chromospheric regions, cannot have any noticeable interaction with the medium. The results of Joy and Sanford may therefore presumably be interpreted in terms of a particular distribution of surface active regions at the time of their observation. Whether or not there may be preferred longitudes for starspots, particularly in the context of binary systems, is a question among those calling for further observational study.

Acknowledgements

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Appendix

Let us denote by $k' = kz_0$ the length of the minor axis of the apparent elliptical outline of the spot when seen in projection. We may then introduce an integral $F^m_{\beta,\gamma}$ by

$$\pi k'^{\beta+\gamma+m1} F^m_{\beta,\gamma} = \int_{d-k'}^{c} (x-d)^m y_e^\beta z_e^\gamma \, dx, \tag{A.1}$$

where

$$y_e = \sqrt{k^2 - \left(\frac{x-d}{z_0}\right)^2}, \qquad z_e = \frac{d(s-x)}{z_0\sqrt{1-k^2}},$$

and $c=s$ for 'partial' cases and $c=d+k'$ for 'annular' cases.

It may then be shown, without much difficulty, that the F-integrals satisfy the three recursion relations

$$(\gamma+1)\frac{\nu^2}{K^2} F^0_{-1,\gamma+1} = (2\gamma+1)\frac{\nu^2}{K} F^0_{-1,\gamma} + \gamma(1-\nu^2) F^0_{-1,\gamma-1}, \tag{A.2}$$

$$F^{m+1}_{\beta,\gamma} = \nu\left(\frac{F^m_{\beta,\gamma+1}}{K} - F^m_{\beta,\gamma}\right), \tag{A.3}$$

$$F^m_{\beta+2,\gamma} = \frac{1}{z_0^2}(F^m_{\beta,\gamma} - F^{m+2}_{\beta,\gamma}), \tag{A.4}$$

where we have abbreviated $K=(\sqrt{1-k^2})/k$.

Hence, if we know only $F^0_{-1,0}$ and $F^0_{-1,1}$, we may set up a line of integrals of the type $F^0_{-1,\gamma}$ $(\gamma > 1)$ using the first of these recursion relations (A.2). Using Equation (A.3) we may then proceed to evaluate integrals of the type $F^m_{-1,\gamma}$ $(m>0)$ in terms of the basic $F^0_{-1,\gamma}$ set. Finally, the third recursion relation enables us to determine any integral of the type $F^m_{\beta,\gamma}$ for odd positive β. Such integrals may then be used in the evaluation of integrals of the type σ^m_n by suitable recursion relations which will be presently set down. Let us first note that

$$F^0_{-1,0} = \frac{z_0}{\pi}\cos^{-1}\nu, \tag{A.5}$$

$$F^0_{-1,1} = \frac{z_0}{\pi} K\left(\cos^{-1}\nu - \frac{\sqrt{1-\nu^2}}{\nu}\right), \tag{A.6}$$

for the 'partial' case ($|\nu|<1$); or simply $F^0_{-1,0}=z_0$, $F^0_{-1,1}=z_0 K$ for the 'annular' case.

In fact, a simple substitution reveals that

$$F^0_{-1,\gamma} = \frac{z_0}{\pi}\left(\frac{K}{\nu}\right)^\gamma \int_0^c (\nu - \cos\phi)^\gamma \, d\phi, \tag{A.7}$$

where the upper label, $c=\pi$ for the 'annular' case, and $\cos^{-1}\nu$ for the 'partial' case.

The form of the F-integrals, the recursion relations they satisfy and their relationship to the σ-integrals suggests some parallel with the I-integrals introduced by Kopal

(1947) in connection with the α-integrals. In fact, it may be established that the F-integrals are formally the same as the I-integrals of even second subscript when multiplied by an appropriate factor and expressed in terms of the same argument – i.e.,

$$F^m_{\beta,\gamma}(\nu) = \frac{(-1)k'^{\gamma}I^m_{\beta,2\gamma}}{z_0^{(\beta+\gamma)}2^{\gamma}(1-k^2)^{\gamma/2}}.$$

The I-integrals of even second subscript were shown (Budding, 1974) not to involve elliptic integrals at any stage, and in fact, to be expressible in terms of simple algebraic and inverse trigonometric expressions. We therefore deduce this simplicity for the network of F-integrals and, correspondingly, the related σ-integrals.

Recursion relations formally similar to those established by Kopal (1947) for α-integrals exist also for the σ-integrals and they may be derived along similar lines. If we first convert to polar variables we may write

$$\pi\sigma^m_n = 2\int_{c_1}^{c_2}(1-r^2)^{n/2}r^{m+1}\int_0^{\theta_1}\cos^m\theta\,\mathrm{d}\theta\,\mathrm{d}r, \tag{A.8}$$

where $c_1=0$ if the spot covers the apparent centre of the star and otherwise $c_1=d-k'$, $c_2=1$ in the 'partial' case or $d+k'$ ($\leqslant 1$) in the 'annular' case and θ_1 is either π in the range where $c_1=0$ and $r<d-k'$, or else is defined by the angle subtended at the centre of the stellar disk between the radius vector to a point on the elliptical outline of a spot and the initial radius. We next notice that

$$\frac{\pi}{z_0}(-1)^m F^m_{-1,0}(\theta_1) = \int_0^{\theta_1}\cos^m\phi\,\mathrm{d}\phi; \tag{A.9}$$

so that

$$\pi m F^m_{-1,0} = \pi(m-1)F^{m-2}_{1,0} + z_0(-1)^m\sin\theta_1\cos^{m-1}\theta_1, \tag{A.10}$$

which allows us to write (as the second term on the right-hand side of Equation (A.10) disappears when $\theta_1=\pi$)

$$m\sigma^m_n + (m-1)\{\sigma^{m-2}_{n+2} - \sigma^{m-2}_n\} = \frac{2}{\pi}\int_{d-k'}^{c_2}(1-r^2)^{n/2}(r\cos\theta_1)^{m-1}\times$$
$$\times\, r\sin\theta_1\, r\,\mathrm{d}r. \tag{A.11}$$

This has the meaning of the line integral

$$\frac{2}{\pi}\int_{d-k'}^{c_2}(z^n_e x^{m-1}_e y_e)r\,\mathrm{d}r,$$

where the term in parentheses is a simple function of the Cartesian coordinates along the elliptical boundary (indicated by the suffix e) at radius vector r from the origin.

Along this boundary we have

$$x^2 - \frac{(x-d)}{z_0^2} = r^2 - k^2; \tag{A.12}$$

so that in place of $r\,\mathrm{d}r$ we may write (taking now x as the independent variable)

$$\left[x\left(1 - \frac{1}{z_0^2}\right) + \frac{d}{z_0^2}\right] \mathrm{d}x = -z_e \frac{K}{v} \mathrm{d}x. \tag{A.13}$$

If we now introduce the auxiliary integral

$$J^m_{\beta,\gamma} = \frac{1}{\pi} \int_{d-k'}^{c_2} x^m y_e^\beta z_e^\gamma \, \mathrm{d}x, \tag{A.14}$$

we find that

$$m\sigma_n^m + (m-1)\{\sigma_{n+2}^{m-2} - \sigma_n^{m-2}\} = 2dH_{1,n}^{m-1}, \tag{A.15}$$

where

$$H_{1,n}^m = \frac{1}{z_0^2}\left[J_{1,n}^m - \frac{d}{(1-k^2)} J_{1,n}^{m+1}\right] = \frac{1}{z_0\sqrt{1-k^2}} J_{1,n+1}^m. \tag{A.16}$$

A second recursion relation for the σ-integrals may be obtained by resorting to the polar coordinate formulation of the double integral. Combining (A.8) and (A.9) we observe that

$$\sigma_n^m = \frac{2}{z_0}(-1)^m \int_{c_1}^{c_2} (1-r^2)^{n/2} r^{m+1} F_{-1,0}^m(\theta_1)\,\mathrm{d}r \tag{A.17}$$

$$= \frac{2(-1)^m}{z_0(n+2)} \int_{c_1}^{c_2} (1-r^2)^{(n+2)/2} \frac{\mathrm{d}}{\mathrm{d}r}(r^m F_{-1,0}^m(\theta_1))\,\mathrm{d}r. \tag{A.18}$$

Replacing n by $n-2$, we have

$$n\sigma_{n-2}^m = \frac{2}{z_0}(-1)^m \int_{c_1}^{c_2} (1-r^2)^{n/2} \frac{\mathrm{d}}{\mathrm{d}r}(r^m F_{-1,0}^m(\theta_1))\,\mathrm{d}r; \tag{A.19}$$

while

$$(n+2)\sigma_n^m = n\sigma_{n-2}^m - \frac{2}{z_0}(-1)^m \int_{c_1}^{c_2} (1-r^2) r^2 \frac{\mathrm{d}}{\mathrm{d}r}(r^m F_{-1,0}^m(\theta_1))\,\mathrm{d}r, \tag{A.20}$$

i.e.

$$n\sigma_{n-2}^m - (m+n+2)\sigma_n^m = \frac{2}{z_0}(-1)^m \int_{c_1}^{c_2} (1-r^2)^{n/2} r^{m+2} \times$$

$$\times \frac{\mathrm{d}F_{-1,0}^m(\theta_1)}{\mathrm{d}r}\,\mathrm{d}r. \tag{A.21}$$

If $c_1=0$ (the spot covers the star centre) $\mathrm{d}F^m_{-1,0}/\mathrm{d}r=0$ for the range $0\leqslant r\leqslant d-k'$, otherwise

$$\frac{\mathrm{d}F^m_{-1,0}}{\mathrm{d}r}=\frac{-z_0}{\pi(-1)^m}\frac{\cos^m\theta_1}{\sin\theta_1}\frac{\mathrm{d}\cos\theta_1}{\mathrm{d}r} \tag{A.22}$$

and

$$\frac{\mathrm{d}\cos\theta_1}{\mathrm{d}r}=\frac{1}{r}\left(\frac{\mathrm{d}x}{\mathrm{d}r}-\frac{x}{r}\right)=\frac{1}{r}\left(-\frac{vr}{Kz_e}-\cos\theta_1\right). \tag{A.23}$$

Using Equation (A.23) we may now rewrite Equation (A.21) in terms of a combination of J-integrals along similar lines to Kopal's (1947) second recursion relation for α-integrals. We obtain

$$(m+n+2)\sigma^m_n-n\sigma^m_{n-2}=2G^m_{-1,n}, \tag{A.24}$$

where we have set

$$G^m_{-1,n}=J^m_{-1,n}-J^m_{-1,n+2}+\frac{K}{v}J^m_{-1,n+1},$$

$$=J^m_{-1,n}-\frac{\sqrt{1-k^2}}{z_0}J^m_{-1,n+1}. \tag{A.25}$$

The auxiliary J-integrals are easily relatable to the F-integrals, previously introduced as a determinable set. We have, in fact,

$$J^m_{\beta,\gamma}=k'^{\beta+\gamma+1}\sum_{j=0}^{m}k'^jd^{m-j}\binom{m}{j}F^j_{\beta,\gamma}. \tag{A.26}$$

We have thus established, in Equations (A.2), (A.3), (A.4), (A.5), (A.6), (A.16) and (A.25), the means whereby any integral of the type σ^m_n may be evaluated if we know σ^0_0 and σ^0_1. Such integrals would find a use when it is required to describe the darkening over a circular area when the brightness of the photosphere at a particular point (x, y, z) varies in some arbitrary dependence to powers of these coordinates, such as according to some gravity-darkening law. This situation could be associated with some distortion of the star from sphericity, perhaps brought about by its being one component of a close binary system. Under such circumstances it has been common practice to express the distortion in terms of spherical harmonic functions of polar coordinates in a form such as

$$r=r_0\{1+\Delta r(\theta,\phi)\}, \tag{A.27}$$

where r_0 has in the preceding discussion, for convenience, been taken to be unity. If now this distortion requires that we take account of a redistribution of emergent flux over the stellar surface, it ought also to imply consideration of distortion from circularity of the area originally presumed circular, i.e. the correction for surficial distortion of the star to the calculated loss of light due to a circular area along the lines discussed so far consists of two parts: a light redistribution component involving the integrals σ_n^m, and a 'boundary correction' component which can be shown to involve the integrals of the normal (undistorted) flux over an increased area, i.e.

$$2\int_{d-k'}^{c_2} \Delta y\,\mathrm{d}x,$$

where the Δy of this integral can be expressed by

$$\Delta y = -\frac{\Delta' r d(x-d)}{z_0\sqrt{k'^2-(x-d)^2}}, \tag{A.30}$$

where $\Delta' r$ is identical with Δr rewritten in the xyz coordinates. The 'boundary correction' term would then substantially involve integrals of the type

$$B^m_{-1,n} = \frac{d}{z_0^2}(J^{m+1}_{-1,n} - dJ^m_{-1,n}) = G^m_{-1,n} - k^2 J^m_{-1,n}. \tag{A.31}$$

The σ-integrals which have formed the subject of this appendix could also find a useful role in the description of 'reflection effects' in close binary systems. In this case it is not a darkening but a brightening which has to be accounted for, though there should be a circular symmetry as far as the incident radiation is concerned. Provided we can express the variation of this incident flux over the illuminated area together with foreshortening effects in its reradiation in terms of the x, y, z coordinates of the integrand, a more generalized account of reflection laws in terms of σ-integrals ought to be possible. A full discussion of this possibility is, however, outside the scope of the present paper.

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