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RAPIDLY ROTATING, POST-NEWTONIAN NEUTRON STARS

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ABSTRACT

Equilibrium models of rapidly rotating neutron stars supported by nonrelativistic, degenerate neutrons are examined in the Parametrized Post-Newtonian (PPN) framework. An energy variational principle is employed to analyze the effects of rapid rotation and relativistic gravity on the mass-density relation and maximum mass of neutron stars. The fractional increase in the mass of stable, equilibrium configurations due to rotation is smaller for neutron stars than for white dwarfs, a consequence of the destabilizing effects of post-Newtonian gravity. Currently viable conservative theories of gravity with high values of the PPN parameter β can have significantly larger masses than predicted by general relativity. Also discussed qualitatively is the evolution of transient, rotating compact objects called "fizzlers," which contract to form pulsars and black holes on secular (gravitational radiation) time scales.

Subject headings: relativity — stars: neutron — stars: rotation

I. INTRODUCTION

Preliminary measurements of the masses of neutron stars are now available. Analysis of the binary systems containing X-ray pulsars in Her X-1 and Vela X-1 indicate that $M \approx 1.3\ M_{\odot}$ for Her X-1 (Middleditch, Mast, and Nelson 1974) and $M \gtrsim 1.4\ M_{\odot}$ for Vela X-1 (Rappaport, Joss, and McClintock 1976). In roughly 5 years a dynamical determination, accurate to 10 percent, of the component masses in the binary system containing the pulsar PSR 1913 + 16 will be possible (Blandford and Teukolsky 1975). Knowing the total mass of this system, we may already establish an upper limit to the component masses, i.e., $M \leq 2.83\ M_{\odot}$ (Taylor *et al.* 1976).

The realistic possibility of observationally determining neutron star masses makes it imperative that theoretical calculations of the allowed mass range and structural features of neutron stars be improved. The allowed mass range for neutron stars depends upon (1) the equation of state, (2) the presence of rotation, and (3) relativistic gravity. Reliable mass limits can be obtained only after the correct local physical properties of neutron star matter *and* the correct global properties of (rapidly) rotating, relativistic configurations are simultaneously incorporated in these calculations. Theoretical work on the determination of neutron star mass limits has focused primarily on improving the equation of state for supranuclear matter (see Canuto 1974, 1975, for a review and references). Alternatively, adopting the equation of state above $\rho = 4.6 \times 10^{14}$ g cm^{-3} which maximizes the mass ($P = \rho c^2$), Rhoades and Ruffini (1973) found that the maximum mass of a nonrotating neutron star cannot exceed $\sim 3.2\ M_{\odot}$ in general relativity.

Since most neutron stars found in nature are likely to be rotating (e.g., radio and X-ray pulsars) and since experimental verification of general relativity is far from complete, the effects of rotation and different theories of relativistic gravity must be considered. A theoretical determination of the maximum mass of a neutron star will depend upon these effects and is crucial in the search for black holes.

The theory of rotating white dwarfs is well advanced. The early work of James (1964) on rotating, degenerate polytropes and the more recent work of Ostriker and Bodenheimer (1968), Ostriker and Tassoul (1969), and Durisen (1975) on differentially rotating, degenerate configurations in Newtonian theory have delineated in great detail the structural and stability properties of rotating white dwarfs. The theory of rotating neutron stars is not as well developed. Not only is the correct equation of state of dense matter still in doubt, but also the effects of rapid rotation in highly relativistic stars (except black holes) are not understood and are technically difficult to analyze. Adopting various approximation schemes and simplifying assumptions, several investigators have begun to analyze these effects in the last 10 years. Slowly rotating configurations in general relativity have been studied numerically by Hartle and his co-workers (see, e.g., Hartle 1967, Hartle and Munn 1975, for a recent discussion and references). Abramowicz and Wagoner (1976) have obtained analytic, though approximate, formulae for the mass of slowly rotating, relativistic neutron stars with uniform density. Rapidly rotating, uniform-density configurations in general relativity have been constructed numerically by Butterworth and Ipser (1975). The work of Butterworth and Ipser is particularly significant. Accurate models of rapidly rotating, fully relativistic fluid bodies are required for a quantitative stability analysis, since nonaxisymmetric instabilities occur at high rotation and radial instabilities occur at high compaction.

Post-Newtonian effects of different relativistic gravity theories have also been examined recently in connection with compact stars. Wagoner and Malone (1975) constructed nonrotating neutron star models in the framework of the Parametrized Post-Newtonian (PPN) formalism. Shapiro and Teukolsky (1976) examined analytically uniformly rotating white dwarfs in the PPN framework. The PPN framework categorizes all metric theories of gravity by their values for 10 "PPN parameters." Even for currently viable theories, observed properties, such as the masses and redshifts of neutron stars and white dwarfs, can differ considerably from general-relativistic predictions.

In this paper we explore in approximate fashion the effects of rapid rotation and relativistic gravity on the equilibrium structure of neutron stars. In particular, we determine the mass-density relations for post-Newtonian, rapidly rotating neutron stars. For simplicity, we employ the equation of state of a nonrelativistic Fermi gas, modified by a correction term which allows the neutrons to become slightly relativistic and which constitutes a small deviation from exact polytropic behavior. We adopt the PPN description of relativistic gravity and derive equations applicable to all metric theories possessing conservation laws. Equilibrium models and stability limits are determined by utilizing the energy variational principle of Zel'dovich and Novikov (1971) for (near) polytropes, generalized to the PPN formalism. The virtue of this approach is that it allows a qualitative understanding of the effects of rotation and relativistic gravity in a mathematically simple, unified framework. The shortcomings of our approximations are clear: the analysis is restricted to a nonrelativistic Fermi gas and to post-Newtonian configurations. Neutron stars near the maximum mass require the full theory of general relativity to obtain numerically accurate models.

We find that the ability of rotation to increase the masses of stable, equilibrium configurations is severely restricted in the case of neutron stars, in contrast to the situation with white dwarfs (where rotation can increase the maximum mass by a factor of two to $\sim 3\ M_{\odot}$ [Ostriker 1971]). We also find that currently viable "conservative" theories of gravity with high values of the PPN parameter β can have significantly larger equilibrium masses than predicted by general relativity. We also discuss, in a qualitative fashion, a generalization of the "standard" scenario for gravitational collapse to a compact object. In particular, we consider the evolution of transient, rotating, compact objects called "fizzlers." These stars slowly collapse to form pulsars and black holes, on secular time scales, by the gravitational radiation of angular momentum.

In § II we develop the methods and equations for determining equilibrium configurations of rotating neutron stars. In § III we review stability criteria for these configurations. In § IV we discuss our numerical results, and in § V we sketch possible evolutionary scenarios for rotating compact stars. In Appendix A we derive the polytropic, post-Newtonian structure constants employed in the energy variational expression. In Appendix B we prove that all polytropes with index $n \leq \frac{3}{2}(\gamma_{\mathrm{ad}} \geq \frac{5}{3})$ are stable to quasi-radial modes in the post-Newtonian approximation.

II. METHOD AND EQUATIONS

We analyze the structure of rotating neutron stars obeying the ideal, degenerate-neutron-gas equation of state by employing an energy variational method. This method is discussed in detail by Zel'dovich and Novikov (1971), whose treatment we follow closely. In this approach, the condition for hydrostatic equilibrium is equivalent to the requirement that the total energy of the star be an extremum for a given rest mass M and angular momentum J. A minimum of the energy corresponds to an equilibrium configuration that is dynamically stable against radial oscillations, while a maximum of the energy corresponds to an unstable equilibrium configuration.

We adopt geometrized units ($G = c = 1$) throughout the remainder of the paper. The total energy E of a uniformly rotating, ellipsoidal configuration supported against collapse by the pressure of nonrelativistic (NR) degenerate neutrons is given to post-Newtonian order by

$$E(M, J, \rho_c) = \tfrac{1}{5}k_1 M(\rho_c/\rho_0)^{2/3} - k_2 M^{5/3}\rho_c^{1/3}g(\lambda) + k_3\lambda J^2 M^{-5/3}\rho_c^{2/3} - k_4 M(\rho_c/\rho_0)^{4/3} - k_5 M^{7/3}\rho_c^{2/3}, \quad (1)$$

where

$$k_1 = 0.796, \qquad k_2 = 0.761, \qquad k_3 = 1.926,$$

$$k_4 = 0.0180, \qquad k_5 = 1.83 + 1.26\gamma - 2.41\beta.$$

In equation (1), ρ_c is the central rest mass density, and $\rho_0 \equiv m_n^4/(3\pi^2\hbar^3) = 6.11 \times 10^{15}\ \mathrm{g\ cm^{-3}}$ is the density at which the Fermi momentum of degenerate neutrons becomes relativistic. The parameters λ and $g(\lambda)$ are rotation parameters, defined below, and are equal to unity in the absence of rotation. The dimensionless constants k_i are polytropic structure constants evaluated above for an $n = \frac{3}{2}$ polytrope (appropriate for a NR degenerate Fermi gas). These constants, representing integrals of Lane-Emden functions over the star, are evaluated for polytropes of arbitrary index n in Appendix A. The PPN parameters γ and β appearing in k_5 allow for comparison of different relativistic theories of gravity. The various terms in equation (1) are discussed below.

a) *Discussion of Terms in the Energy Expression*

The first two terms in equation (1) represent the Newtonian internal and gravitational energies, respectively, of an $n = \frac{3}{2}$ polytrope. The third term is the rotational kinetic energy associated with the uniform rotation of a polytropic configuration. The fourth and fifth terms are the first-order corrections to the internal and post-Newtonian gravitational energies, respectively, evaluated for an $n = \frac{3}{2}$ polytrope.

The internal energy per unit mass $\pi(\rho)$ of an ideal Fermi neutron gas is given by

$$\pi(\rho) = \frac{3}{8x^3}\{x(1 + 2x^2)(1 + x^2)^{1/2} - \ln [x + (1 + x^2)^{1/2}]\} - 1 \tag{2}$$

(Zel'dovich and Novikov 1971) where $x \equiv (\rho/\rho_0)^{1/3}$ and ρ is the local rest mass density. To lowest order in x, we have, approximately,

$$\pi(\rho) \approx \tfrac{3}{10}(\rho/\rho_0)^{2/3} - \tfrac{3}{56}(\rho/\rho_0)^{4/3}, \quad (\rho/\rho_0) \ll 1\ . \tag{3}$$

The above expression, integrated over an $n = \frac{3}{2}$ polytrope, has been employed in evaluating the first and fourth terms in equation (1).

The oblateness parameter λ satisfies $\lambda = (c/a)^{2/3}$, where c and a are the polar and equatorial diameters, respectively, of the spheroidal core. The ellipsoidal deformation function $g(\lambda)$ is given by

$$g(\lambda) = \lambda^{1/2}(1 - \lambda^3)^{-1/2} \cos^{-1}(\lambda^{3/2}) \tag{4}$$

(Landau and Lifshitz 1962), and measures the weakening of gravity, in configurations of given M and ρ_c, due to rotational flattening. In writing equation (1), we follow Zel'dovich and Novikov (1971) and assume that in a rotating configuration the density is constant on similar ellipsoids of revolution. This approximation, which implies that the law of variation of the gravitational energy in the deformation of a star is the same as in an ellipsoidal deformation of a sphere of incompressible fluid, is adequate for the central regions of the star which contain most of the mass.

The expression for the rotational kinetic energy in equation (1) is strictly valid only for uniform rotation. The condition of uniform rotation, however, confines the ratio T/W, where T is the rotational kinetic energy of the star and W is the Newtonian gravitational binding energy, to lie in the narrow range $0 \leq T/W \leq 0.0353$ for an $n = \frac{3}{2}$ polytrope. The maximum value of this ratio is reached when the centrifugal force on the surface at the equator equals the gravitational force. Larger values of T/W lead to differential rotation in the outer layers of the star. Numerical studies of differentially rotating polytropes by Ostriker and Bodenheimer (1968) indicate that the above kinetic energy expression may nevertheless be employed in a first approximation to describe stable, axisymmetric, differentially rotating configurations. Differential rotation in stable stars is never very extreme ($\Omega_{\text{equator}}/\Omega_{\text{center}} > 0.2$), and most of the mass is located in a dense core which rotates nearly as a solid body.

The last term in the energy expression gives the contribution of relativistic gravity at post-Newtonian order. The constant k_5 appearing in this post-Newtonian term is derived in Appendix A for arbitrary polytropic gas spheres, and is given in equation (1) for $n = \frac{3}{2}$. In determining k_5 we consider the geometrical effect of gravity on volume measure, the effective mass of internal and gravitational energies, and the nonlinear nature of the gravitational field. We determine k_5 in the PPN Framework of Nordtvedt and Will (see Will 1973, for a review and references), thereby making minimum assumptions regarding the correct relativistic theory of gravity. At post-Newtonian order, any metric theory of gravity is defined completely by its values for 10 PPN parameters. Confining our analysis to theories which conserve energy, momentum, angular momentum, and center-of-mass motion ("conservative theories") restricts the general theory to only three free parameters (Lee, Lightman, and Ni 1974): γ, β, and ζ_w. Since solar system experiments to date require $|\zeta_w| < 1 \times 10^{-2}$ (Will 1973), we set $\zeta_w = 0$. The parameter γ measures space curvature while β measures the nonlinearity of gravity. General relativity predicts $\gamma = \beta = 1$, giving $k_5 = 0.681$ for an $n = \frac{3}{2}$ polytrope. Solar system experiments place the following limits on γ and β:

$\gamma = 1.030 \pm 0.022$ (NRAO deflection experiment of Fomalont and Sramek 1975);

$\beta = 1.14(+0.2, -0.3)$ (perihelion shift plus time-delay experiments; see Will 1973 for discussion).

In accordance with the rather large experimental uncertainty in the parameter β, we examine in § IV the effect of varying β in evaluating k_5.

We point out that equation (1) is a valid, self-consistent approximation to the total energy E of an equilibrium configuration only when $\rho/\rho_0 \ll 1$ and $M/R \ll 1$. For $\rho \geq \rho_0$, equation (3) must be replaced by a corresponding expression for relativistically degenerate neutrons and the first and fourth terms in equation (1) must be recalculated. When M/R becomes large and approaches unity, application of the post-Newtonian approximation breaks down, and an exact description of gravity must be used. We will nevertheless treat equation (1) as an exact expression in the following analysis and indicate regions in parameter space where it no longer remains physically reliable.

b) The Variational Principle and Equilibrium Equations

According to the energy method, equilibrium is determined by extremizing the energy E given by equation (1) with respect to λ and ρ_c for fixed M and J. The two equations $\partial E/\partial\lambda|_{M,J} = 0 = \partial E/\partial\rho_c|_{M,J}$ thus determine the equilibrium rest mass of the star as a function of J and ρ_c. The critical central density $\rho_{\max}$ which separates radially stable from unstable equilibrium states is obtained by simultaneously solving a third equation, $\partial^2 E/(\partial\rho_c)^2|_{M,J} = 0$. In a fully relativistic treatment limited to slow rotation, Hartle and Munn (1975) have shown that radial instability sets in very near the density at which the mass attains a maximum for given J.

The first equilibrium condition, $\partial E/\partial\lambda = 0$, determines λ as a function of the ratio T/W:

$$g'(\lambda) = \frac{k_3}{k_2}\frac{J^2\rho_c^{1/3}}{M^{10/3}} = \frac{T}{W}\frac{g(\lambda)}{\lambda}, \tag{5}$$

which yields, from equation (4), the relation

$$T/W = \frac{1}{2}\left[1 + \frac{3\lambda^3}{(1-\lambda^3)} - \frac{3\lambda^{3/2}}{(1-\lambda^3)^{1/2}\cos^{-1}\lambda^{3/2}}\right]. \tag{6}$$

The second equilibrium condition $\partial E/\partial\rho_c = 0$ yields a quadratic equation for the equilibrium rest mass as a function of ρ_c and T/W, with solution

$$M = \rho_0^{-1/2}[-a_1 \pm (a_1^2 + a_2)^{1/2}]^{3/2}, \tag{7}$$

where

$$a_1 = \frac{k_2 g(\lambda)}{4k_5}(\rho_c/\rho_0)^{-1/3}(1 - 2T/W), \quad a_2 = \frac{k_1}{5k_5} - 2\frac{k_4}{k_5}(\rho_c/\rho_0)^{2/3}.$$

The upper sign and lower signs are chosen according to whether $k_5 > 0$ or $k_5 < 0$, respectively. Equations (4), (5), (6), and (7) are four coupled equations for the four unknowns, λ, $g(\lambda)$, T/W, and M, for given ρ_c and J. In practice, it is simplest to first specify T/W, from which λ can be obtained by solving equation (6). Specification of ρ_c then allows calculation of M from equation (7), and subsequent computation of J from equation (5).

For each value of J there exists a maximum rest mass configuration. The mass and density of this configuration are obtained by solving the equation $\partial^2 E/\partial\rho_c^2|_{M,J} = 0$ simultaneously with the equilibrium solution, equation (7), yielding

$$M_{\max} = \rho_0^{-1/2}(8k_4/k_2 g)^{3/8} y^{9/8}, \tag{8a}$$

$$\rho_{\max}/\rho_0 = (k_2 g/8k_4)^{3/4} y^{3/4}. \tag{8b}$$

Here the parameter y satisfies the cubic equation

$$c_1 y^3 + c_2 y^2 + c_3 y - c_4 = 0, \tag{8c}$$

$$c_1 = (4k_5/3k_2 g)^2, \qquad c_2 = 2c_1^{1/2}(1 - 4T/3W),$$

$$c_3 = c_2^2/4c_1, \qquad c_4 = 2k_1^2/(225 k_2 k_4 g).$$

As in the computation of the equilibrium masses, the determination of $M_{\max}$ and $\rho_{\max}$ is most easily obtained by specifying a particular value of T/W and then solving for J.

Although the expression for the energy, equation (1), is accurate to post-Newtonian order, variation of equation (1) yields a rest mass accurate only to Newtonian order in M/R.

The inertial mass of the configuration M_i is related to the rest mass M by the relation $M_i = M + E(M, J, \rho_c)$. The inertial mass gives the total mass-energy of the star. The critical density at which quasi-radial modes change stability can be determined more exactly by locating the maximum value of M_i as a function of ρ_c for fixed J (Hartle and Munn 1975). In our calculations, M never exceeds M_i by more than a few percent, and the maximum value of M_i occurs very nearly at the same density as the maximum value of M. All numerical results are plotted in terms of M_i and not M, but the distinction is not significant in our post-Newtonian approximation scheme. Solutions to the above equations, together with a detailed discussion of their properties and range of validity, are presented in § IV.

III. NONAXISYMMETRIC INSTABILITIES

In Newtonian theory, axisymmetric, rotating equilibrium configurations exhibit secular and dynamic instabilities of nonradial modes for sufficiently large values of T/W. For Maclaurin spheroids and uniformly rotating polytropes, secular instability of a nonaxisymmetric toroidal ($L = 2$) mode for $T/W \gtrsim 0.14$ has been demonstrated in the presence of viscosity (Tassoul and Ostriker 1970; Press and Teukolsky 1973) and of gravitational radiation reaction (Chandrasekhar 1970). In the absence of viscosity, the above modes are unstable on a dynamic time scale if $T/W \gtrsim 0.26$. For $0.14 \lesssim T/W \lesssim 0.26$ the instability requires the presence of dissipation and grows

TABLE 1

ROTATION PARAMETERS

T/W	λ	$g(\lambda)$
0.000	1.000	1.000
0.020	0.952	1.000
0.035	0.915	0.998
0.040	0.904	0.998
0.060	0.859	0.995
0.080	0.815	0.992
0.100	0.773	0.987
0.120	0.732	0.981
0.140	0.693	0.974
0.160	0.655	0.966
0.180	0.617	0.956
0.200	0.581	0.945
0.220	0.545	0.933
0.240	0.519	0.923
0.260	0.476	0.903
0.280	0.441	0.885

on a dissipative time scale. The values of T/W at both the secular and dynamical points of instability are remarkably insensitive to the mass, angular momentum distribution, and (polytropic) equation of state.

Growth times for gravitational radiation to drive the secular instability are short for degenerate dwarfs and (post-Newtonian) neutron stars; hence equilibrium configurations of these objects are probably limited to the range $0 \leq T/W \leq 0.14$. Using the e-folding time obtained by Chandrasekhar (1970) for gravitational radiation to drive the secular instability, Friedman and Schutz (1975) show that neutron stars are unstable and lose their excess angular momentum in seconds whenever $T/W \gtrsim 0.14$ (see § V). Unstable white dwarfs spin down, by emitting gravitational radiation, to stability or to collapse in times that vary from weeks to 10^3 years. For $T/W \gtrsim 0.26$, the unstable modes grow on dynamic time scales which are typically $\lesssim 10$ s for rotating white dwarfs and $\lesssim 10^{-3}$ s for neutron stars.

Nonradial stability criteria for relativistic stellar models (i.e., high-mass neutron stars) have not yet been determined. It is not clear how the secular instability limit of $T/W = 0.14$ will be modified by general relativity (since there exists no unambiguous partition of energy into rotational kinetic and gravitational potential energy in relativity). It is significant, however, that relativistic stars which develop ergoregions are unstable (Friedman 1975).

IV. SOLUTIONS AND DISCUSSION

a) Validity of the Approximations

Equations (4)–(8) have been solved numerically, and the results are summarized in Tables 1 and 2 and Figures 1–5. As pointed out in § II*a*, the approximation of NR neutron degeneracy breaks down when $\rho_c \gtrsim \rho_0 = 6.11 \times 10^{15}$ g cm^{-3}; and the application of post-Newtonian gravity theory is invalid when $M/R \gtrsim 0.1$. Here R is the

TABLE 2

MAXIMUM MASSES FOR POST-NEWTONIAN, ROTATING NEUTRON STARS IN GENERAL RELATIVITY

J (10^{49} g cm^2 s^{-1})	T/W	$M_{\max}$ ($M_\odot$)	$\rho_{\max}$ (10^{15} g cm^{-3})	M/R
0.00	0.000	1.08	7.41	0.20
0.21	0.020	1.11	7.57	0.20
0.30	0.035	1.14	7.69	0.21
0.33	0.040	1.15	7.74	0.21
0.42	0.060	1.19	7.89	0.21
0.53	0.080	1.24	8.04	0.22
0.64	0.100	1.28	8.18	0.22
0.76	0.120	1.33	8.34	0.23
0.89	0.140	1.38	8.47	0.24
1.04	0.160	1.44	8.61	0.24
1.19	0.180	1.49	8.73	0.25
1.37	0.200	1.55	8.85	0.26
1.57	0.220	1.62	8.95	0.26
1.76	0.240	1.68	9.08	0.27
2.02	0.260	1.76	9.12	0.28

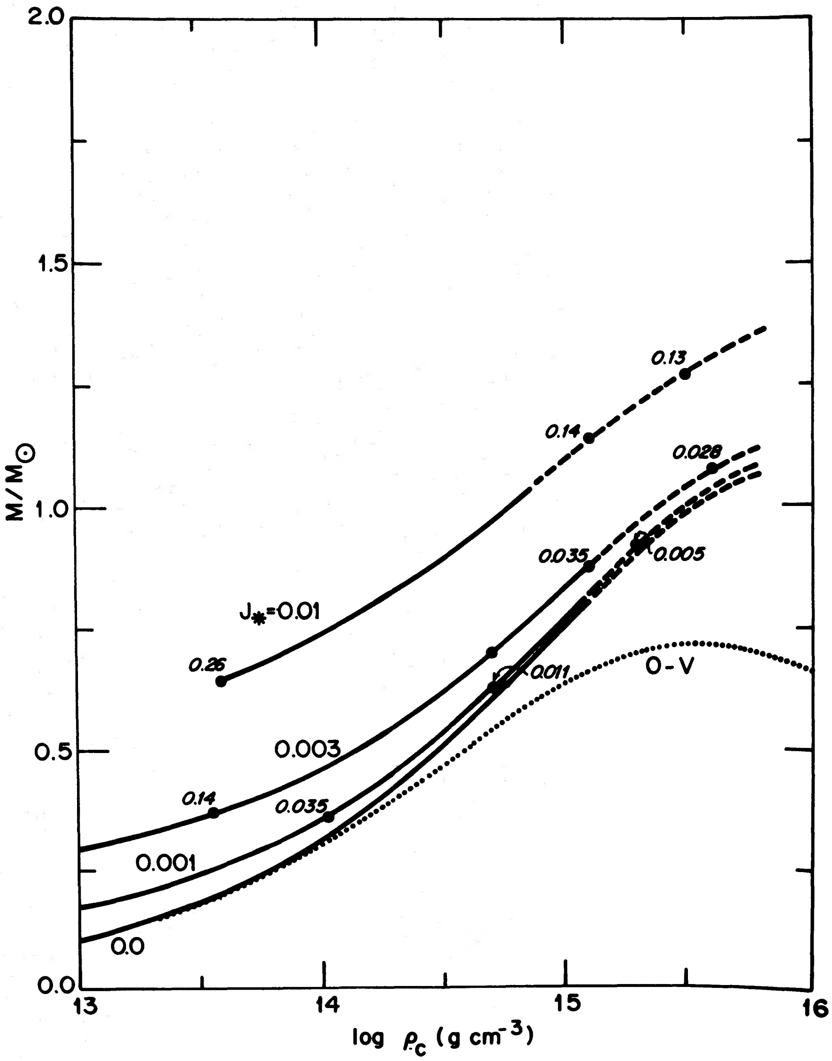

FIG. 1.—Equilibrium configurations for fixed J (with $\gamma = \beta = 1$). Angular momentum is expressed in units of $J_* = J\rho_0 G^2/c^5 = J/(8.9 \times 10^{50}\ \mathrm{g\ cm^2\ s^{-1}})$. Large dots label values of T/W, with the last dot on each curve specifying the minimum value. Dashed portions indicate regions where $M/R > 0.1$. The dotted curve is the Oppenheimer-Volkoff equilibrium sequence for nonrotating configurations.

radius of a nonrotating sphere with the same mass, volume, and polytropic index as the rotating configuration and is given by the $n = \frac{3}{2}$ polytrope relation

$$R = (1.1267)(M/\rho_c)^{1/3}\,. \tag{9}$$

The radius of a rotating configuration, $R_e(\Theta)$, increases toward the equator. For a uniformly rotating configuration ($0 \le T/W < 0.0353$), $R_e(\Theta)$ satisfies the Roche equation

$$\frac{R_e(\Theta)}{R_0} = 1 + \frac{1}{2}\left(\frac{\Omega^2 \sin^2 \Theta}{M}\right) R_e^{\,3}(\Theta)\,,$$

where the angular frequency Ω is related to J by

$$\Omega = \frac{J}{(0.2045)MR^2}$$

for an $n = \frac{3}{2}$ polytrope, Θ is the polar angle, and $R_0 = \lambda R$ is the radius measured along the rotation axis of the star.

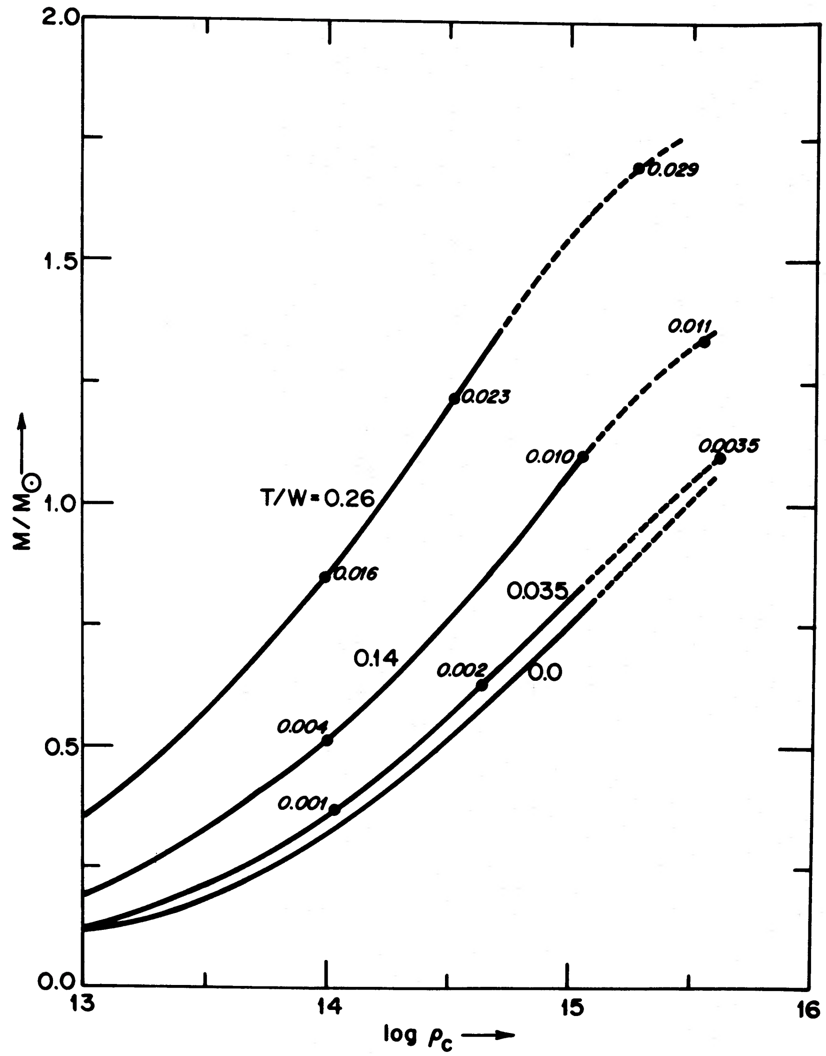

FIG. 2.—Equilibrium configurations for fixed T/W (with $\gamma = \beta = 1$). Large dots label the values of the angular momentum in units of J_*, with the last dot on each curve specifying the maximum value.

In reality the adopted equation of state is probably invalid above $\rho_c \gtrsim 2 \times 10^{14}$ g cm^{-3}, at which point detailed calculations indicate that the equation of state becomes stiffer than an ideal Fermi gas (see Canuto 1974, 1975). We indicate by dashed lines in Figures 1–4 all regions in which $M/R \geq 0.1$ ($\rho_c < \rho_0$ when $M/R = 0.1$). In the low-density regime, $\rho_c \lesssim 2 \times 10^{14}$ g cm^{-3}, the adopted approximations are valid and our numerical results are quantitatively reliable. At higher densities our results are only qualitatively correct. In particular, an accurate determination of the maximum mass and central density just prior to the onset of radial instability is not possible in our scheme, since this occurs in the regime $\rho_c > \rho_0$ and $M/R > 0.1$. A crude estimate of the validity of the polytropic, post-Newtonian approximations employed here is obtained by comparing our $J = 0$ configurations for general relativity with the Oppenheimer-Volkoff (1938) fully relativistic calculations involving no approximations to the degenerate Fermi neutron gas equation of state (see Fig. 1).

We treat rotation as a small correction term in equation (1), restricting $T/W \lesssim 0.26$ and neglecting the effective gravitational mass of rotational kinetic energy. For extremely relativistic stars, the contribution of rotational energy to the gravitational mass is significant and can even reduce the ellipticity of the configuration with increasing J (Butterworth and Ipser 1975).

b) *Influence of Rotation on Equilibrium Configurations*

In our approximation rotation exhibits two principal effects, both serving to increase the equilibrium mass at a given central density. The first effect is to flatten the configuration, thereby weakening Newtonian gravity

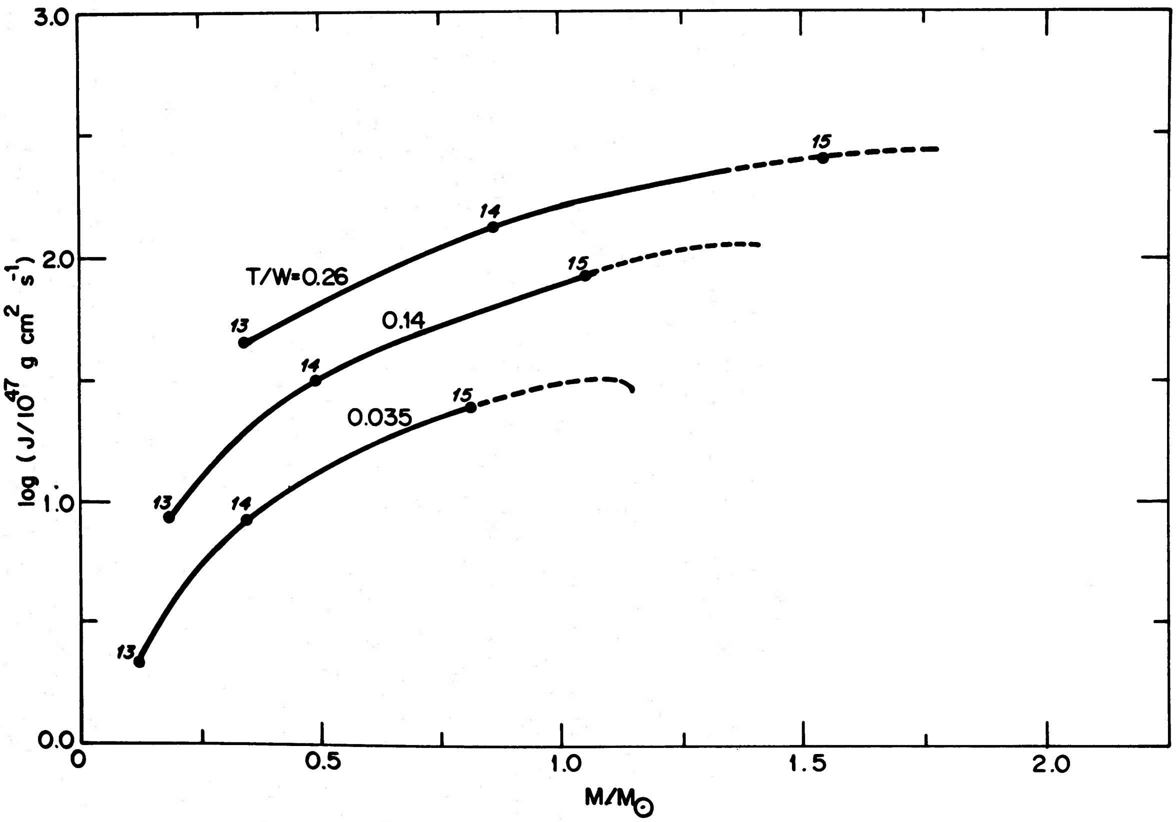

FIG. 3.—Equilibrium configurations in the (J, M)-plane for fixed T/W (with $\gamma = \beta = 1$). Dots label values of $\log_{10} \rho_c$ (g cm^{-3}) along curves.

(i.e., $g < 1$ in eq. [1]). The second effect is the addition of positive kinetic energy, thereby increasing the effective outward "pressure." Weakened Newtonian gravity and enhanced effective "pressure" result in a larger equilibrium mass at a given density. This behavior is clear from the expression for the equilibrium mass, equation (7), and from Figures 1 and 2. Rotation increases the maximum mass as seen in Table 2.

It is also evident from Figures 1–3 that for fixed J, the ratio T/W has a *minimum* value along a sequence of equilibrium configurations, while for fixed T/W, J has a *maximum* value. This behavior results from the extreme sensitivity of T/W (and J) to the mass M along an equilibrium sequence:

$$T/W \propto \frac{\lambda}{g(\lambda)} \frac{J^2 \rho_c^{1/3}}{M^{10/3}} \,. \tag{10}$$

Minima in T/W and maxima in J occur just prior to the critical density associated with the maximum mass along an equilibrium sequence. This result is consistent with the fully relativistic calculations of Abramowicz and Wagoner (1975) along the ascending branch of their equilibrium curves. We also find that for constant M, T/W increases monotonically with J.

c) *Influence of the Equation of State*

For the adopted equation of state, $P/\rho = \rho(d\pi/d\rho) = (\frac{1}{5})(\rho/\rho_0)^{2/3} - (\frac{1}{14})(\rho/\rho_0)^{4/3}$, the second term effectively lowers the adiabatic index, $\gamma_{\mathrm{ad}} \equiv d \ln P/d \ln \rho$ below $\frac{5}{3}$ for finite ρ/ρ_0. As is shown in Appendix B, when $\gamma_{\mathrm{ad}} \geq \frac{5}{3}$, the post-Newtonian equations do *not* admit a maximum mass for finite density along a sequence (J = constant) of equilibrium configurations. Instead, the sequence asymptotically approaches a maximum mass as the central density approaches infinity, mimicking the behavior of classical Newtonian, degenerate configurations. However, introducing the first-order correction term to the equation of state (the fourth term in eq. [1]) produces a turnover in the M-ρ curve and a maximum mass near $\rho \sim \rho_0$, as in the exact treatment of Oppenheimer and Volkoff.

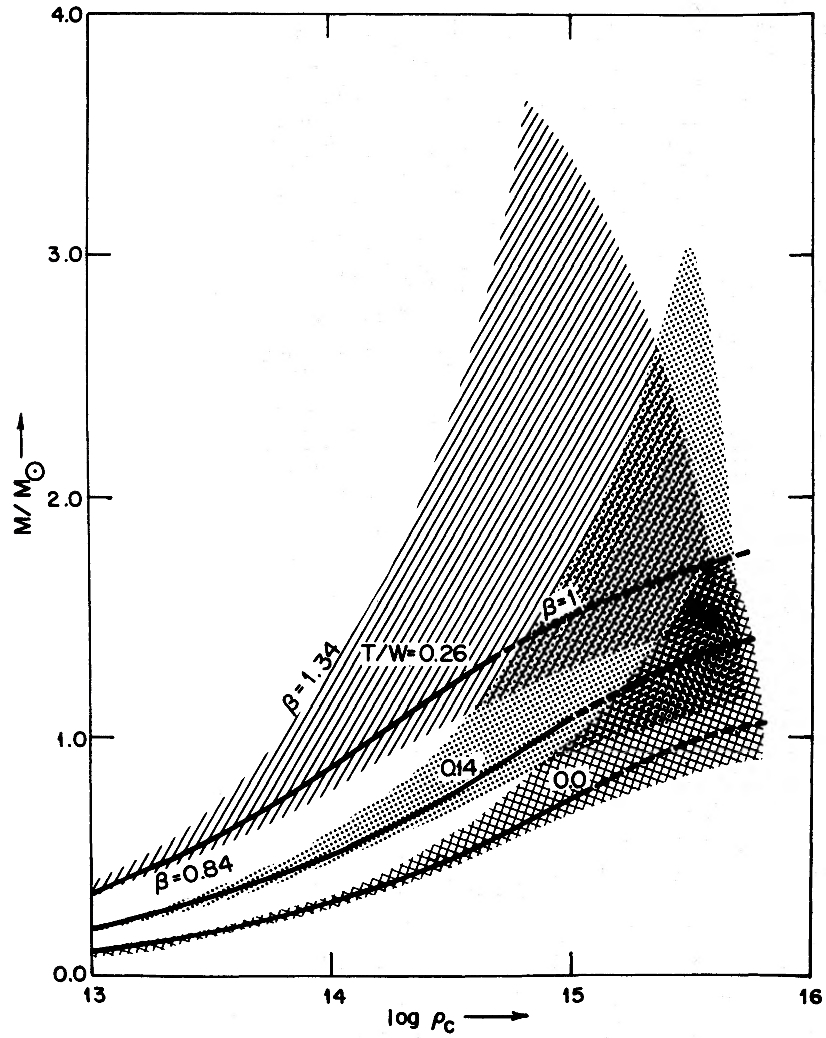

FIG. 4.—Equilibrium configurations, at fixed T/W, for currently viable conservative theories of gravity. Bounding each general relativity curve ($\beta = 1$) is the allowed mass range consistent with present experimental limits on the PPN parameter β.

Apparently, post-Newtonian gravity alone is not sufficiently strong to induce pulsational instability unless the equation of state is sufficiently soft ($\gamma_{ad} < \frac{5}{3}$).

d) *Post-Newtonian Gravitational Effects*

The influence of relativistic gravity on equilibrium configurations enters to PPN order via the parameter k_5. In accordance with the strict limits placed by experiments on the parameter γ, we set $\gamma = 1$ (the GR value) and treat k_5 as a function of β alone:

$$k_5 = 3.09 - 2.41\beta, \qquad 0.84 < \beta < 1.34. \tag{11}$$

As β increases, k_5 decreases, i.e., gravitational forces are weakened. When $\beta > 1.28$, k_5 becomes negative and post-Newtonian gravity becomes weaker than Newtonian gravity. The weakening of gravitational forces results in a larger equilibrium mass for fixed J and central density. The experimentally allowed range of equilibrium masses is illustrated in Figure 4 for fixed value of T/W. Even for only moderately relativistic configurations ($\rho_c \leq 10^{15}$ g cm^{-3}, $M/R \lesssim 0.1$) current experimental limits on β cannot yet rule out stable equilibrium configurations containing nearly *twice* the general-relativistic mass at $T/W \sim 0.14$. For higher densities (or higher T/W), the discrepancy in predicted equilibrium masses becomes more pronounced. It is clear that even at PPN order, different currently viable conservative theories of gravity predict significantly different masses for rotating neutron stars.

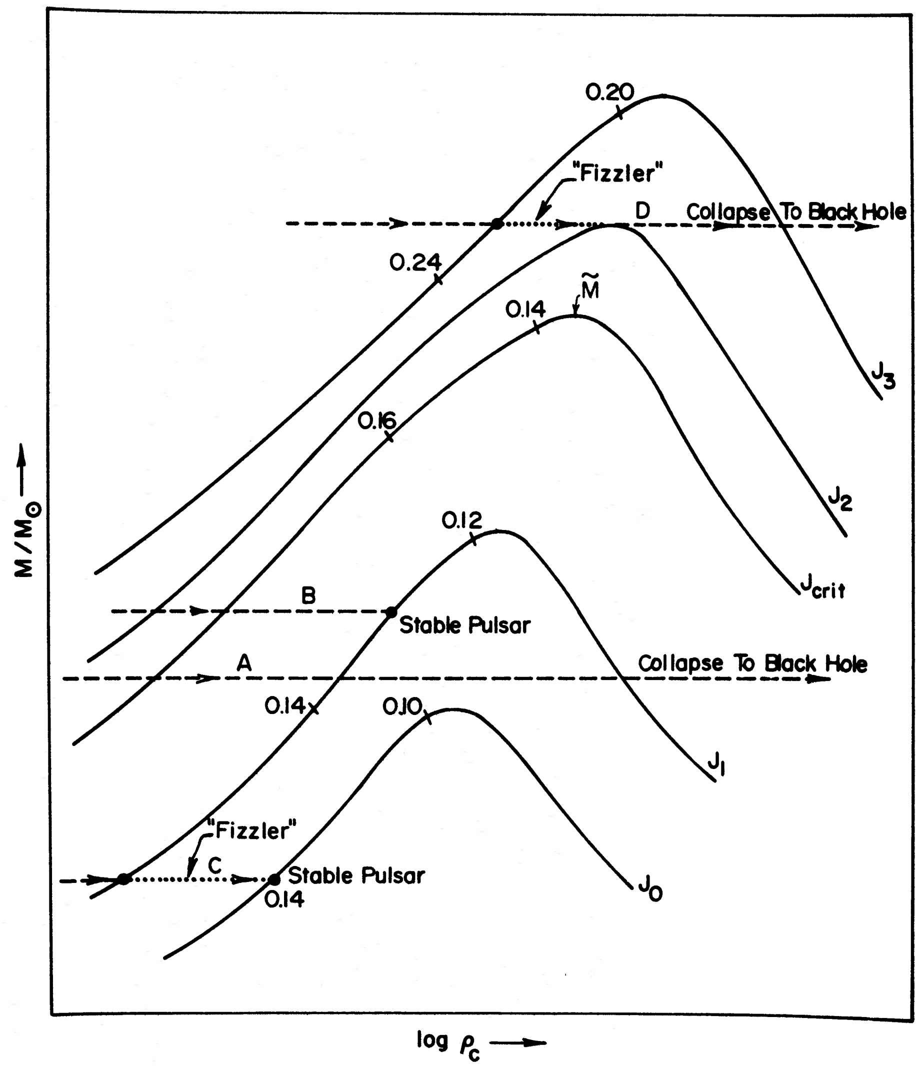

FIG. 5.—An evolutionary scenario for compact objects. Dashed lines indicate evolution on a dynamical time scale; dotted lines indicate evolution on a dissipative time scale (via gravitational radiation), and large dots indicate equilibrium configurations. See text for definition of J_{crit}, $\tilde{M}$, and discussion.

e) The Maximum Mass of Rotating Compact Stars

The maximum mass of an equilibrium configuration of given angular momentum is determined by solving simultaneously the two equations $\partial E/\partial \rho_c|_{M,J} = \partial^2 E/\partial {\rho_c}^2|_{M,J} = 0$. The solutions $M_{\max}(J)$ are given in Table 2 for $\gamma = \beta = 1$. We emphasize again that the maximum mass determined, in this fashion, from equation (1) is only approximate, since the post-Newtonian approximation is invalid for $\rho_c \sim \rho_{\max}$ (see § IV*a*). The density $\rho_{\max}$ in Table 2 marks the onset of radial instability at each J.

Increasing the angular momentum cannot increase the mass of a neutron star indefinitely, since the star eventually becomes unstable to nonaxisymmetric perturbations. Let $(T/W)_{\text{crit}}$ (~ 0.14 in the Newtonian limit) be the maximum value of T/W prior to the onset of nonradial instability. If J_{crit} is the angular momentum of that sequence of equilibrium configurations whose minimum value of T/W is $(T/W)_{\text{crit}}$, then $\tilde{M} = M_{\max}(J_{\text{crit}})$ is the maximum mass of a rotating neutron star in our approximation.

It is significant that the fractional increase in the maximum mass of a compact star due to rotation,

$$f = [\tilde{M} - M_{\max}(J = 0)]/M_{\max}(J = 0)\,,$$

is considerably smaller for a neutron star than for a white dwarf, i.e., $f_{\text{ns}} \ll f_{\text{wd}}$. For white dwarfs, our analysis of the post-Newtonian energy expression for a relativistic, degenerate electron gas (see Shapiro and Teukolsky 1975) yields $f_{\text{wd}} \sim 80$ percent, or $\tilde{M}_{\text{wd}} \sim 2.6\, M_\odot$, for carbon dwarfs destabilized by neutronization. This result is in excellent agreement with more accurate numerical calculations of Durisen (1975). For neutron stars, we find $f_{ns} \sim 28$ percent. We do not know how more exact calculations will modify the result. Inclusion of the full nonlinear theory of gravity will enhance gravitational forces and probably lower f_{ns} further. (Inserting an $n = \frac{3}{2}$ polytropic equation of state in the fully relativistic, uniform-density analysis of Abramowicz and Wagoner 1976, we obtain $f_{ns} \lesssim 10\%$.) Even at post-Newtonian order, relativistic gravity serves to limit the increase in mass of a neutron star due to rotation, while it has little effect for white dwarfs. Combining equations (8a,b,c) for $M_{max}(J)$ and $\rho_{max}(J)$, one obtains

$$M_{max}(J) = \left(\frac{4}{15}\frac{k_1}{k_2 g}\right)^{3/2} (\rho_{max}/\rho_0)^{1/2}\left[1 - \frac{4}{3}\frac{T}{W} + \frac{4}{3}\frac{k_5}{k_2 g} M_{max}{}^{2/3}\rho_{max}{}^{1/3}\right]^{-3/2}$$

The inclusion of rotation, i.e., $T/W > 0$, clearly drives the maximum mass upward, while the inclusion of post-Newtonian gravity, i.e., $k_5 > 0$ ($\gamma = \beta = 1$), lowers the mass. The post-Newtonian term (order M/R) is negligible for white dwarfs, but significant for neutron stars, where it drives f_{ns} from 65 percent ($k_5 = 0$) to 28 percent ($k_5 = 0.681$). *Relativistic gravity in stars of higher compaction competes more successfully with rotation in the determination of the maximum mass.*

In addition, the fractional increase f_{ns} *decreases* for *harder* equations of state at PPN order since (1) the numerical coefficient multiplying T/W decreases and (2) the post-Newtonian term increases with increasing stiffness. Consequently, equations of state which maximize the mass of nonrotating neutron stars may not maximize the mass of highly rotating configurations. It is clear that a reliable determination of the mass enhancement due to rotation awaits a more detailed numerical treatment of fully relativistic, rapidly rotating neutron stars and corresponding stability criteria.

V. GRAVITATIONAL COLLAPSE OF COMPACT OBJECTS: POSSIBLE SCENARIOS

From the above discussion of equilibrium configurations, mass limits, and stability criteria for post-Newtonian, rapidly rotating neutron stars, we can construct different scenarios (depending upon initial conditions) for the collapse of cold, degenerate matter to form compact ($\rho > 10^{13}$ g cm^{-3}) stars. Let the initial, collapsing configuration have mass M, angular momentum J_I, and let $M_{max}(J)$, J_{crit} and $\tilde{M}$ be defined as in § IV*e*. The following cases, illustrated in Figure 5, are then possible:

Case A: $M > M_{max}(J_I)$, (e.g., $J_I = J_0$ in Fig. 5). The collapsing star is too massive to attain neutron star equilibrium. Its collapse proceeds, on a dynamical time scale, to the formation of a black hole.

Case B: $M < M_{max}(J_I)$ and $T/W < 0.14$ (e.g., $J_I = J_1$ in Fig. 5). The star attains a stable equilibrium appropriate for its mass and angular momentum, and forms a pulsar.

Case C: $M < M_{max}(J_I)$, $T/W > 0.14$, and $M < \tilde{M}$ (e.g., $J_I = J_1$ in Fig. 5). The star attains equilibrium but is secularly unstable to nonaxisymmetric perturbations. Gravitational radiation drives the instability, and the star radiates angular momentum, thus decreasing J. As the star loses angular momentum, it contracts, moving horizontally to the right in the (M, ρ)-plane along a sequence of equilibrium configurations of successively lower J and T/W. Eventually, the angular momentum is reduced to a value J_F such that $T/W \approx 0.14$, at which point the star gains stability, collapse is halted, and a stable pulsar forms. The secularly unstable evolution from J_I to J_F proceeds on a gravitational radiation time scale $\tau_{GR} \gg \tau_D$. The term "fizzler" has been suggested by Gold (1974) to denote cold, degenerate objects which possess too much angular momentum to collapse directly to neutron stars on a dynamical time scale.

Case D: $M < M_{max}(J_I)$, and $M > \tilde{M}$ (e.g., $J_I = J_3$ in Fig. 5). By definition of $\tilde{M}$, $T/W > 0.14$ when the star reaches dynamical equilibrium. As in case C the star "fizzles," decreasing its angular momentum and contracting. Unlike case C, however, the configuration never satisfies $T/W = 0.14$. Instead, J decreases to J_F where $M_{max}(J_F) = M$, at which point the star is *radially* unstable and collapses to a black hole on the dynamical time scale.

In the above discussion we have assumed that gravitational radiation does not carry off significant mass compared with M and that the axisymmetric configurations depicted in Figure 5 are not qualitatively different from moderately triaxial fizzlers. Two possibilities not depicted in Figure 5 are (1) fission and (2) collapse to nonradiating, triaxial configurations (e.g., the Dedekind ellipsoids). A close binary system formed by the fission of a collapsing compact configuration would probably decay very rapidly via gravitational radiation and then form a single star with lower J. In the absence of dissipation other than gravitational radiation, an unstable Maclaurin spheroid can be driven to a nonrotating, nonradiating, triaxial Dedekind ellipsoid, with angular momentum in internal circulation (Chandrasekhar 1969; Miller 1974). Such a configuration does not evolve further. However, the presence of any finite viscosity (e.g., turbulent viscosity) would then drive the secular instability and could lead to an evolutionary sequence qualitatively similar to the one described.

a) Estimation of Time Scales and Critical Parameters

Taking the *Newtonian* value $T/W \sim 0.14$ for secular instability our calculations give $J_{crit} \sim 1.0 \times 10^{49}$ g cm^2 s^{-1} and $\tilde{M} \equiv M_{max}(J_{crit}) \sim 1.4\, M_\odot$. Friedman and Schutz (1975) (corrected version) obtain the following gravitational radiation (i.e., "fizzler") time scales, using results of Chandrasekhar (1970) and the equation of state of Baym, Pethick, and Sutherland (1971):

$$\tau_{GR} \sim 10^5 \text{ s} \quad \text{when } M = M_\odot,\ T/W \sim 0.15\,,$$

$$\tau_{GR} \sim 1 \text{ s} \quad \text{when } M = M_\odot,\ T/W \sim 0.24\,,$$

$$\tau_{GR} \sim 1 \text{ yr} \quad \text{when } M = 0.15\,M_\odot,\ T/W \sim 0.15\,,$$

$$\tau_{GR} \sim 10^3 \text{ s} \quad \text{when } M = 0.15\,M_\odot,\ T/W \sim 0.24\,.$$

These time scales can be compared with the dynamical time scale, $\tau_D = 2\pi(R^3/M)^{1/2}$, which we compute from the rotating neutron star models of Baym, Pethick, and Sutherland (1971):

$$\tau_D \sim 4 \times 10^{-3} \text{ s} \quad \text{when } M = 0.15\,M_\odot\,,$$

$$\tau_D \sim 5 \times 10^{-4} \text{ s} \quad \text{when } M = 1.00\,M_\odot\,.$$

Since τ_{GR} is very short, the "fizzler" epoch of a collapsing neutron star is extremely short lived, although the corresponding epoch for a rapidly rotating white dwarf can vary from a few weeks to 10^3 years. The fizzler epoch of a neutron star might be detected by observing the emitted gravitational radiation from the object. Since τ_{GR} may exceed τ_{dyn}, the characteristic frequency of this radiation may be considerably smaller than that produced in the standard free-fall collapse. In addition, pulsar-like radio emissions (with $\dot{P} < 0$) may also be detectable from a collapsing fizzler. The longer fizzler epoch of a white dwarf may in some cases be directly observable.

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APPENDIX A

THE POLYTROPIC STRUCTURE CONSTANTS, k_i

I. THE NEWTONIAN ENERGY TERMS

In this section we compute the dimensionless structure constants k_i ($i = 1, 2, 3, 4$), appearing in the first four terms of the energy equation of a spherically symmetric, static, polytropic gas distribution.

a) k_1: The internal energy of a gas sphere of mass M, central density ρ_c, internal pressure $P = b\rho^{1+1/n}$, and internal energy per unit mass $\Pi(\rho) = nP/\rho$ is given by

$$E_{\text{int}} = \int_0^M \Pi(\rho)dm = k_1 M b \rho_c^{1/n}\,,$$

where

$$k_1 = \frac{n}{\mu_1}\int_0^{\zeta_1} \Theta^{n+1}\zeta^2 d\zeta = \frac{\mu_1(n+1)n}{\zeta_1(5-n)}\,.$$

In the above expression ρ_c is the central density, n is the polytropic index, ζ_1 is the first zero of the Lane-Emden function Θ defined by $\rho = \rho_c\Theta^n$, Θ_1' is the value of the derivative of Θ at ζ_1, and $\mu_1 = -\zeta_1^2\Theta_1'$. In equation (1), $b = 1/(5\rho_0^{2/3})$, $\rho_0 = m^4/(3\pi^2\hbar^3)$, $n = \frac{3}{2}$, $\zeta_1 = 3.654$, $\mu_1 = 2.714$, and $\Theta_1' = -0.203$.

b) k_2: the Newtonian gravitational energy, $-W$, of a polytropic gas sphere is given by

$$W = \int_0^M \frac{mdm}{r} = k_2 M^{5/3}\rho_c^{1/3}\,,$$

where

$$k_2 = \frac{3(4\pi\mu_1)^{1/3}}{\zeta_1(5-n)}\,.$$

c) k_3: The rotational kinetic energy of a uniformly rotating polytrope with angular momentum J is

$$T = J^2/2I\,,$$

where the moment of inertia I is given by

$$I = \frac{2}{3}\int_0^M r^2 dm \,.$$

Evaluating the above expressions in terms of Lane-Emden functions, we find

$$T = k_3 J^2 M^{-5/3} \rho_c^{2/3} \,,$$

where

$$k_3 = 3\mu_1^{5/3}(4\pi)^{2/3}\Big/\left(4\int_0^{\zeta_1} \Theta^n \zeta^4 d\zeta\right)$$

[see Zel'dovich and Novikov for a discussion of the rotational distortion coefficients λ and $g(\lambda)$].

d) k_4: The first-order correction term to the internal energy per unit mass of a nonrelativistic Fermi neutron gas is $\Pi'(\rho) = -3(\rho/\rho_0)^{4/3}/56$. This leads to a correction in the internal energy of the star given by

$$E_{\rm int}' = \int_0^M \Pi'(\rho) dm = -k_4 M(\rho_c/\rho_0)^{4/3} \,,$$

where

$$k_4 = \frac{3}{56}\int_0^{\zeta_1} \zeta^2 \Theta^{7/2} d\zeta/\mu_1 \,,$$

and $n = \frac{3}{2}$.

II. THE POST-NEWTONIAN ENERGY TERM

In this section we compute the PPN correction to the total energy of a polytrope of index n. We generalize the analysis of Zel'dovich and Novikov (1971), who calculated to post-Newtonian order the difference ΔE between the energy E of a momentarily static, spherical mass distribution in general relativity and the corresponding Newtonian expression for the energy E_N. They subsequently applied their calculation to determine the lowest-order relativistic correction to the energy of an equilibrium $n = 3$ polytrope.

The (conserved) energy of a spherically symmetric, momentarily static configuration is given to PPN order by the expression

$$E = \int_0^\infty 4\pi r^2 \rho dr[\Pi - \tfrac{1}{2}U + \tfrac{1}{2}(2\beta - 6\gamma - 1)U^2 + (3\gamma - 1)U\Pi] \tag{A1}$$

(Duncan and Anand 1974), where ρ is again the rest-mass density and

$$U(r) = \int_0^\infty 4\pi r'^2 \rho dr'(|r - r'|)^{-1} \,. \tag{A2}$$

In equation (A1), the coordinate label r is the isotropic radial marker appearing in the standard PPN metric (Will and Nordtvedt 1972). The instantaneous proper volume dV of a thin mass shell between r and $r + dr$ is obtained to PPN order from the metric, giving

$$dV = (1 + 2\gamma U)^{3/2} 4\pi r^2 dr \approx (1 + 3\gamma U) 4\pi r^2 dr \,. \tag{A3}$$

Expressing the energy as an integral over the proper volume V of the star, we may write

$$E = \int_V \rho dV[\Pi - \tfrac{1}{2}U + \tfrac{1}{2}(2\beta - 3\gamma - 1)U^2 - U\Pi] \,. \tag{A4}$$

The Newtonian expression for the energy is

$$E_N = \int_V \Pi \rho dV - \int_M \frac{m'}{r'} dm' \,, \tag{A5}$$

where

$$dm' = \rho dV, \qquad m' = \int_V \rho dV, \qquad r' = (3V/4\pi)^{1/3} \,.$$

Here m' is the instantaneous "Newtonian" mass (i.e., the rest mass), and r' is the "Newtonian" or Euclidean radius. The function $\rho(V)$ is the same in Newtonian and PPN theory. The difference in energy $\Delta E = E - E_N$ can be regarded as the PPN correction to the Newtonian expression and will be computed below to order $(M^2/R)(M/R) \sim M^{7/3}\rho_c^{2/3}$.

From equations (A4) and (A5) we find

$$\Delta E = \int_V \rho dV \left[\frac{1}{2}(2\beta - 6\gamma - 1)U^2 - U\Pi + \frac{m'}{r'} - \frac{m}{r}(1 - 3\gamma U)\right], \tag{A6}$$

where we define

$$m = \int_0^r 4\pi r^2 \rho dr\,.$$

To sufficient accuracy we may write

$$\frac{m'}{r'} - \frac{m}{r} = \frac{m' - m}{r'} - \frac{m(r' - r)}{r'r}\,, \qquad m' - m = 3\gamma\int_0^r 4\pi r^2 \rho U dr\,, \qquad r' - r = \frac{3\gamma}{4\pi r^2}\int_0^r 4\pi r^2 U dr\,. \tag{A7}$$

Substituting equations (A7) into equation (A6) we obtain, after considerable manipulation, the final energy correction in terms of a sum of five integrals:

$$\Delta E = I_1 + I_2 + I_3 + I_4 + I_5\,, \tag{A8}$$

where

$$I_1 = -\int_M \Pi m dm/r\,, \qquad I_2 = \tfrac{1}{2}(2\beta - 2\gamma - 1)\int_M m^2 dm/r^2\,,$$

$$I_3 = -\int_M \left(\textstyle\int \Pi dm\right) dm/r\,, \qquad I_4 = (4\beta - 2 - \gamma)\int_M \left(\textstyle\int m dm/r\right) dm/r\,,$$

$$I_5 = -\gamma\int_M \left(\textstyle\int m r dr\right) m dm/r^4\,.$$

The above integrals are taken over the mass of the star. We can everywhere identify the density, radius, and mass appearing in these integrals with the corresponding Newtonian quantities; the resulting expression for ΔE will be correct to PPN order. The integrals determined above agree with those obtained by Zel'dovich and Novikov (1971) in the limit $\gamma = \beta = 1$.

We now evaluate ΔE for an equilibrium configuration obeying a polytropic equation of state with index n, for which we may use, to lowest order, the hydrostatic equilibrium condition

$$\frac{1}{\rho} dP/dr = -m/r^2 = dU/dr \tag{A9}$$

and

$$\Pi = nP/\rho\,. \tag{A10}$$

We obtain, after several integrations by parts, the simplifying relations

$$I_3 = -\frac{(n + 5)}{(5 - n)} I_1 - \frac{6n}{(5 - n)(2\beta - 2\gamma - 1)} I_2\,,$$

$$I_4(4\beta - 2 - \gamma)^{-1} = \frac{5(1 + n)}{n(5 - n)} I_1 + \frac{2(4 + n)}{(5 - n)(2\beta - 2\gamma - 1)} I_2\,,$$

$$I_5 = \gamma/n I_1\,.$$

Substitution of the above relations into equation (A8) yields

$$\Delta E = I_1\left[\frac{10(2\beta - 1)(1 + n) - 2n(n + 3\gamma)}{n(5 - n)}\right] - I_2\left[\frac{6n + (5 - n)(1 + 2\gamma - 2\beta) - 2(4 + n)(4\beta - 2 - \gamma)}{(5 - n)(2\beta - 2\gamma - 1)}\right]. \tag{A11}$$

Substituting the Lane-Emden functions into I_1 and I_2, we obtain

$$I_1 = \frac{(4\pi)^{2/3}}{\mu_1^{7/3}} \frac{n}{n+1} \int_0^{\zeta_1} \Theta^{n+1}\Theta'\zeta^3 d\zeta M^{7/3}\rho_c^{2/3}, \tag{A12}$$

and

$$\frac{I_2}{[\frac{1}{2}(2\beta - 2\gamma - 1)]} = \frac{(4\pi)^{2/3}}{\mu_1^{7/3}} \int_0^{\zeta_1} \Theta^n \zeta^4 \Theta'^2 d\zeta M^{7/3}\rho_c^{2/3}. \tag{A13}$$

Inserting the above relations into equation (A11), we obtain, finally,

$$\Delta E = -k_5 M^{7/3}\rho_c^{2/3}, \tag{A14}$$

where

$$k_5 = -\frac{[10(2\beta - 1)(n+1) - 2n(n+3\gamma)]}{(n+1)(5-n)} \mathscr{I}_n(1) + \frac{[6n + (5-n)(1+2\gamma-2\beta) - 2(n+4)(4\beta - 2 - \gamma)]}{2(5-n)} \mathscr{I}_n(2),$$

and where

$$\mathscr{I}_n(1) = \frac{(4\pi)^{2/3}}{\mu_1^{7/3}} \int_0^{\zeta_1} \Theta^{n+1}\Theta'\zeta^3 d\zeta, \qquad \mathscr{I}_n(2) = \frac{(4\pi)^{2/3}}{\mu_1^{7/3}} \int_0^{\zeta_1} \Theta^n \Theta'^2 \zeta^4 d\zeta.$$

APPENDIX B

PROOF THAT ALL POST-NEWTONIAN ROTATING POLYTROPES WITH ADIABATIC INDEX $\gamma_{\rm ad} \geq \frac{5}{3}$ ($n \leq \frac{3}{2}$) ARE RADIALLY STABLE

For arbitrary adiabatic index, $\gamma_{\rm ad}$ (abbreviated to γ in this section), the energy expression of a uniformly rotating polytrope in the post-Newtonian approximation has the form (cf. eq. [1])

$$E = AMx^{3(\gamma-1)} - BM^{5/3}x + CJ^2M^{-5/3}x^2 - DM^{7/3}x^2, \tag{B1}$$

where $x = \rho_c^{1/3}$ and A, B, C, and D are constant. Equilibrium configurations are obtained by the variation equation

$$\partial E/\partial x|_{J,M} = 0 = 3(\gamma - 1)Mx^{(3\gamma-4)} - BM^{5/3} - 2x(CM^{7/3} - DJ^2M^{-5/3}), \tag{B2}$$

or, equivalently,

$$y_1(x) = y_2(x), \tag{B3}$$

where

$$y_1(x) \equiv 3(\gamma - 1)Mx^{(3\gamma-4)} - BM^{5/3}, \tag{B4a}$$

$$y_2(x) \equiv 2x(CM^{7/3} - DJ^2M^{-5/3}). \tag{B4b}$$

The (M, ρ_c)-curve can turn over and exhibit a maximum only if equation (B2) has *two* solutions; i.e., y_1 and y_2 must intersect twice, for fixed J and M. For $\gamma = \frac{5}{3}$, equation (B2) reduces to a linear equation in x with at most *one* positive solution. For $\gamma > \frac{5}{3}$, the slope of y_1 increases with x while that of y_2 is constant (cf. eqs. [B4]). This result, together with the additional fact that $y_2 > y_1$ at $x = 0$, indicate that there is exactly one intersection of the curves at positive x. Thus, for constant J configurations, there exists *no* maximum mass for $\gamma \geq \frac{5}{3}$; moreover, all equilibrium configurations for $\gamma \geq \frac{5}{3}$ are stable against radial pulsations for arbitrarily large central densities.

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