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RADIATIVE TRANSFER THROUGH A FLOWING REFRACTIVE MEDIUM

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ABSTRACT

We discuss the propagation of photons in a nondispersive refractive medium with arbitrary velocity and gravity fields. Photons move along the geodesics associated with the optical metric given by Gordon in the limit where geometrical optics applies. The known properties of geodesics then simplify the task of finding the world lines of photons in a variety of problems. The formalism for describing photon propagation, summarized here, makes it possible to derive a transfer equation for a flowing refractive medium which is formally identical to the relativistic equation of radiative transfer. However, the theory is phenomenological in that it presumes that the index of refraction is specified.

Subject heading: radiative transfer

I. INTRODUCTION

In most astrophysical applications of radiative transfer, one is concerned with reasonably high-frequency radiation and relatively rarefied media so that the index of refraction is very nearly unity. Hence refractive effects are normally not considered important in transfer theory. But examples do arise, such as the passage of radio waves through a plasma or even of optical radiation through a dense atmosphere like that of Venus, in which refraction is significant, and appropriate modifications of the transfer equation are then needed. For the case of static media in the absence of gravity such modifications have been discussed in detail, and Harris (1965) has given a careful analysis of the ingredients of the appropriate transfer theory.

It is clear that refraction enters the transfer equation through modification of the photon streaming operator which contains terms involving both the velocity and the acceleration of a photon in the medium considered. In a medium with differential refraction, photon acceleration will occur; and, to describe such effects, one must introduce the appropriate equations of motion of the photon through the medium. This part of the problem can be handled most simply if we take advantage of the fact that in a refracting medium, with the introduction of an appropriate metric, the motion of photons is described by an equation like that for geodesics in ordinary spacetime (Anderson and Spiegel 1972). Such a metric was introduced by Gordon (1923) half a century ago in a discussion of Maxwell's equations and light propagation in material media, and Ehlers (1967) has discussed the formulation of geometrical optics in this context. The optical metric is just what is needed for transfer theory in a refractive medium.

We wish to describe here how this geometrical approach to transfer theory brings the apparatus of general relativity into play and permits us to formally treat media with arbitrary velocity and gravitational fields in a relatively simple way. The advantage gained is manifest in the description of the motion of photons in media with differential motion and refraction. This problem is otherwise a very complicated affair as can be seen from Synge's (1956) discussion for general velocity fields or the explicit solutions for special velocity fields recently provided by Lerche (1974*a*, *b*, *c*).

Our plan here is to begin with the assumption that the dispersion relation is known for the stationary medium in the absence of gravity. If this relation holds in the local inertial rest frame of the matter, it may readily be made covariant, and in doing this we come directly to an expression for the optical metric in § II. In § III we begin by assuming that photons obey Hamiltonian dynamics, which is to say that we postulate a geometrical optics. It then follows that photons move along the geodesics associated with the optical metric. Then, using known properties of geodesics, we show how results such as those of Lerche can be obtained quite simply from this formalism. Having assembled these results, we are in a position to write down the transfer equation for a moving refracting medium.[1] We do this in § IV and discuss there some consequences of the transfer equation. Finally, we conclude with the mention of difficulties inherent in such a phenomenological approach.

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[1] A similar formulation of the transfer equation has been suggested in a recent preprint by Gerbal.

II. DISPERSION RELATION FOR A MOVING MEDIUM

In making use of the transfer equation we describe the radiation field as a gas of photons. We assume that geometrical optics is valid and that a photon is represented by a wave packet. The status of this approximation in transfer theory is discussed for a static medium by Harris (1965). The world lines of the photons in this approach are the rays of geometrical optics and, when geometrical optics is applicable, the equations of motion of photons can be obtained once the dispersion relations of the waves in the medium are known. We may readily determine the dispersion relation in a medium with an arbitrary velocity field if we know its form in the local inertial rest frame, $\mathscr{L}$, of the medium. For the propagation of electromagnetic waves in a locally isotropic medium (with no polarization effects) the dispersion relation in the frame $\mathscr{L}$ is[2]

$$2D \equiv \omega_0{}^2 - n^{-2}\boldsymbol{k}_0{}^2 = 0\,, \tag{2.1}$$

where ω_0 is the angular frequency of a wave, $\boldsymbol{k}_0$ is its propagation vector, n is the index of refraction of the medium, and all of these quantities are measured in $\mathscr{L}$. More generally, for propagation in an anisotropic medium, we have, in $\mathscr{L}$ a dispersion relation like[3]

$$2D = \omega_0{}^2 - \mu_{ij}k_{0i}k_{0j}\,. \tag{2.2}$$

These dispersion relations are evidently valid for stationary, homogeneous media. In the geometrical optics limit they apply to inhomogeneous media with the refractive parameters (n or μ_{ij}) evaluated locally. When there is differential motion, these relations hold in the local inertial rest frame $\mathscr{L}$ if it is supposed that n is independent of derivatives of the velocity field of the matter. If this were not true but we knew the dependence of n on the local state of motion, we could also proceed in the manner described below.

When these dispersion relations hold in $\mathscr{L}$, we can obtain the form of D in an arbitrary observer frame once we have expressed the quantities ω_0 and k_0 in terms of the corresponding quantities, ω and $\boldsymbol{k}$, in that frame. To this end we construct a propagation four-vector $k^\mu = (\omega, \boldsymbol{k})$ and, from the velocity $\boldsymbol{v}$ of the medium in the observer frame, another four-vector $U^\mu = \gamma(1, \boldsymbol{v})$ where $\gamma = (1 - \boldsymbol{v}^2)^{-1/2}$. Since, in $\mathscr{L}$, $U^\mu = (1, 0, 0, 0)$, while the spacetime metric is locally $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, it follows that

$$\omega_0 = U^\mu k_\mu\,, \tag{2.3}$$

where

$$k_\mu = g_{\mu\nu}k^\nu\,.$$

Next, to construct $\boldsymbol{k}_0$, we introduce a set of three spacelike mutually orthonormal vectors $e_i{}^\mu$ that are also orthogonal to U^μ. Hence they satisfy the conditions

$$g_{\mu\nu}U^\mu e_i{}^\nu = 0\,, \qquad g_{\mu\nu}e_i{}^\mu e_j{}^\nu = -\delta_{ij}\,, \tag{2.4}$$

where δ_{ij} is the usual Kronecker symbol. As is readily verified, $e_i{}^\mu = \delta_i{}^\mu$ in $\mathscr{L}$, and therefore we may write

$$k_{0i} = e_i{}^\mu k_\mu\,. \tag{2.5}$$

Equations (2.3) and (2.5) together form the covariant expression,

$$k_{0\nu} = e_\nu{}^\mu k_\mu\,,$$

where $e_0{}^\mu = U$.

If we introduce the foregoing expressions for ω_0 and k_{0i} into equation (2.2), we obtain

$$2D = \tilde{g}^{\mu\nu}k_\mu k_\nu = 0\,, \tag{2.6}$$

where

$$\tilde{g}^{\mu\nu} = U^\mu U^\nu - \mu_{ij}e_i{}^\mu e_j{}^\nu\,. \tag{2.7}$$

For a locally isotropic medium,

$$\mu_{ij} = n^{-2}\delta_{ij}\,; \tag{2.8}$$

hence

$$\tilde{g}^{\mu\nu} = U^\mu U^\nu + n^{-2}(g^{\mu\nu} - U^\mu U^\nu)\,. \tag{2.9}$$

For reasons that will soon be clear, we shall denote the $\mu\nu$ component of the inverse of $\tilde{g}^{\mu\nu}$ by the symbol $\tilde{g}_{\mu\nu}$ and introduce it as a new metric onto the spacetime manifold. The metric $\tilde{g}_{\mu\nu}$ is a generalization to the anisotropic case of the optical metric first introduced by Gordon (1923) in his discussion of Maxwell's equations in material media. In the present context, the optical metric is simply defined by the relation

$$\tilde{g}^{\mu\nu}\tilde{g}_{\nu\rho} = \delta^\mu{}_\rho\,. \tag{2.10}$$

[2] Throughout this work we adopt units in which c and $\hbar$ are unity, except where otherwise noted.

[3] Latin indices take the values 1, 2, 3, and Greek indices range through 0, 1, 2, 3; summation convention applies to both.

Then, if μ_{ij}^{-1} is the inverse of μ_{ij},

$$\tilde{g}_{\mu\nu} = U_\mu U_\nu - \mu_{ij}^{-1} e_{\mu i} e_{\nu i} . \tag{2.11}$$

In this expression the covectors U_μ and $e_{\mu i}$ are obtained from their contravector counterparts U^μ and $e_i{}^\mu$ by lowering indices with the physical metric $g_{\mu\nu}$.

We note that though k_μ is a null-vector relative to the optical metric (see also Ehlers 1966) it is in general not null relative to the physical metric, so that for $n^2 > 1$, k_μ is spacelike.

III. RAY THEORY

The description of electromagnetic radiation in terms of the transfer equation rests on the assumption that the radiation field may be described by photons; this means, as we have said, that we are working to the accuracy of geometrical optics. Of course, the refractive index is related to both refraction and scattering, and the effect of random fluctuations of the refractive index on scales much less than the mean wavelength associated with the photons may be described by a scattering coefficient (Pekeris 1947). Refraction is ascribed only to slowly varying refractive indices. Thus there is a phenomenological aspect to the problem in that we suppose that processes like absorption and scattering can be separated from refraction and introduced into transfer theory in the customary way. The slowly varying component of the refractive index then enters only in the part of the transfer equation that deals with the free streaming of photons. We shall assume that this separation has been made and shall not go into the difficulties of this question except to mention that they have been discussed in the case of initial value problems by Burridge and Papanicoulau (in preparation). Rather, we shall assume straightaway that the photon concept makes sense in a refractive medium and that the motion of photons is adequately described by geometrical optics. In that case, the photons obey Hamiltonian dynamics as Hamiltonian himself clarified. The general theory has been described in several books (e.g., Kline and Kay 1965; Luneburg 1966; Synge 1954) while useful summaries by Keller (1974) and by Weinberg (1962) are also available. A very concise discussion of the classical origins of the equivalence of ray theory and particle dynamics has been given by Eckart (1950). We shall quote here some essential results from this classical work and then give its formulation in the optical space introduced in the previous section.

The relation between the wave and particle pictures is that the rays of wave theory are the paths of the particles; in the spacetime description adopted here the rays are the world lines of the particles. The mathematical link between the two pictures is provided by the dispersion relation which we formally write as

$$D(x^\mu, k_\mu) = 0 , \tag{3.1}$$

where, as in § II, k_μ is the propagation vector. The explicit appearance of the coordinate x^μ in the dispersion relation arises in geometrical optics through the dependence on the parameters of the medium which in turn may depend weakly on x^μ. If more than one kind of wave is possible, more than one relation of the form (3.1) may be needed.

In the particle picture, D is the Hamiltonian while x^μ and k_μ are the canonically conjugate dynamical variables. Both x^μ and k_μ are assumed to be functions of a parameter λ which varies monotonically along the world line or ray. Hamilton's equations have the form

$$\frac{dx^\mu}{d\lambda} = \frac{\partial D}{\partial k_\mu} , \qquad \frac{dk_\mu}{d\lambda} = -\frac{\partial D}{\partial x^\mu} ; \tag{3.2}$$

and with the dispersion relation (2.11) we find

$$\dot{x}^\mu = \tilde{g}^{\mu\nu} k_\nu , \tag{3.3}$$

where the dot denotes differentiation with respect to λ. Equations (3.3) and (2.6) imply that

$$\dot{x}^\mu k_\mu = 0 , \tag{3.4}$$

so that if k_μ is spacelike, $\dot{x}^\mu$ is timelike; while if k_μ is timelike (as it is when $n < 1$), $\dot{x}^\mu$ is spacelike. This latter result points up a well-known difficulty arising from the identification of wave packets with photons in material media (Sommerfeld 1914; Brillouin 1914). The velocity $\dot{x}^\mu$, though parallel to the direction of energy propagation, is in fact distinct from the velocity with which signals propagate. This distinction is made clear in, for example, the discussion of Stratton (1941).

If we eliminate k_μ from equation (3.2) to obtain an equation for $x^\mu(\lambda)$, we find, after some simple algebra,

$$\ddot{x}^\mu + \tilde{\Gamma}_{\rho\sigma}{}^\mu \dot{x}^\rho \dot{x}^\sigma = 0 , \tag{3.5}$$

where[4]

$$\tilde{\Gamma}_{\rho\sigma}{}^\mu = \tfrac{1}{2}\tilde{g}^{\mu\nu}(\tilde{g}_{\rho\nu,\sigma} + \tilde{g}_{\nu\sigma,\rho} - \tilde{g}_{\rho\sigma,\nu}) \tag{3.6}$$

[4] We use throughout a comma to denote ordinary differentiation, a semicolon to denote covariant differentiation in real physical space, and a colon to indicate covariant differentiation in optical space.

is the Christoffel symbol of the second kind constructed from $\tilde{g}_{\mu\nu}$. This shows that, for the dispersion relations we are considering, the Hamiltonian equations are those for a geodesic in a spacetime with metric $\tilde{g}_{\mu\nu}$. In fact, equation (3.5) is that of a null geodesic since along it,

$$\tilde{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0\,, \tag{3.7}$$

which follows from equations (2.6), (2.10), and (3.3). Many of the results about null geodesics in general relativity therefore become useful here. We know, for example, that equation (3.5) follows from a variational principle, namely,

$$\delta \int_A^B (\tilde{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu) d\lambda = 0\,. \tag{3.8}$$

The Euler-Lagrange equations that follow directly from this principle are

$$-2\frac{d}{d\lambda}(\tilde{g}_{\mu\rho}\dot{x}^\mu) + \tilde{g}_{\mu\nu,\rho}\dot{x}^\mu \dot{x}^\nu = 0\,, \tag{3.9}$$

and for some purposes this is more convenient than the usual geodesic equation (3.5). Thus, if $\tilde{g}_{\mu\nu}$ is independent of a particular coordinate, x^ρ, we see from equation (3.9) that $\tilde{g}_{\mu\rho}\dot{x}^\mu$ is constant along each geodesic. This is just a special case of the result that if κ^μ is a Killing vector of $\tilde{g}^{\mu\nu}$; that is, if κ^μ satisfies

$$\kappa_{\mu:\nu} + \kappa_{\nu:\mu} = 0\,, \tag{3.10}$$

then

$$\kappa_\mu \dot{x}^\mu = \text{constant} \tag{3.11}$$

along each ray or null geodesic (see, for instance, Anderson 1966). The constant of the motion provided by equation (3.11) is just $\tilde{g}_{\mu\rho}\dot{x}^\mu$ when $\tilde{g}_{\mu\nu}$ is independent of x^ρ since then it may be shown that $\kappa^\mu = \delta_\rho{}^\mu$. More generally, if it is found that if there are three independent Killing vectors associated with $\tilde{g}_{\mu\nu}$, then the three corresponding constants of the motion implied by equation (3.11) together with condition (3.7) permit the solution of the geodesic equation to be reduced to quadratures, as the examples of § IV illustrate.

In closing this section, we recall that the canonical theory is here written in terms of the conjugate variables x^μ and k_μ. These are taken as basic; and in the notation used above, $\tilde{k}_\mu = k_\mu$. However, $\tilde{k}^\mu = \tilde{g}^{\mu\nu}k_\nu$ is distinct from k^μ. This example hopefully will clarify the notation. It is also instructive to note that $\tilde{U}_\mu = \dot{x}_{\mu\nu}U^\nu$ has the same components as $U_\mu = g_{\mu\nu}U^\nu$, as a short calculation will verify.

IV. SOME EXAMPLES

a) The Optical Christoffel Symbol

We have seen in the previous two sections that the motion of photons through a refracting medium with arbitrary velocity and gravitational fields reduces simply to geodesic motion in the limit of geometrical optics. In general, the skein of geodesics in optical space produced by complex motions such as turbulence would be difficult to unravel, but the problem is in principle no worse than that of complicated motion in physical space. However, in practice, complex motions of the medium do complicate the geometry of optical space, and to illustrate this we may exhibit the form of the Christoffel symbol for an arbitrary velocity field.

Suppose we consider a refracting medium with an arbitrary velocity field U^μ. We would like to see how the geometry of optical space depends on the quantities involved in the irreducible representation of $U_{\mu;\nu}$. Hence we introduce the standard decomposition (see, e.g., Anderson 1966)

$$U_{\mu;\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \tfrac{1}{3}\theta h_{\mu\nu} + \dot{U}_\mu U_\nu\,, \tag{4.1}$$

where

$$h_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu \tag{4.2}$$

is a projection operator,

$$\theta = U^\mu{}_{;\mu} \tag{4.3}$$

is the dilatation,

$$\sigma_{\mu\nu} = \tfrac{1}{2}h_\mu{}^\rho h_\nu{}^\sigma(U_{\rho;\sigma} + U_{\sigma;\rho}) - \tfrac{1}{3}\theta h_{\mu\nu} \tag{4.4}$$

is the shear,

$$\omega_{\mu\nu} = \tfrac{1}{2}h_\mu{}^\rho h_\nu{}^\sigma(U_{\rho;\sigma} - U_{\sigma;\rho}) \tag{4.5}$$

is the vorticity, and

$$\dot{U}^\mu = U^\mu{}_{;\sigma}U^\sigma \tag{4.6}$$

is the acceleration of the medium.

For illustrative purposes, let the medium be isotropic in $\mathscr{L}$ so that equation (2.9) applies, and let n be a constant.

Then, using formulae (2.7), (2.10), (2.13), and (3.6), together with the foregoing representation of $U_{\mu;\nu}$, we may obtain after a certain amount of straightforward calculation,

$$\tilde{\Gamma}_{\mu\nu}{}^{\tau} = \Gamma_{\mu\nu}{}^{\tau} + (1 - n^2)U^{\tau}(\sigma_{\mu\nu} + \tfrac{1}{3}\theta h_{\mu\nu}) + \frac{1 - n^2}{n^2} h^{\tau\sigma}(U_{\mu}\omega_{\sigma\nu} + U_{\nu}\omega_{\sigma\nu} + U_{\rho}U_{\sigma}\dot{U}_{\nu}) , \tag{4.7}$$

where $\Gamma_{\mu\nu}{}^{\tau}$ is the Christoffel symbol constructed from the physical metric. For this relatively simple case we see that the geometry of the photon world lines is explicitly determined by the complexities of the flow and by any gravitational fields which may be present. Even if the physical space is flat, the velocity fields can induce a large curvature in optical space.

b) The Classical Limit

In many applications, the differential motions encountered are not relativistic; we may take advantage of this by reintroducing c, the speed of light, into the equations and letting c tend to infinity. Thus we replace x^{μ} by $(t, \boldsymbol{x}/c)$, and k^{μ} by $(\omega, c\boldsymbol{k})$. For illustrative purposes we take the specific case of a locally isotropic medium and omit gravitational fields.

It should be realized that, even in the classical limit, optical space is curved and that the equations of motion of photons are still expressed by equations (3.5) and (3.6). The simplification of classical theory lies in the fact that in the expression of $\tilde{g}_{\mu\nu}$ we are to retain only the leading terms in inverse powers of c.

From equations (2.8) and (2.11) we find that

$$\tilde{g}_{\mu\nu} = \frac{1}{n^2} g_{\mu\nu} + \left(1 - \frac{1}{n^2}\right) U_{\mu}U_{\nu} \tag{4.8}$$

where for the present illustration $g_{\mu\nu}$ is the Minkowski metric and $U^{\mu} = \gamma(1, \boldsymbol{v}/c)$. To compute the leading terms in the expansion of $\tilde{g}_{\mu\nu}$ we make the presence of c explicit with the replacement of n by c/v_p where v_p is the phase velocity. The expansion of $\tilde{g}_{\mu\nu}$ then gives

$$\tilde{g}_{00} = 1 - \boldsymbol{v}^2/v_p{}^2 + O(c^{-2}), \quad \tilde{g}_{0i} = cv_i/v_p{}^2 + O(c^{-1}), \quad \tilde{g}_{ij} = -c^2\delta_{ij}/v_p{}^2 + O(1) . \tag{4.9}$$

This expresses the structure of optical space in the classical limit; and the quantity $\tilde{g}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$, which appears in the variational principle (3.8) and is to be used in the basic equation (3.7), becomes

$$\tilde{g}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = (1 - v^2/v_p{}^2)\dot{t}^2 + 2v_i\dot{x}_i\dot{t}/v_p{}^2 - \dot{x}_i\dot{x}_i/v_p{}^2 . \tag{4.10}$$

We may note also that in this particular example there is no dispersion in $\mathscr{L}$, hence the group and phase velocities are equal; the group velocity, of course, is the velocity associated with the motion of photons. Even when $\boldsymbol{v} = 0$, we can discuss photon propagation as motion in a curved space. In this case only the three-space geometry is affected by refraction, and the present formulation reduces to the simpler one given by Luneburg (1966).

c) Plane Shear Flow

To provide explicit applications of the above formalism to cases of refracting media with relativistic differential velocities, we consider simple flow fields. The problem of the passage of radiation through a medium with a plane shearing flow and a constant index of refraction has already been treated by Lerche (1974*a*), and this provides a good illustration of the application of the geometrical metric.

We assume that the medium has the velocity four-vector

$$U^{\mu} = \gamma(1, V, 0, 0), \quad \gamma = (1 - V^2)^{-1/2}, \tag{4.11}$$

where V is the speed in the x-direction and is a given function of y, the transverse coordinate. From expression (4.8) we find that

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} \gamma^2(1 - n^2V^2) & -\gamma^2(1 - n^2)V & 0 & 0 \\ -\gamma^2(1 - n^2)V & \gamma^2(V^2 - n^2) & 0 & 0 \\ 0 & 0 & -n^2 & 0 \\ 0 & 0 & 0 & -n^2 \end{pmatrix} . \tag{4.12}$$

Since $\tilde{g}_{\mu\nu}$ is independent of t, x, and z, it follows from the considerations outlined in § III that $g_{0\mu}\dot{x}^{\mu}$, $g_{1\mu}\dot{x}^{\mu}$, and

$g_{3\mu}\dot{x}^\mu$ are all constants of the motion. This may be expressed by the relations

$$\gamma^2(1 - n^2V^2)\dot{x}^0 - \gamma^2 V(1 - n^2)\dot{x}^1 = a_0\,, \tag{4.13a}$$

$$\gamma^2 V(1 - n^2)\dot{x}^0 + \gamma^2(V^2 - n^2)\dot{x}^1 = a_1\,, \tag{4.13b}$$

and

$$-n^2\dot{x}^3 = a_3\,, \tag{4.13c}$$

where a_1, a_2, a_3 are constants along the world lines of the photons. Equation (3.7), in the present example, is

$$\gamma^2(1 - n^2V^2)(\dot{x}^0)^2 - 2\gamma^2 V(1 - n^2)\dot{x}^0\dot{x}^1 + \gamma^2(V^2 - n^2)(\dot{x}^1)^2 - n^2[(\dot{x}^2)^2 + (\dot{x}^3)^2] = 0\,. \tag{4.14}$$

To solve this system we must specify initial conditions. For example, suppose that at $y = 0$ an observer comoving with the fluid sends photons in the positive y-direction. This means that $V(0) = 0$ and, if we choose the origin of the path parameter so that $\lambda = 0$ on $y = 0$, then $\dot{x}^1(0) = 0$ and $\dot{x}^3(0) = 0$. With these conditions, which are also those adopted by Lerche (1974*a*), we can proceed to solve the system of equations. But for the present purposes it will suffice simply to find the paths of the photons.

With the initial condition $x^3(0) = 0$ we learn from equation (4.13c) that $\dot{x}^3 = 0$ on the entire world line. The ray is then confined to the (x, y)-plane and its form is specified by its slope

$$\tan\theta = \dot{x}^2/\dot{x}^1\,, \tag{4.15}$$

where θ is the angle between the ray and the x-axis, as seen in the frame of the comoving observer at $y = 0$. The initial conditions applied to equation (4.13b) show that $a_1 = 0$ along the ray, and this permits us to express $\dot{x}^0$ in terms of $\dot{x}^1$. On introducing this relation into equation (4.14) we find that

$$\tan\theta = +\frac{(n^2 - V^2)^{1/2}}{\gamma V(n^2 - 1)}\,. \tag{4.16}$$

The choice of sign in this expression is determined by equations (4.13a) and (4.13b). Application of the initial conditions gives $a_0 = \dot{x}^0(0) = \omega$. Then we may solve for $\dot{x}^1$ and find that it is $(n^2 - 1)V\omega/n^2$. With the convention $\omega > 0$ we can then select the appropriate root of equation (4.14). Equation (4.16) then agrees with Lerche's result.

We observe that if $V = n$, the ray becomes antiparallel to the x-axis and we have a turning point. In fact, geometrical optics is no longer valid and a deeper analysis is needed to see how much radiation is actually transmitted. If we take n as fixed, then the problem is mathematically clear. However, it seems likely that, in the neighborhood of the turning point, radiation will feed back on the matter and complications will arise. For example, momentum exchange between matter and radiation should be especially pronounced near the turning point, and this will modify the geodesics in optical space. Feedbacks affecting n may also occur. Such difficulties are assumed not to arise in the transfer theory discussed in § V, but it seems clear that such questions are likely to be of interest in a variety of problems, as Lerche has suggested.

The example just considered may be extended to the case where the shearing medium is a plasma (Lerche 1974*c*). For zero temperature, the dispersion relation has the general form

$$2D \equiv \tilde{g}^{\mu\nu}k_\mu k_\nu - \omega_p{}^2 = 0\,, \tag{4.17}$$

where ω_p is a constant and is equal to the plasma frequency. Here, even in optical space, k is not null, and the photons behave like particles with finite rest mass. Nevertheless, the same kind of formalism applies. In particular, D plays the role of a Hamiltonian, and the rays described by equations (3.2) are geodesics in optical space. In the present application equation (3.7) has $\omega_p{}^2$ on the right-hand side instead of zero. Accordingly, the zero on the right-hand side of equation (4.14) is replaced by $\omega_p{}^2$ though equations (4.13) remain unaltered. The only fundamental difference is that the ray is no longer a null geodesic and the scale of the path parameter must be chosen appropriately. This is resolved by the initial condition on $\dot{x}^0$, $\dot{x}^0(0) = \omega$. If the other initial conditions are retained, we find $a_0 = \omega$, $a_1 = 0$, $a_3 = 0$, along a ray. One can then solve for the components of $\dot{x}^\mu$, and the results are

$$\dot{x}^0 = \frac{\gamma^2\omega(n^2 - V^2)}{n^2}\,, \tag{4.18a}$$

$$\dot{x}^1 = \frac{\gamma^2\omega V(n^2 - 1)}{n^2}\,, \tag{4.18b}$$

$$\dot{x}^2 = \frac{1}{n}\,[\gamma^2\omega^2(n^2 - V^2)/n^2 - \omega_p{}^2]^{1/2}\,, \tag{4.18c}$$

$$\dot{x}^3 = 0\,. \tag{4.18d}$$

These results agree with those of Lerche (1974*c*) for a cold plasma when $\omega_p^2/\omega^2 = 1 - n^2$, and we find

$$\tan\theta = \pm\frac{[V^2(n^2-1) + n^2(1-n^2V^2)]^{1/2}}{\gamma V(n^2-1)}$$

while the condition for a turning point is

$$V^2 = \frac{n^2}{(n^2-1)^2} = \frac{(\omega^2-\omega_p^2)\omega^2}{\omega_p^4}\,.$$

d) *Expansion*

As a final example we consider photons propagating through an expanding medium. To isolate the effects of expansion we suppose that the properties of the medium remain constant in physical time and that the physical space is flat. The expansion is plane with a velocity

$$U^\mu = \gamma(1, V, 0, 0)\,, \qquad \gamma = (1 - V^2)^{-1/2}\,, \tag{4.19}$$

where V is in the x-direction and is an arbitrary function of x $(=x^1)$ alone.

The metric $\tilde{g}_{\mu\nu}$ has the same form as in the previous example but it is now independent of t, y, and z so that $\tilde{g}_{0\mu}\dot{x}^\mu$, $g_{2\mu}\dot{x}^\mu$, and $\tilde{g}_{3\mu}\dot{x}^\mu$ are the constants of the motion. Hence we have

$$\gamma^2(1-n^2V^2)\dot{x}^0 - \gamma^2V(1-n^2)\dot{x}^1 = a_0\,, \tag{4.20a}$$

$$n\dot{x}^2 = a_2\,, \tag{4.20b}$$

$$n\dot{x}^3 = a_3\,. \tag{4.20c}$$

Equation (3.7) now reads

$$\gamma^2(1-n^2V^2)(\dot{x}^0)^2 - 2\gamma^2V(1-n^2)\dot{x}^0\dot{x}^1 + \gamma^2(V^2-n^2)(\dot{x}^1)^2 - n^2[(\dot{x}^2)^2 + (\dot{x}^3)^2] = 0\,. \tag{4.21}$$

At $x = 0$ we set $\dot{x}^2/\dot{x}^1 = \tan\theta_0$, $\dot{x}^3 = 0$, $V = 0$, and $n = n_0$. Some algebra then leads to the result

$$\tan^2\theta = \frac{n_0^2\sin^2\theta_0}{n^2 - \gamma^2(1-n^2V^2)n_0^2\sin^2\theta_0}\,. \tag{4.22}$$

Evidently, if a ray starts out parallel to the flow, it remains so. If $n > 1$, there is a tendency to align the motion of the photons with the direction of the expansion. To see this, consider n to be constant. Then $\gamma^2(1 - n^2V^2) \le 1$ (for $n > 1$); and if (as we have supposed) V increases with x, then $\theta \to 0$ and $V \to 1$. For $n < 1$, the tendency is to turn photons away from the flow. For constant n, in this case, the photons (according to geometrical optics) move orthogonally to the direction of expansion when V^2 has increased to $\cos^2\theta_0/(1 - n^2\sin^2\theta_0)$.

These illustrations of kinetic trapping of refracted radiation raise interesting possibilities. What happens when radiation from a source is trapped by relativistic motions in the ambient material? Of course, there will be some leaking in general, but there is also the possibility that radiation pressure will build up and produce instabilities leading to eruptive phenomena. Refracted radiation generally has low frequency, and the situation in kinematic trapping has some similarities to the instabilities of the zero frequency case, i.e., that of a magnetic field which is amplified by being wound up and thus strengthened until it buckles.

V. TRANSFER THEORY

a) *Transfer Equation*

If one knows the equations of motion of the photons, one can readily write down an equation of continuity for the one-photon distribution function, or phase-space density. This task is straightforward in relativistic transfer theory (Lindquist 1966; Anderson and Spiegel 1972); and, if we work in optical space, it can be equally readily carried out for a refractive medium. Though, as we have formulated the problem here, we may deal with relativistic motions and arbitrary gravitational fields, the use of geodesic equations for photon motions is convenient also for nonrelativistic problems with refraction and flow.

Consider an eight-dimensional phase space with coordinates x^μ and $\tilde{k}^\mu$. The variations of x^μ and $\tilde{k}^\mu$ along a ray are governed by equations (3.2). These equations may be rewritten as

$$\dot{x} = \tilde{k}^\mu, \qquad \dot{\tilde{k}}^\mu = -\tilde{\Gamma}_{\rho\sigma}{}^{\mu}\tilde{k}^\rho\tilde{k}^\sigma\,. \tag{5.1}$$

We assume the existence of a dispersion relation of the form (4.17) and introduce an invariant element of "volume"

$d\tilde{K}$ in the hypersurface $D(x, \tilde{k}) = 0$ in (optical) momentum space such that

$$d\tilde{K} = (-\tilde{g})^{1/2}\delta^{+}[D(x, \tilde{k})]d^4\tilde{k}\,, \tag{5.2}$$

where δ^+ is the positive-frequency delta function satisfying

$$\delta^{+}(x^2 - a^2) = \frac{1}{x}\,\delta(x - a)\,, \tag{5.3}$$

$\tilde{g} = \det \tilde{g}_{\mu\nu}$ and, for $\tilde{g}_{\mu\nu}$ given by equation (2.13),

$$\tilde{g} = g \det \mu_{ij}{}^{-1}\,, \tag{5.4}$$

with $g = \det g_{\mu\nu}$. From definition (5.2) we see that $d\tilde{K}$ is a scalar.

We next introduce in optical configuration space a timelike hypersurface with directed element $d\tilde{S}_\mu = (-\tilde{g})^{1/2}\epsilon_{\mu\nu\rho\sigma}d_1x^\nu d_2x^\rho d_3x^\sigma$ where d_ix^μ are three linearly independent infinitesimal displacements lying in the hypersurface and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor density of weight -1. We note that $d\tilde{S}_\mu$ is a vector and not a vector density. It is related to the usual hypersurface element $dS_\mu = (-g)^{1/2}\epsilon_{\mu\nu\rho\sigma}d_1x^\nu d_2x^\rho d_3x^\sigma$ by

$$(-\tilde{g})^{1/2}d\tilde{S}_\mu = (-g)^{-1/2}dS_\mu\,. \tag{5.5}$$

Now let $d\tilde{N}$ be the number of photon world lines having $\tilde{k}^\mu$ in $d\tilde{K}$ and crossing $d\tilde{S}^\mu$. Then we define a one-photon distribution function $\tilde{f}(x, \tilde{k})$ such that

$$d\tilde{N} = \tilde{f}(x, \tilde{k})\tilde{k}^\mu d\tilde{S}_\mu d\tilde{K}\,. \tag{5.6}$$

To see that this definition of $\tilde{f}$ is a generalization of the usual one, suppose that the medium is isotropic and that equation (2.1) holds. Then, since $\tilde{k}^\mu = \tilde{g}^{\mu\nu}k_\nu$, $\tilde{k}^0 = \omega$ and $\tilde{k}^i = \tilde{g}^{ij}k_j = n^{-2}k_i$ in $\mathscr{L}$. We also have $\tilde{g} = -n^6$ in $\mathscr{L}$, so that with the help of equations (2.1) and (5.3) we find $d\tilde{K} = (n^3\omega)^{-1}\delta(\omega - |\boldsymbol{k}|n^{-1})d\omega d^3\boldsymbol{k}$. Likewise $\tilde{k}^\mu d\tilde{S}_\mu = n^3\omega dV$ in $\mathscr{L}$, where dV is a proper volume. Then, if we set $\tilde{f}(x, \tilde{k})|_{\omega=|\boldsymbol{k}|/n} = f(t, \boldsymbol{x}, \boldsymbol{k})$, we see that equation (5.6) reduces to the usual definition of the classical one-photon distribution function.

Consider now a beam of photons of small cross section in physical space and with momenta in $d\tilde{K}$. Let λ be the path parameter along the pencil. We wish to write an equation for $d\tilde{f}/d\lambda$ allowing for losses and gains from the beam due to absorption, scattering, and emission. We express this in the general form

$$\frac{d\tilde{f}}{d\lambda} = \rho(\tilde{\alpha} - \tilde{\beta}\tilde{f})\,. \tag{5.7}$$

Here $\tilde{\alpha}$ and $\tilde{\beta}$ are functions of x and $\tilde{k}$ and are related to the invariant emission and absorption coefficients of Thomas (1930) and thus to the usual emission and absorption coefficients as we shall discuss presently.

Since

$$\frac{d\tilde{f}}{d\lambda} = \dot{x}^\mu\frac{\partial\tilde{f}}{\partial x^\mu} + \dot{\tilde{k}}^\mu\frac{\partial\tilde{f}}{\partial\tilde{k}^\mu}\,, \tag{5.8}$$

we find, with the help of equations (5.1), that

$$\tilde{k}^\mu\frac{\partial\tilde{f}}{\partial x^\mu} - \tilde{\Gamma}_{\rho\sigma}{}^\mu\tilde{k}^\rho\tilde{k}^\sigma\frac{d\tilde{f}}{\partial\tilde{k}^\mu} = \rho(\tilde{\alpha} - \tilde{\beta}\tilde{f})\,\cdot \tag{5.9}$$

Thus, the equation of transfer for a refracting medium, when expressed in optical coordinates, is formally identical to the standard one for a nonrefractive medium (Lindquist 1966; Anderson and Spiegel 1972).

To compare equation (5.9) with the equation found by Harris for the specific intensity in a static medium with a time-independent refractive index we take $\tilde{g}_{\mu\nu} = \operatorname{diag}(1, -n^2, -n^2, -n^2)$. Then

$$\tilde{k}^\mu = \dot{x}^\mu = (\omega, n^{-2}\boldsymbol{k})\,, \tag{5.10}$$

where ω is constant along a ray if n is independent of time. If we use this fact, set $\boldsymbol{k} = n\omega\boldsymbol{l}$, replace $\tilde{f}$ by $f(\boldsymbol{x}, t, \omega, \boldsymbol{l})$, and $\tilde{\alpha}$ and $\tilde{\beta}$ by $\alpha(\boldsymbol{x}, t, \omega, \boldsymbol{l})$ and $\beta(\boldsymbol{x}, t, \omega, \boldsymbol{l})$, we have

$$\omega\frac{df}{\partial t} + \frac{1}{n}\,\omega\boldsymbol{l}\cdot\nabla f + \omega\frac{dl_i}{dt}\frac{\partial f}{\partial l_i} = \rho(\alpha - \beta f)\,\cdot \tag{5.11}$$

Now, if we use the expression Harris suggested for the specific intensity,

$$I = \omega k^2 f = \omega^3 n^2 f\,, \tag{5.12}$$

equation (5.10) will agree with his equation (30) if we make the identifications

$$\alpha = \omega\xi\,, \qquad \beta = \omega a\,, \tag{5.13}$$

where ξ and a are quantities defined by Harris. In the discussion of the radiative stress-tensor (below) we shall see that relation (5.12) between I and f holds in $\mathscr{L}$. From this we conclude that equations (5.13) hold in $\mathscr{L}$. Since α and β are scalars, we can use our knowledge of them in $\mathscr{L}$ to determine them in any frame.

b) Photon Number Current

For some purposes, one often works with just the low-order moments of the transfer equation. These usually represent particular conservation laws, and the lowest order moment of the usual transfer equation is the conservation law for photon number. If we take the lowest moment of equation (5.9), which is to simply integrate it over momentum space, we find after some elementary manipulations,

$$\tilde{N}^{\mu}{}_{:\nu} = \rho\int(\tilde{\alpha} - \tilde{\beta}\tilde{f})d\tilde{K}\,, \tag{5.14}$$

where

$$\tilde{N}^{\mu} = \int \tilde{k}^{\mu}\tilde{f}d\tilde{K}\,. \tag{5.15}$$

For a purely scattering medium, the right-hand side of equation (5.14) vanishes and we have

$$(-\tilde{g})^{1/2}\tilde{N}^{\mu}{}_{:\mu} = [(-\tilde{g})^{1/2}\tilde{N}^{\mu}]_{,\mu} = 0\,. \tag{5.16}$$

This expresses the conservation of $\tilde{N}^{\mu}$. Of course, $\tilde{N}^{\mu}$ is not the ordinary number current since the volume elements in the optical and physical spaces are not identical. However, with pure scattering, we expect the usual number current to be conserved, and to verify this we proceed as follows.

If we integrate equation (5.6) over $d\tilde{K}$, we obtain the total number of photon world lines intersecting the element of optical hypersurface $d\tilde{S}^{\mu}$. In view of expression (5.15) this is just $\tilde{N}^{\mu}d\tilde{S}_{\mu}$. By analogy, we define the total number of photon world lines crossing the element of hypersurface in physical space to be $N^{\mu}dS_{\mu}$ and we interpret the object N^{μ} as the photon-number flux. Then, using equation (5.5), we obtain

$$N^{\mu} = (\tilde{g}/g)^{1/2}\tilde{N}^{\mu}\,. \tag{5.17}$$

Equation (5.16) becomes

$$[(-g)^{1/2}N^{\mu}]_{,\mu} = 0\,, \tag{5.18}$$

which is the usual expression of the conservation of photon number.

c) Radiative Stress Tensor

The next higher moment of the transfer equation, obtained by multiplying by $\tilde{k}^{\nu}$ and then integrating over $d\tilde{K}$, is

$$\tilde{T}^{\mu\nu}{}_{:\nu} = \rho\int \tilde{k}^{\mu}(\tilde{\alpha} - \tilde{\beta}\tilde{f})d\tilde{K}\,, \tag{5.19}$$

where

$$\tilde{T}^{\mu\nu} = \int \tilde{k}^{\mu}\tilde{k}^{\nu}\tilde{f}d\tilde{K}\,. \tag{5.20}$$

Equation (5.19) expresses the way in which energy and momentum are exchanged between the matter and radiation, but there are difficulties of interpretation. As with the number current, we would like to find the analog of $\tilde{T}^{\mu\nu}$ in physical space and identify this as the stress (or stress-energy) tensor. To do this we may try to proceed in analogy with the method we used to define N^{μ}.

Suppose we consider the total four-momentum of the world lines crossing the element of hypersurface $d\tilde{S}^{\mu}$. In optical space we find this by multiplying equation (5.6) by $\tilde{k}^{\nu}$ and integrating to obtain

$$d\tilde{P}^{\mu} = \int(\tilde{k}^{\mu}d\tilde{N})d\tilde{K} = \tilde{T}^{\mu\nu}d\tilde{S}_{\nu}\,. \tag{5.21}$$

If we take (5.21) as a definition of $\tilde{T}^{\mu\nu}$, we are led to define $T^{\mu\nu}$ through the analogous relation

$$dP^{\mu} = T^{\mu\nu}dS_{\nu}\,, \tag{5.22}$$

where dP^{μ} is the total four-momentum of the world lines intersecting dS_{ν}. However, there are two ways to use this definition to compute $T^{\mu\nu}$ from the expression $dN = fk^{\mu}dS_{\mu}dK$. That is, according as one assigns k or $\tilde{k}$ as the pseudo-momentum, one finds $\int (k^\mu dN)dK$ or $\int \tilde{k}^\mu dNdK$ for dP^μ. There result two possible expressions for $T^{\mu\nu}$ which we shall distinguish by subscripts. We obtain either

$$T_M{}^{\mu\nu} = (\tilde{g}/g)^{1/2} \int k^\mu \tilde{k}^\nu \tilde{f} d\tilde{K} \tag{5.23}$$

or

$$T_A{}^{\mu\nu} = (\tilde{k}/g)^{1/2} \int \tilde{k}^\mu \tilde{k}^\nu \tilde{f} dk\,. \tag{5.24}$$

We have stressed this dichotomy since it appears to be closely related to the somewhat controversial problem of defining the stress-energy tensor for an electromagnetic field in the presence of matter in bulk (Pauli 1958; Ehlers 1967). Expressions (5.23) and (5.24) correspond respectively to the Minkowski and Abraham forms of the electromagnetic stress tensor. The present discussion indicates that the choice depends, at least in the context of transfer theory, on how one interprets the photon in classical (but relativistic) physics. In fact, what is physically important is that the total stress tensor (matter plus radiation) be conserved. If one has a model for the matter, then the choice of radiative tensor can be decided. But, as we have already noted, the present theory is phenomenological in assuming that n is fixed in advance, and to couple it to the matter-dynamics would require a theory for n. Attempts to do this must presumably build from a microscopic theory. Interestingly enough, some recent work along these lines (de Groot 1969) gives yet another form for $T^{\mu\nu}$. In transfer theory we could conceivably recover even this form by identifying an appropriate momentum of photons. This is all by the way here, since one is really interested in approximating solutions to the transfer equation. From these, one can then calculate any physically meaningful property of the radiation field, at least to the extent that geometrical optics makes sense. Our aim in presenting these remarks on $T^{\mu\nu}$ is that it may give another way of seeing the origin of the disagreement in attempts to define $T^{\mu\nu}$. For a useful discussion of the macroscopic theory, see the recent paper by Robinson (1975).

Despite these difficulties, we can exploit the fact that, in $\mathscr{L}$, $k^0 = \tilde{k}^0$ and hence $T_A{}^{0\mu} = T_M{}^{0\mu}$ to calculate unambiguously the radiative flux $\boldsymbol{F}$ in $\mathscr{L}$. From definition (5.22) we see that

$$F^i = T_A{}^{0i} = T_M{}^{0i}\,. \tag{5.25}$$

Thus for the computation of the radiative flux, the ambiguities in the definition of $T^{\mu\nu}$ do not present a problem. Moreover, we can also verify that relation (5.12) is correct.

In terms of the specific intensity the flux is

$$F^i = \int l^i I(\omega, \boldsymbol{l}) d\omega d\Omega\,,$$

where $\boldsymbol{l}$ is a unit vector in the direction of propagation and $d\Omega$ is an element of solid angle. If we now make use of the expression in the discussion following equation (5.5) and set $\boldsymbol{k} = \omega n \boldsymbol{l}$, we find that

$$T^{0i} = \int n^2 \omega^3 f l^i d\omega d\Omega\,. \tag{5.26}$$

This result together with equations (5.22) and (5.23) then leads to the relation (5.12).

VI. CONCLUSION

The outline of transfer theory presented here began with the premise that a prescription is known for calculating the index of refraction. This prescription could in principle require a knowledge of the radiation field, which would introduce a nonlinearity into the problem, but we have here assumed that the refractive index is specified as a function of space and time. We have also supposed that the dispersion relation is quadratic in the propagation four-vector. Given these basic requirements, we added the assumption that the photons involved in the transfer process obey Hamilton's equations, with the dispersion relation providing the form of the Hamiltonian. The essence of the procedure was the generalization of the dispersion relation for a (locally) static and gravity-free medium to one with arbitrary velocity and gravitational fields. This led to the result that there is a metric for which the associated geodesics are the photon world lines. Once this is known, the equation of transfer can be written down for a refractive medium in analogy with the relativistic theory of radiative transfer. As we proposed earlier (Anderson and Spiegel 1972), this procedure is convenient for handling transfer problems involving refraction even when there are no relativistic aspects to the problem.

Except possibly for the last step, the joining of propagation problems to transfer theory, the discussion simply draws together a number of previously (though not always widely) known results. The cornerstone of these is Gordon's (1923) discussion of Maxwell's equations for a moving refractive medium. As Ehlers (1967) has discussed, one can formulate on this basis a theory of geometrical optics. In starting at the other end, as it were, and assuming that Hamilton's equations describe the motion of photons, one is led quite simply to Gordon's optical metric, as we have seen. But from either point of view, one is limited to slowly varying refractive indices and, when turning points occur, perhaps to the usual kind of transition formulae associated with the WKB solution. However, it would clearly be of interest to look further into these questions at the deeper level of physical optics. On the other hand, phenomena like diffraction around a black hole and the metric irregularities of a turbulent cosmology seem worthy of investigation at the present level of approximation.

A problem that remains to be explored is that of handling the transfer of polarized radiation through a birefringent medium. An example might be propagation of radio waves through a dense gas permeated by a strong magnetic field. In such a medium different polarizations can have different dispersion relations and hence give rise to different optical metrics. As the different polarizations would be mixed by scattering, separate transfer equations with separate metrics would not be possible. Does a generalization of the material in § V exist to deal with this problem?

Another problem, which was foreshadowed at the end of § V, arises if we wish to couple the dynamics of the medium to that of radiation. If there is refraction, the radiative transfer is best formulated in optical space, but the matter equations are naturally expressed in physical space. In such problems one often prefers to work with stress tensors, and as we saw in § V, there is more than one way to compute the radiative stress tensor in physical space. If, however, we are given a model for the matter dynamics, we must define a radiative stress tensor such that the total stress-energy is conserved. This in turn means that we would be led to seek an appropriate definition for the momentum of a photon in a refractive medium. Any approach at this level would be phenomenological, but for the purposes of many problems of astrophysical interest such an approach would suffice to reveal many qualitative aspects of the solutions.

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