

EVOLUTION OF CRYSTALLIZING PURE ^{12}C WHITE DWARFS

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ABSTRACT

We describe the first results of a quantitative exploration of white dwarf evolution for models incorporating an accurate dense plasma equation of state and a full treatment of partial ionization and convection in the envelope. We discuss in detail the results for a $1 M_{\odot}$, pure ^{12}C star. The cooling curve and luminosity function deviate appreciably from the behavior predicted by Mestel's cooling theory above $\log(L/L_{\odot}) \approx -0.5$ due to neutrino energy losses and below $\log(L/L_{\odot}) \approx -3.5$ due to Debye cooling. Crystallization occurs at $\Gamma \approx 160$, and the effects produced by the release of latent heat are evident. The combined effect of increased heat capacity due to Coulomb interactions and release of latent heat during crystallization increases the stellar lifetime by nearly a factor of 3 in the phases preceding Debye cooling, as expected. Deep convective cooling appears less important than previously thought. Agreement between the theoretical luminosity function and the observational functions derived by Weidemann from the data of Luyten and of Eggen and Greenstein is generally good, and white dwarf lifetimes determined from the cooling curve are consistent with the ages of white dwarfs in clusters. Resolution of differences between the theoretical discovery function and the number of known white dwarfs at very high and low luminosities may provide new information concerning the distribution and origin of the white dwarfs. The direct effects of crystallization are too small to permit observational detection at present. Debye cooling may offer an indirect test, however, because its effect is dramatic and roughly composition independent. For this, unfortunately, an accurate observational luminosity function down to at least $\log(L/L_{\odot}) \approx -5$ appears needed.

Subject headings: dense matter — interiors, stellar — stellar evolution — white dwarf stars

I. INTRODUCTION

As originally pointed out by Mestel (1952; see also Schwarzschild 1958), the study of white dwarf evolution is basically the investigation of a cooling problem. Degenerate electrons provide the pressure support for the star but do not contribute significantly to the heat capacity, while the ions contribute negligible pressure, but provide nearly all the reservoir of thermal energy. The thin, nondegenerate envelope of the star "insulates" the hot core from cold interstellar space, and leakage of the residual thermal energy through this insulating envelope constitutes the observable stellar luminosity. For these reasons, quantitative computations of the cooling times of white dwarfs depend sensitively upon knowledge both of the dense plasma interior (which is needed to determine the available thermal energy) and of the thermal structure of the envelope (which is required to determine the rate at which this thermal energy is lost).

Recent progress in understanding the dense plasma interior followed recognition of the importance of Coulomb interactions (Kirzhnits 1960; Abrikosov 1960; Salpeter 1961), which lead eventually to crystallization of the white dwarf (Mestel and Ruderman 1967; Van Horn 1968). A parallel advance in understanding followed the recognition (Böhm 1968) that in sufficiently cool white dwarfs deep convection can break through the insulating radiative layer into the degenerate core, thus permitting relatively efficient convective energy transport through the nondegenerate envelope. A detailed understanding of white dwarf evolution thus requires knowledge of the interplay among various effects of Coulomb interactions in the interior, including crystallization, and deep convection in the envelope.

In this paper we describe the first results of an attempt to explore this interplay quantitatively through construction of evolutionary models employing a recently developed, accurate, dense plasma equation of state (Lamb 1974) and incorporating a full treatment of partial ionization and convection in the envelope (Fontaine 1973;

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Fontaine and Van Horn 1975). Here we report and discuss our results for a $1 M_{\odot}$ pure ^{12}C star. Our choice of composition is based in part on observational (Wegner 1973; Bues 1973) and theoretical (Salpeter 1971) indications that the cores of many white dwarfs may be rich in ^{12}C and in part upon the fact that the homogeneous composition cleanly displays the effects in which we are interested and avoids the additional complexities and uncertainties inherent at present in equations of state for mixtures.

In § II, we summarize the details of our calculations together with a few of the most important features of the dense plasma equation of state. The $1 M_{\odot}$ pure ^{12}C evolutionary sequence is described in § III, and in § IV we compare our results with previous work and with observation.

II. CONSTRUCTION OF MODELS

a) The Computer Program

Our models have been calculated with a computer program based on a combination of the evolutionary code developed by Kutter and Savedoff (1969)¹ and the envelope code developed by Fontaine (1973). We have modified the treatment of the outer boundary condition, equation of state, neutrino energy losses, and opacities, and these changes are described below.

The evolutionary calculations employ the boundary conditions

$$\text{at } m = 0 : L_r = 0, \quad r = 0; \quad \text{at } m = M - M_e : P = P_e, \quad T = T_e. \quad (1)$$

Here L_r is the luminosity at radius r , M_e is the mass of the envelope, and P_e and T_e are the pressure and temperature at the base of the envelope. We have chosen a value $M_e = 10^{-6} M_{\odot}$, which ensures that all convection zones lie within the envelope and that ionization is complete at the base of the envelope. In the envelope subprogram, a gray atmosphere integration starting at optical depth $\tau = 10^{-4}$ is carried inward either to $\tau = 10$ or to the point at which convection first begins, whichever occurs first. The inward integration is then continued using the four stellar interior equations, with the assumption of constant luminosity throughout the envelope. Convection is treated using the standard mixing-length theory (Böhm-Vitense 1958), and the effects of partial ionization are fully included in the envelope equation of state (cf. § IIc below). Successive inward integrations are carried out until a fractional accuracy of better than 5 percent in P_e is achieved at the base of the envelope. Further details of the envelope subprogram are given by Fontaine (1973) and Fontaine and Van Horn (1975).

Both P_e and T_e are supplied by the envelope subprogram through the "method of triangles" described by Kippenhahn, Weigert, and Hofmeister (1967). Triangle sides of lengths $\Delta \log L = 0.1$ and $\Delta \log R = 0.05$ were used. We have checked the accuracy of the scheme by calculating envelope models corresponding to the (L, R) values of 10 converged models spanning the evolutionary sequence and comparing the exact values of (P_e, T_e) obtained in this way with the values given by the evolutionary program (using linearized boundary conditions). In all cases the maximum errors in $\log P_e$ and $\log T_e$ were less than 0.01.

During model calculations we restricted the maximum increment between mass shells of the logarithmic Schwarzschild and Härm (1965) variables to 0.1. With this criterion, 100–120 mass shells were sufficient at all stages of the star's evolution. Convergence was monitored by requiring that the maximum correction to $\log T$ at any shell in the model be less than 10^{-5} after 10 or less iterations.

b) Interior Equation of State

Our dense-plasma equation of state, employed in the fully ionized interior, closely follows the work of Salpeter (1961), Kovetz and Shaviv (1970), and Shaviv and Kovetz (1972). We compute the phase transition by exact thermodynamics (cf. Landau and Lifshitz 1969, p. 257), given our approximations for the equations of state in the solid and liquid phases. In both phases we include electron exchange (Kovetz, Lamb, and Van Horn 1972) and Thomas-Fermi (Salpeter 1961) contributions. The description of the ion liquid phase includes ideal gas, Coulomb liquid (Hansen 1973), and quantum liquid (Wigner 1932) contributions, while the description of the ion crystal phase includes Madelung energy and phonon (Cohen and Keffer 1955; Kugler 1969) contributions appropriate for a body-centered cubic (bcc) lattice. We treat screening, which plays a role in the low-density (non-relativistic electron) regime, in a manner similar to that used by Kovetz and Shaviv (1970), but employ a semirelativistic Thomas-Fermi model.

Our model calculations employ a tabular form of this equation of state, with thermodynamic quantities given at a grid of points equally spaced in the natural stellar interior variables $\log P$ ($\Delta \log P = 0.2$) and $\log T$ ($\Delta \log T = 0.1$). We determine needed values by two-dimensional, 4-point Aitken-Lagrange interpolation. We have checked the accuracy of the interpolation scheme at 50 points covering the equation of state table, with special attention given to the crystallization boundary, and find the resulting estimated fractional errors in the thermodynamic quantities to be less than 10^{-5} .

¹ We have retained the equation of energy conservation employed by Kutter and Savedoff (1969, eq. [1]), even though it lacks a term involving the time derivative of the chemical composition (Strittmatter *et al.* 1970). This is satisfactory for our purposes since our models avoid ^{12}C ignition, and the contribution of the missing term is thus entirely negligible.

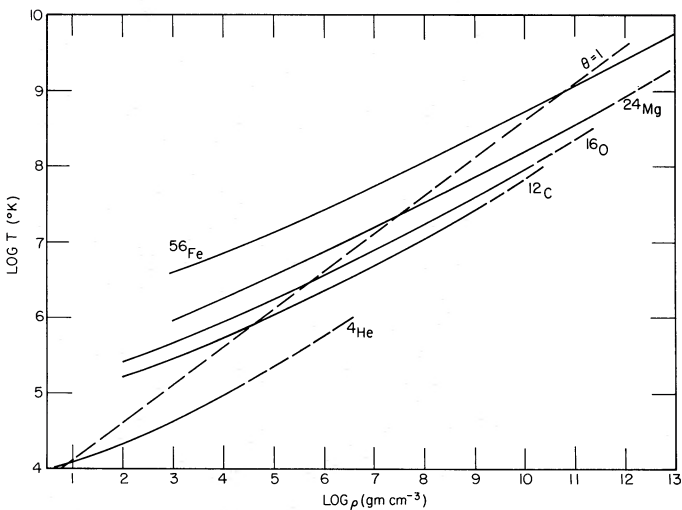


FIG. 1.—Crystallization curves for pure ^4He , ^{12}C , ^{16}O , ^{24}Mg , and ^{56}Fe . The dashed portion of each curve corresponds to the regime $P(\text{quantum liquid})/P(\text{ideal gas}) > 1$ for that composition (see § IIb). Ion quantum effects are important to the right of the line $\theta = 1$. At low densities the crystallization curves have been terminated near the regions of partial ionization for the respective compositions.

Below we very briefly summarize the most important results from our equation of state calculations in order to provide background needed for the discussion of our evolutionary model calculations. Full details of the equation of state are given in Lamb (1974), and a complete discussion will be reported elsewhere (Lamb and Van Horn 1975).

In Figure 1 the crystallization curves for compositions of pure ^4He , ^{12}C , ^{16}O , ^{24}Mg , and ^{56}Fe are shown, and in Table 1 numerical values for the variation of several thermodynamic properties along the ^{12}C crystallization curve are given. We find that crystallization occurs when the dimensionless parameter $\Gamma = (Ze)^2/\langle r \rangle kT$, the ratio of Coulomb potential to thermal energies (Brush, Sahlin, and Teller 1966) is 160 ± 15 (see also Pollock and Hansen 1973). Here $(4\pi/3)\langle r \rangle^3 = n = \rho/AH$. Note that the line $\theta = 1$ —where $\theta \equiv \Theta/T = \hbar\Omega_p/kT$, Θ is the Debye temperature, and $\Omega_p = [4\pi n(Ze)^2/AH]^{1/2}$ is the ion plasma frequency—intersects the crystallization curves. Ion quantum effects consequently play an important role in determining the location and properties of the phase transition at densities typical of more massive white dwarfs. In both Figure 1 and Table 1, the criterion $P(\text{quantum liquid})/P(\text{ideal}) > 1$ denotes the point beyond which our treatment of ion quantum effects becomes inadequate, where $P(\text{quantum liquid})$ and $P(\text{ideal})$ are the ionic quantum liquid and ideal gas contributions to the pressure.

For all compositions, we find a latent heat per ion, q , near $-kT$ and a relative density discontinuity, $\delta\rho/\rho$, near 10^{-4} . These values are in good agreement with the estimates of Van Horn (1968) and with the later calculations of Shaviv and Kovetz (1972). In Figure 2 the ionic, electronic, and total heat capacities for ^{12}C are displayed as a function of temperature at the density $\rho = 10^{7.5}$ appropriate to the center of a $1 M_\odot$ white dwarf. We note two important features in these curves. First, the ionic heat capacity is relatively large in the dense liquid regime due

TABLE 1
PROPERTIES ALONG ^{12}C CRYSTALLIZATION CURVE

log T	log ρ	log P	ψ	Γ	θ	$-q/kT$	$\delta\rho/\rho(\times 10^4)$
5.2.....	2.130	15.745	31.50	115.7	0.2869	0.7311	22.44
5.4.....	2.928	17.239	67.64	134.7	0.4537	0.8433	10.81
5.6.....	3.638	18.485	126.7	146.6	0.6486	0.9096	6.057
5.8.....	4.304	19.622	220.5	154.2	0.8809	0.9512	3.638
6.0.....	4.943	20.693	364.3	159.0	1.160	0.9778	2.307
6.2.....	5.565	21.713	571.5	161.6	1.497	0.9956	1.566
6.4.....	6.174	22.679	839.0	162.7	1.904	1.010	1.173
6.6.....	6.771	23.585	1131	162.4	2.391	1.027	0.9941
6.8.....	7.357	24.432	1391	160.7	2.962	1.056	0.9473
7.0.....	7.928	25.228	1576	157.2	3.607	1.107	0.9942
7.2.....	8.480	25.980	1670	151.4	4.293	1.188	1.127
7.4*	9.008	26.692	1681	143.3	4.977	1.302	1.352
7.5*	9.264	27.035	1660	138.5	5.304	1.369	1.501

* $P(\text{quantum liquid})/P(\text{ideal}) > 1$.

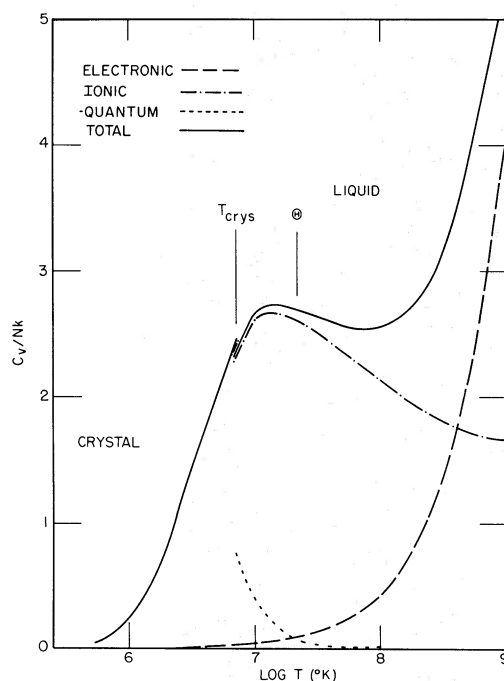


FIG. 2.—The electronic, ionic, and total heat capacities (per ion) as a function of temperature at $\log \rho = 7.5$. The dashed curve shows the negative contribution of the quantum liquid correction to the ionic heat capacity. Here T_{crys} and Θ are the crystallization temperature and Debye temperature.

to the Coulomb interactions; thus only a small discontinuity exists in the heat capacity across the phase transition (see Mestel and Ruderman 1967). Second, the rapid decrease in the heat capacity of the crystal phase as T decreases below Θ is clearly evident.

c) Envelope Equations of State

We use two envelope equations of state in the course of our model calculations. Throughout the white dwarf regime [that is, all models with $\log (L/L_{\odot}) < -1$], we have employed the Lawrence Livermore Laboratory (LLL) equation of state (Graboske, Harwood, and Rogers 1969) modified at very low temperatures and very high pressures using a Thomas-Fermi model as described by Fontaine (1973). We regard this equation of state as the best description of a partially (temperature and pressure) ionized gas currently available; a detailed discussion is given in Fontaine, Graboske, and Van Horn (1975). At higher luminosities, corresponding to higher envelope temperatures, LLL equation of state data are not available. In this regime we have used the Cox-Stewart equation of state for a Weigert V mixture [$X(^{12}\text{C}) = 0.999$] constructed according to the procedure outlined in Cox and Stewart (1969). Comparison of the pressures given by these two equations of state shows that at the high temperatures where we change over from one equation of state to the other during the evolutionary calculations the fractional difference is typically of order 1 percent. The transition is thus satisfactorily smooth and introduces no serious error into the calculations.

The thermodynamic quantities used in the envelope calculations are obtained by two-dimensional, 3-point interpolation in either the Cox-Stewart or LLL equation of state tables. Fontaine (1973) has carried out accuracy checks of this interpolation scheme and has found the resulting estimated fractional errors in the thermodynamic quantities to be less than 10^{-3} .

d) Carbon Burning and Neutrino Energy Losses

Recent experimental results (Patterson, Winkler, and Zaidins 1969; Mazarakis and Stephens 1972) indicate that large uncertainties remain in the carbon-burning rate, and theoretical calculations by Michaud (1972) using the latest data give a specific nuclear energy production rate, ϵ_n , that is ~ 5 times lower than that given by Reeves (1966). In our evolutionary calculations, we have employed Kutter and Savedoff's (1969) approximation for the Reeves rate, reduced by a factor of 5. With this approximation we find that our $1 M_{\odot}$, pure ^{12}C models just avoid ignition (defined as $\epsilon_n > \epsilon_{\nu}$ at any mass-shell, where ϵ_{ν} is the specific rate of neutrino energy loss). Thus, although we allow for the contribution of nuclear burning to the thermal energy balance of the star in its pre-white dwarf evolution, we find it to be unimportant by the time the white dwarf stage is reached and consistent with our use of a pure ^{12}C equation of state throughout the evolution.

TABLE 2
COEFFICIENTS FOR BREMSSTRAHLUNG
NEUTRINO ENERGY LOSS FORMULA

$a = 0.5167$	$c_0 = 0.4775$
$b_0 = 7.577$	$c_1 = 1.707$
$b_1 = -2.801$	$c_2 = -9.090$
$b_2 = 0.2670$	$c_3 = 5.250$

Recent developments in gauge theories of the weak and electromagnetic interactions and the existence of neutral currents (Weinberg 1974, and references therein) are expected to produce only moderate changes in the rates of neutrino energy loss which play a role in white dwarf cooling (Dicus 1972; Bahcall, Treiman, and Zee 1974). Our calculations therefore assume the conventional universal Fermi weak interaction (Feynman and Gell-Mann 1958; Sudarshan and Marshak 1958), and we include neutrino energy losses from photon, plasmon, pair, recombination, and bremsstrahlung neutrino reactions. Urca neutrino energy losses are negligible for the densities and temperatures we consider (Beaudet, Salpeter, and Silvestro 1972) and therefore were omitted.

The rates for photon, plasmon, pair, and recombination neutrino losses were taken from Beaudet, Petrosian, and Salpeter (1967). For bremsstrahlung losses, we use an expression based on the analytic interpolation formula given by De Zotti (1972). This expression has the form

$$\epsilon_\nu (\text{ergs g}^{-1} \text{s}^{-1}) = 7.870 \times 10^{-3} (2\mu_e) Z \times 10^{f(x,y)}, \quad (2)$$

where

$$f(x, y) = a_0 x - (b_0 + b_1 x + b_2 x^2)^{1/2} + y g(x, y), \quad (3)$$

$$g(x, y) = c_0 \arctan [c_1(x - y) + c_2] + c_3. \quad (4)$$

Here $x = \log(\rho/\mu_e)$, $y = \log T_8$, μ_e is the mean molecular weight per electron, and T_8 is the temperature measured in units of 10^8 K. The eight coefficients in equations (3) and (4) are given in Table 2. Comparison of this expression with the analytic high-density results of Festa and Ruderman (1969)² and Flowers (1973), with the intermediate-density numerical results of Cazzola, De Zotti, and Saggion (1971), and with the analytic low-density results of Gandel'man and Pinaev (1959) shows differences of less than 10 percent in all regimes where bremsstrahlung energy losses exceed or are comparable to losses from other neutrino processes.

Our bremsstrahlung calculations do not include a correction for ion correlations. For ^{12}C these reduce the bremsstrahlung rate by a factor of 6 at most (Festa and Ruderman 1969; De Zotti 1972; Flowers 1973). Festa and Ruderman (1969) and Cazzola *et al.* (1971) have suggested that for compositions of heavy nuclei, bremsstrahlung losses may dominate other neutrino energy loss processes in the intermediate-density, low-temperature regime. For ^{12}C we find that in just this regime bremsstrahlung losses are exceeded by recombination losses, in agreement with the remarks of Beaudet *et al.* (1967). For heavier nuclei, recombination losses ($\propto Z^6$) dominate bremsstrahlung losses ($\propto Z$) even more strongly.

e) Opacities

For the radiative opacity, we employ two-dimensional, 2-point interpolation and extrapolation in the table calculated for the Weigert V mixture [$X(^{12}\text{C}) = 0.999$] by Cox and Stewart using the method outlined in Cox and Stewart (1969). Within the table, this procedure is expected to give accuracies of 20–30 percent.

For the conductive opacity, we employ a two-dimensional polynomial expression in the rectangular region $-5 \leq \log \rho \leq 6$, $3.5 \leq \log T \leq 8$. This polynomial represents a least-squares fit to the conductive opacity for ^{12}C calculated by Hubbard and Lampe (1969) for part of this region, and to the conductive opacity expression given by Van Horn (1969) throughout the remainder. It gives a typical accuracy of 10 percent. The results of Kovetz and Shaviv (1973) for the electrical and thermal conductivities in the region of relativistic electron degeneracy became available after our calculations were essentially complete. Our calculations use a simple extrapolation from nonrelativistic values for this regime; because the electron conductivity is already so large, different treatments of the relativistic regime have an insignificant effect on the thermal structure of the star.

This treatment of radiative and conductive opacities gives reasonable total opacities and smoothly bridges a "gap" centered near $\log \rho = 3$, $\log T = 6$, where to our knowledge no detailed opacity calculations presently exist. Further details are given in Fontaine (1973) and in Lamb (1974).

f) Treatment of the Crystallization Boundary

We have found it satisfactory to treat crystallization completely within the equation of state. That is, when a particular mass shell m_j crosses the crystallization boundary, as determined by the corresponding values of (P_j, T_j) ,

² Equations (17) and (19) of this paper contain a misprint; thus equation (19) should read (see also Flowers 1973, eqs. [44]–[46])

$$F(\beta) = 0.14 \left\{ B_1 B_2 - B_3 (\beta^2 - 1) \left[\frac{2}{3} + \frac{1}{2} \left(\beta^2 - \beta \frac{(\beta^2 + 1)}{2} \ln \frac{\beta + 1}{\beta - 1} \right) \right] \right\}.$$

we merely calculate the thermodynamic quantities needed at this mass shell according to the crystal, rather than the liquid, equation of state. In contrast to our earlier expectations, it has proven unnecessary to insert a separate double mass-shell at the phase boundary. However, in calculating models which were undergoing crystallization, we have sometimes found that stable oscillations in the position of the crystallization boundary, which are an artifact of the numerical difference scheme, set in during the iterations if the boundary lies too near a mass shell. Among 27 models that were undergoing crystallization, five displayed these oscillations. The maximum amplitudes of the oscillations (in $\log T$) lay between 6×10^{-4} and 8×10^{-3} for all of these models. Similar oscillations have been encountered at the outer boundary of advancing convection zones where a discontinuity in composition (molecular weight) creates a discontinuity in density (Heney, Forbes, and Gould 1964). In the present calculations we have adopted a brute force approach: if oscillations were encountered in a particular model, the iteration procedure was terminated without further attempts to converge that model. The evolutionary calculations were then continued with the next model in the sequence. Because the oscillations (and hence the errors) were small and occurred infrequently, this procedure produced satisfactory evolutionary tracks in all the variables of interest.

III. RESULTS

Tables 3 and 4 summarize the properties of 16 selected models spanning the evolutionary sequence of the $1 M_{\odot}$, pure ^{12}C star from $\log (L/L_{\odot}) = 3.325$ through $\log (L/L_{\odot}) = -5.517$. Figure 3 shows the luminosity of the star as a function of age, and Figure 4 displays the internal structure during approximately this same period. Table 5 lists six important epochs indicated in these and later figures and identifies the corresponding models in the summary tables.

a) Pre-White Dwarf Evolution

The starting model for the $1 M_{\odot}$ sequence was approximately a polytrope of index $n = 3$ with central conditions $\log \rho_c = 1.724$, $\log T_c = 7.463$, $\log P_c = 16.880$, electron degeneracy parameter $\psi_c = -3.86$, radius $\log R = 10.829$, and age $\tau = 5.34 \times 10^4$ years (arbitrarily assigned). During the pre-white dwarf evolutionary phase, carbon burning consumed only 2.7×10^{-3} of the stellar mass. Thus the $1 M_{\odot}$ sequence can be regarded as representative of the most massive white dwarfs that have not undergone substantial carbon burning (recall § II d). Kutter and Savedoff (1969), Beaudet and Salpeter (1969), and most recently Boozer, Joss, and Salpeter (1973) have discussed the pre-white dwarf evolution of carbon-rich stars extensively. We therefore focus our attention on the phase of rapid neutrino cooling which follows the epoch of maximum luminosity and which includes energy losses from bremsstrahlung neutrinos for the first time (a more complete discussion of the pre-white dwarf phase is given in Lamb 1974).

In Figure 5 we compare the relative importance of the five processes of neutrino energy loss included in our calculations. Photoneutrino losses dominate through the epoch of maximum stellar luminosity. However, as cooling begins, plasma neutrinos dominate as a combined result of the decreasing temperature and of the strong density-dependence of this process. This dominance continues until L_{ν} falls below L [near $\log (L/L_{\odot}) \approx -0.5$], and during most of this period the plasma neutrino luminosity exceeds but parallels L . The temperature in the star is never high enough to make pair neutrino losses important, and recombination neutrino losses are also negligible. Below $\log (L/L_{\odot}) \approx 2.5$ bremsstrahlung losses exceed photoneutrino losses and become the second most important energy loss mechanism, yet through the epoch at which $L_{\nu} = L$ they contribute at most 5–10 percent of the total neutrino losses and do not dominate the plasma neutrino losses until $L_{\nu} \approx 10^{-2} L$. We expect bremsstrahlung neutrinos to make an even smaller relative contribution in lower-mass pure carbon stars because of their lower central densities.

It is interesting to note that the different neutrino emission mechanisms dominate in quite distinct regions within the star. At $\log (L/L_{\odot}) \approx 0.3$ bremsstrahlung losses first exceed plasma losses at the center. As cooling continues, bremsstrahlung losses dominate in an increasing fraction of the star until at the epoch at which $L_{\text{brems}} = L_{\text{plasma}}$, bremsstrahlung losses dominate interior and plasma losses dominate exterior to $q = 0.8$.

Early in the pre-white dwarf phase, neutrino energy losses and gravitational contraction combine to produce a temperature inversion at the center of the star (Vila 1966; Savedoff, Van Horn, and Vila 1969). During the subsequent evolution, the resulting temperature maximum moves outward through the star until it reaches $q = 0.93$ near the epoch at which $\log (L/L_{\odot}) \approx 0$. Here the temperature maximum finally disappears. Because the largest contributions to both plasma and bremsstrahlung losses originate near the temperature maximum, its disappearance produces the sharp fall in both of these neutrino luminosities that is clearly evident in Figure 5.

b) White Dwarf Evolution

Before beginning a detailed discussion of our results for the white dwarf phase, it is useful to review briefly the qualitative features of this stage of evolution and to note the relative roles of thermal energy, gravitational contraction, and Coulomb interactions (Mestel and Ruderman 1967; Van Horn 1971). We can also consider crystallization within this framework and compare the roles of the latent heat of crystallization and of the residual gravitational contraction due to the change in density at the phase transition boundary. These considerations form the basis for our further discussions.

TABLE 3
PROPERTIES OF SELECTED MODELS

Model	1	2	3	4	5	6	7
$\log \tau$	5.708	5.766	5.905	6.481	6.976	7.953	8.496
$\log (L/L_{\odot})$	3.325	2.271	1.343	-0.167	-0.504	-1.368	-2.099
$\log T_{\text{eff}}$	5.465	5.325	5.123	4.764	4.683	4.472	4.291
$\log R$	9.102	8.855	8.794	8.757	8.752	8.742	8.738
$-\Omega (\times 10^{-50})$	3.396	4.331	4.508	4.616	4.643	4.389	4.709
$\log P_c$	24.167	24.531	24.595	24.633	24.643	24.660	24.667
$\log T_c$	8.326	8.105	8.013	7.844	7.725	7.395	7.119
$\log \rho_c$	7.165	7.427	7.471	7.498	7.506	7.518	7.524
ψ_c	33.49	74.27	96.07	146.0	193.5	419.3	795.3
Γ_c	4.129	8.398	10.74	16.17	21.40	46.19	87.60
θ_c	0.070	0.1590	0.2067	0.3147	0.4177	0.9054	1.721
$\log P_{1/2}$	23.345	23.740	23.085	23.844	23.854	23.871	23.878
$\log T_{1/2}$	8.644	8.192	8.042	7.833	7.719	7.393	7.119
$\log \rho_{1/2}$	6.582	6.870	6.917	6.946	6.953	6.966	6.971
$\log R_{1/2}$	8.509	8.422	8.407	8.398	8.396	8.391	8.390
$\psi_{1/2}$	7.892	32.48	48.54	81.31	106.6	229.0	433.6
$\Gamma_{1/2}$	1.269	4.482	6.564	10.86	14.20	30.37	57.30
$\theta_{1/2}$	0.0174	0.0685	0.1022	0.1709	0.2241	0.4818	0.9107
$\log P_e$	15.975	16.942	17.174	17.313	17.332	17.365	17.380
$\log T_e$	7.326	7.362	7.269	7.089	7.046	6.927	6.795
$\log \rho_e$	0.966	1.897	2.226	2.544	2.604	2.737	2.838
$\log R_e$	9.060	8.843	8.786	8.751	8.747	8.739	8.735
ψ_e	-5.13	-3.100	-1.995	-0.504	-0.161	0.804	1.950
Γ_e	0.3544	0.6666	1.063	2.054	2.374	3.458	5.041
$\log P_{\text{conv}}$	9.173	9.881	10.422	12.602	12.862	14.375	14.559
$\log T_{\text{conv}}$	5.815	5.838	5.847	6.191	6.209	6.376	6.323
$\log \rho_{\text{conv}}$	-4.300	-3.602	-3.062	-1.250	-1.007	0.351	0.609
$\log (1 - R_{\text{conv}}/R)$	-3.350	-3.487	-3.416	-2.985	-2.974	-2.840	-2.904
$\log (1 - q_{\text{conv}})$	-12	-12	-12	-10.693	-10.454	-8.981	-8.813
ψ_{conv}	-12.1	-10.6	-9.43	-6.38	-5.88	-3.27	-2.50

NOTE.—The age of the star is τ and the total gravitational potential energy is Ω ; the other symbols have their usual meanings. The subscripts denote the mass shell nearest the following locations within the star: “c” $\Leftrightarrow q = 0$ (center), “ $\frac{1}{2}$ ” $\Leftrightarrow q = \frac{1}{2}$, “e” $\Leftrightarrow q = 1 - 10^{-6}$ (interior-envelope boundary), “conv” $\Leftrightarrow$ inner boundary of the innermost convection zone.

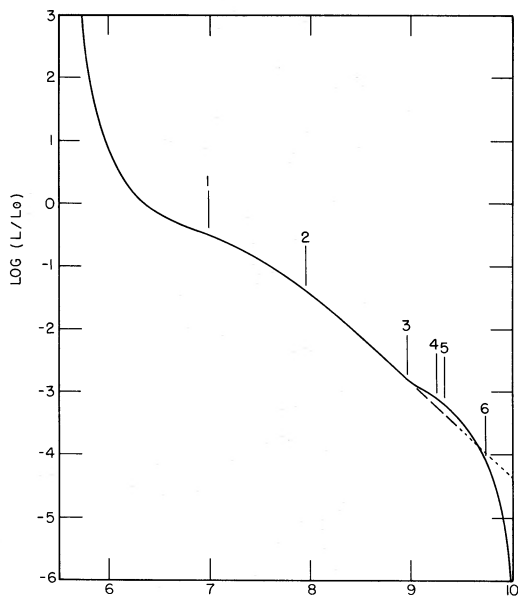


FIG. 3.—The cooling curve, luminosity versus age, for a $1 M_{\odot}$ pure ^{12}C white dwarf. The epochs numbered 1–6 correspond to those listed in Table 5. The dashed curve corresponds to a sequence of models in which crystallization was artificially suppressed; the dotted curve is an extension of this sequence which is shown merely as a reference line.

TABLE 3—Continued

8	9	10	11	12	13	14	15	16
8.913	8.962	9.111	9.320	9.431	9.567	9.742	9.908	9.959
-2.716	-2.788	-2.952	-3.208	-3.349	-3.628	-4.123	-4.890	-5.517
4.138	4.120	4.079	4.015	3.981	3.911	3.787	3.596	3.439
8.736	8.736	8.735	8.734	8.734	8.733	8.733	8.733	8.733
4.718	4.719	4.721	4.724	4.726	4.727	4.729	4.729	4.729
24.670	24.670	24.671	24.672	24.673	24.674	24.674	24.674	24.674
6.883	6.852	6.782	6.652	6.561	6.414	6.152	5.682	5.222
7.526	7.527	7.527	7.528	7.528	7.529	7.529	7.529	7.529
137.4	1474	1737	2346	2888	4059	7428	21880	63130
151.1	162.4	190.8	257.5	317.6	445.8	815.0	2405	6937
2.971	3.194	3.753	5.068	6.250	8.777	16.05	47.35	136.6
23.882	23.882	23.883	23.884	23.885	23.885	23.885	23.886	23.886
6.883	6.852	6.781	6.651	6.561	6.414	6.151	5.682	5.222
6.974	6.974	6.974	6.975	6.976	6.976	6.976	6.977	6.977
8.389	8.389	8.389	8.388	8.388	8.388	8.388	8.388	8.388
748.7	803.5	946.6	1278	1574	2211	4047	11920	34390
98.89	106.2	125.1	168.8	207.9	291.6	534.4	1575	4541
1.573	1.690	1.990	2.688	3.310	4.644	8.509	24.44	72.34
17.386	17.387	17.388	17.390	17.391	17.393	17.394	17.394	17.395
6.658	6.641	6.598	6.494	6.405	6.284	6.062	5.625	5.178
2.907	2.914	2.928	2.955	2.971	2.987	3.002	3.013	3.017
8.734	8.733	8.733	8.733	8.732	8.732	8.732	8.732	8.732
3.354	3.556	4.074	5.528	7.027	9.577	16.46	45.87	129.3
7.320	7.653	8.541	11.08	13.77	18.41	31.06	85.67	240.5
14.529	14.515	14.884	15.811	16.149	15.859	15.173	14.277	14.152
6.237	6.224	6.250	6.241	6.171	6.023	5.743	5.238	4.744
0.685	0.686	1.034	1.903	2.197	2.049	1.681	1.255	1.236
-3.018	-3.047	-3.092	-3.092	-3.063	-3.210	-3.534	-4.042	-4.182
-8.851	-8.864	-8.499	-7.576	-7.238	-7.532	-8.217	-9.114	-9.239
-2.04	-1.99	-1.22	1.00	2.83	3.14	3.17	4.95	15.4

TABLE 4
ADDITIONAL PROPERTIES OF SELECTED CRYSTALLIZING MODELS

Model	9	10	11	12	13	14	15	16*
$\log P_{\text{crys}}$	24.649	24.362	23.884	23.514	22.780	21.483	18.929	17.395
$\log T_{\text{crys}}$	6.852	6.782	6.651	6.561	6.413	6.149	5.666	5.178
$\log \rho_{\text{crys}}$	7.512	7.308	6.975	6.724	6.239	5.424	3.897	3.017
$\log R_{\text{crys}}$	7.595	8.173	8.388	8.497	8.591	8.685	8.728	8.732
q_{crys}	4.22×10^3	0.174	0.499	0.727	0.898	0.991	$1 - 1.50 \times 10^{-5}$	$1 - 1.00 \times 10^{-6}$
ψ_{crys}	1452	1375	1278	1167	887.5	523.7	161.4	129.3
Γ_{crys}	160.5	161.3	168.8	171.3	166.0	163.1	153.6	240.5
θ_{crys}	3.139	2.916	2.688	2.477	1.992	1.432	0.5098	0.0602

NOTE.—The subscript “crys” denotes the mass shell nearest the crystallization boundary.

* All quantities for this model refer to the interior-envelope boundary.

TABLE 5
WHITE DWARF CHRONOLOGY

Epoch	Model No.	Description
1.....	5	$L_v = L$
2.....	6	$c_{v \text{ elec}} = c_{v \text{ ion}}$ at $q = \frac{1}{2}$
3.....	9	Onset of crystallization
4.....	10	$\Theta_{\text{bcc}}/T = 1$ at $q = \frac{1}{2}$ *
5.....	11	Convection reaches the degenerate core ($\psi = 0$)
6.....	14	Crystallization reaches $q = 0.99$

* $\Theta_{\text{bcc}} = 0.444\Theta$ is the Debye temperature for a bcc lattice (Carr 1961).

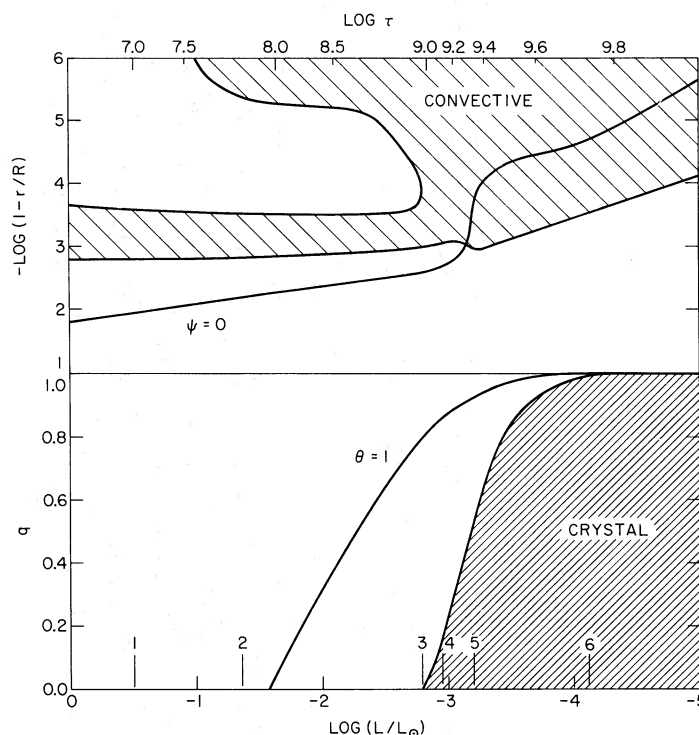


FIG. 4.—Changes in the internal structure of the star as a function of luminosity (or age). The epochs numbered 1–6 correspond to those listed in Table 5. Note the appearance of a second, outer convection zone near $\log(L/L_\odot) \sim -1.2$, the onset of crystallization (epoch 3), and contact of the convective region with the degenerate core (epoch 5). We have determined the curve $\psi = 0$ assuming an ideal electron gas so that its behavior entering the convective region may not be accurate.

For the purposes of this qualitative review we shall assume that the electrons behave as a nonrelativistic degenerate ideal Fermi gas; the results carry through in a more general treatment with only minor corrections. We also use the form of the ion equation of state appropriate to the crystal phase:

$$P_i = P_p + P_l, \quad P_l = \frac{1}{3}u_l\rho, \quad P_p = \frac{1}{2}u_p\rho. \quad (5)$$

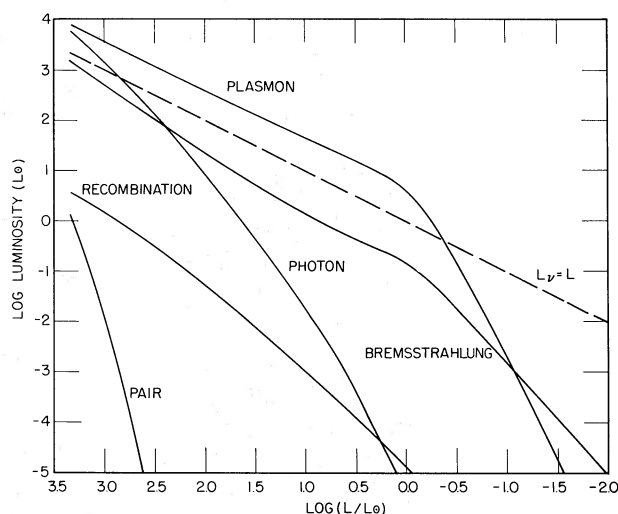


FIG. 5.—Luminosities of neutrino energy loss processes during the phase of rapid neutrino cooling as a function of photon luminosity. The dashed line corresponds to a neutrino luminosity equaling the photon luminosity.

Here P_i is the total ion pressure, P and u are the pressure and specific internal energy, and the subscripts l and p denote the static lattice and phonon contributions. For the internal energies we use the high-temperature crystal approximation:

$$u_l \simeq -\Gamma \frac{kT}{AH}, \quad u_p = 3 \frac{kT}{AH} \quad (\theta \gg 1). \quad (6)$$

These expressions also represent the dense liquid phase to a very good approximation (cf. § IIb and Fig. 2).

Prior to the onset of crystallization, the luminosity of the star is given by

$$L = - \int_0^M \left(\epsilon_v + T \frac{ds}{dt} \right) dm = -L_v - \int_0^M \left[\frac{du}{dt} + P \frac{d(\rho^{-1})}{dt} \right] dm, \quad (7)$$

where L_v is the neutrino luminosity, s is the specific entropy, and we have neglected nuclear burning and associated composition changes. The second term in the integral on the far right represents the work, $\int P dv$, done on the star by gravitational contraction. This can be shown as follows (cf. Mestel and Ruderman 1967). We first note that a differential change in the gravitational potential energy of a white dwarf, $\delta\Omega$, can be written in the following equivalent forms:

$$\delta\Omega = - \int_0^M Gm\delta(r^{-1})dm = - \int_0^M \frac{dP}{dm} (4\pi r^2 \delta r) dm = \int_0^M P\delta(\rho^{-1})dm. \quad (8)$$

This derives from the fact that a star is an equilibrium configuration, for which the first adiabatic variation of the total energy necessarily vanishes. Now in analogy with Van Horn (1971), we write $u = u(\rho, T)$. Then

$$\delta E \equiv \int_0^M \left(\frac{\partial u}{\partial T} \Big|_{\rho} \delta T + \frac{\partial u}{\partial \rho} \Big|_T \delta \rho \right) dm = \int_0^M \left[c_v \delta T + \left(T \frac{\partial P}{\partial T} \Big|_{\rho} - P \right) \delta(\rho^{-1}) \right] dm \equiv \delta E_{\text{thermal}} + \delta E_{\text{grav}} - \delta\Omega, \quad (9)$$

where c_v is the specific heat capacity. This demonstrates that in any first variation from the zero-temperature "black dwarf" configuration ($T = c_v = 0$), the thermal, density, and total variations of the total energy *all* vanish [that is, $\delta(E + \Omega) = \delta E_{\text{thermal}} = \delta E_{\text{grav}} = 0$].

Using equation (9), the expression for the luminosity of the star becomes

$$(L + L_v)\delta t = - \int_0^M c_v \delta T dm - \int_0^M T \frac{\partial P}{\partial T} \Big|_{\rho} \delta(\rho^{-1}) dm = -\delta E_{\text{thermal}} - \delta E_{\text{grav}}. \quad (10)$$

The fraction, δE_{grav} , of $\delta\Omega$ that contributes to the luminosity of the star is thus

$$\delta E_{\text{grav}} = \int_0^M T \left(\frac{\partial P_e}{\partial T} \Big|_{\rho} + \frac{\partial P_i}{\partial T} \Big|_{\rho} \right) \delta(\rho^{-1}) dm = \frac{15}{4} \left\langle \frac{1}{Z\psi} \right\rangle \delta\Omega, \quad (11)$$

where the degeneracy parameter $\psi \approx \epsilon_{\text{Fermi}}/kT \gg 1$ for a white dwarf, and we have neglected the first term in the integral since the thermal correction to the degenerate electron pressure is $O(\psi^{-2})$. Here and below the average is defined by the equation. Thus although $\delta\Omega$ is large, most of the work done by gravitational contraction is expended in raising the internal (Fermi) energy of the electrons, while only a fraction $O(1/Z\psi)$ becomes energy δE_{grav} which contributes to the luminosity of the star.

Let us next compare the energy δE_{grav} with the thermal energy content of the star, $\delta E_{\text{thermal}}$. We define

$$\delta E_{\text{Coul}} \equiv \int_0^M \delta u_l dm = \int_0^M \frac{\partial u_l}{\partial \rho} \Big|_T \delta \rho dm = \int_0^M \frac{P_i}{P} P \delta(\rho^{-1}) dm \approx \frac{5}{6} \langle \Gamma/Z\psi \rangle \delta\Omega. \quad (12)$$

The virial theorem gives

$$3 \int_0^M \frac{P}{\rho} dm = \int_0^M \frac{Gm}{r} dm. \quad (13)$$

Using the fact that the thermal correction to the degenerate electron pressure is $O(\psi^{-2})$ and that $\partial u_l / \partial T \Big|_{\rho} = \partial u_p / \partial \rho \Big|_T = 0$ to identify δu_p with $c_v \delta T$, we can combine equations (5), (12), and (13) to give

$$\frac{3}{4} \delta E_{\text{thermal}} + \delta E_{\text{grav}} = \frac{1}{2} \delta (E_{\text{Coul}} + \Omega). \quad (14)$$

From this expression and our earlier results, we then have

$$\delta E_{\text{thermal}} \sim \delta\Omega, \quad \delta E_{\text{Coul}} \sim \langle \Gamma/Z\psi \rangle \delta\Omega, \quad \delta E_{\text{grav}} \sim \langle 1/Z\psi \rangle \delta\Omega. \quad (15)$$

Thus Coulomb interactions contribute a correction $O(\Gamma/Z\psi)$ to the potential energy of the star, while $\delta E_{\text{grav}}/\delta E_{\text{thermal}} \approx O(1/Z\psi)$. This demonstrates the fact that it is essentially correct to speak of a white dwarf as radiating at the expense of its thermal energy content.

We now turn to a discussion of crystallization. We note first that equation (11) can be equally well applied to the gravitational energy $\delta E_{\delta\rho/\rho}$ released by the discontinuous change in density across the phase boundary. In addition, the energy released as latent heat during crystallization is

$$\delta E_{\text{latent heat}} = \int_0^M \delta u_{\text{latent heat}} dm = \int_0^M \frac{q}{AH} dm. \quad (16)$$

Comparing equations (11) and (16), and noting from our earlier work that $q \approx -kT$ and $\delta\rho/\rho \approx 10^{-4}$ (see Table 1), we have

$$\delta E_{\delta\rho/\rho}/\delta E_{\text{latent heat}} \approx \frac{3}{2} \left\langle \frac{\delta\rho}{\rho} \frac{kT}{-q} \right\rangle \approx 10^{-4}. \quad (17)$$

The gravitational energy released by the slight increase in density accompanying the phase transition is thus negligible compared with the latent heat. It is also interesting to note that

$$\delta E_{\delta\rho/\rho}/\delta E_{\text{latent heat}} \sim \delta E_{\text{grav}}/\delta E_{\text{thermal}} \sim O(1/Z\psi). \quad (18)$$

From these results, the total luminosity of a white dwarf undergoing crystallization is given by

$$L + L_\nu = \frac{d}{dt} (E_{\text{thermal}} + E_{\text{grav}} + E_{\text{latent heat}} + E_{\delta\rho/\rho}) \equiv f(L/L_\odot) \frac{k}{AH} M \frac{dT_c}{dt}. \quad (19)$$

Here we have introduced the cooling function $f(L/L_\odot)$ (cf. Van Horn 1968), which gives the effective heat capacity per ion of the star in dimensionless units. In Figure 6a we show the cooling function throughout the white dwarf

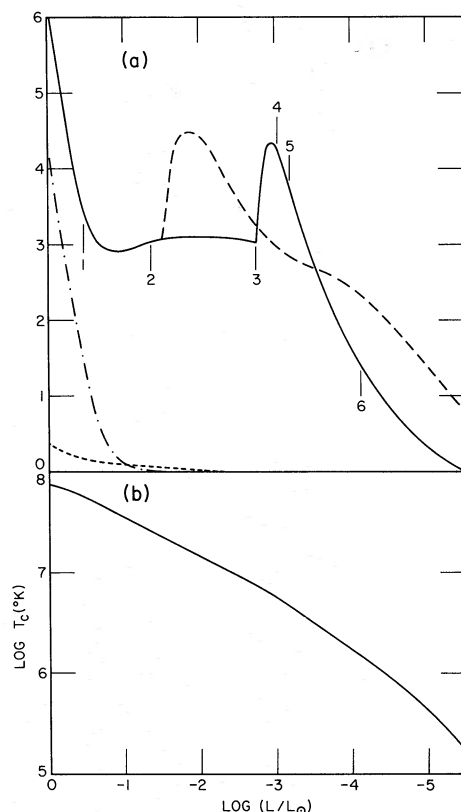


FIG. 6.—(a) The curves correspond to the following quantities: —, $f(L/L_\odot)$ for a $1 M_\odot$ pure ^{12}C white dwarf; ---, $f(L/L_\odot)$ for a $1.081 M_\odot$ white dwarf, taken from Van Horn (1968); -·-, $L_\nu(AH/kM)/(dT_c/dt)$; ···, $(dE_{\text{grav}}/dt)(AH/kM)/(dT_c/dt)$. The epochs numbered 1–6 correspond to those listed in Table 5. (b) Central temperature versus luminosity.

phase, calculated numerically from 46 models spanning this interval. We also show the two quantities $L_\nu(AH/kM)/(dT_c/dt)$ and $(dE_{\text{grav}}/dt)(AH/kM)/(dT_c/dt)$ which measure neutrino energy losses and the energy supplied by gravitational contraction, respectively. Several models undergoing crystallization were only partially converged, as discussed in § II f, and this introduced some noise into $f(L/L_\odot)$, which we have removed by two passes with a first degree, three point least-squares smoothing function (Hildebrand 1956, p. 258). In Figure 6b we show $\log T_c(L/L_\odot)$.

Figures 4, 6, and 3 illustrate the correlation between the internal structure, the cooling function, and the luminosity of the star. In these figures, we can identify the following epochs, listed in Table 5. As L_ν falls below the photon luminosity L (epoch 1), the time scales for the decrease of both the thermal energy content and the luminosity of the star become longer. As the heat capacity of ions begins to dominate that of the electrons (epoch 2), $f(L/L_\odot)$ approaches the value 3, characteristic of a high-temperature crystal. With the onset of crystallization (epoch 3) the cooling function develops a sharp rise as latent heat supplies additional energy, and for the same reason the luminosity decline suddenly slows. As the latent heat release subsequently begins to slow, Debye cooling becomes important (epoch 4) and the thermal energy content of the star begins to fall rapidly. Soon thereafter deep convection reaches the degenerate core (epoch 5), contributing even further to the increasingly rapid decline of the central temperature and luminosity. Figures 3 and 6 show dramatically the combined effects of Debye and convective cooling and imply ages of $\tau < 10^{10}$ years for white dwarfs more massive than $1 M_\odot$, as previously indicated by the calculations of Östriker and Axel (1968). Although the relative importance of the two processes is somewhat uncertain, the comprehensive study of white dwarf convection carried out by Fontaine (1973; see also Fontaine and Van Horn 1975) indicates that Debye cooling probably dominates.

The increase in the heat capacity, C_v , due to Coulomb interactions ($C_v/Nk \sim 3/2 \rightarrow \sim 3$) indicated in Figure 6 (recall also Fig. 2) approximately doubles the white dwarf lifetime in comparison with earlier estimates as first pointed out by Mestel and Ruderman (1967) and Van Horn (1968). We can also estimate the additional increase in lifetime due to the release of latent heat directly from the data shown in Figure 3: from comparison of the solid and dashed curves, we find $\delta\tau/\tau \approx 0.30$ ($\delta \log \tau \approx 0.11$). This agrees with the approximate ratio $\delta E_{\text{latent heat}}/\delta E_{\text{thermal}} = \frac{1}{3}$ and indicates that the release of latent heat increases the white dwarf lifetime by 60–70 percent relative to Mestel's (1952) original theory as expected (Van Horn 1968).

It is also of interest to estimate the sensitivity of the luminosity at the onset of crystallization to changes in the equation of state. For this purpose, we use as reference the small difference in Madelung energies between a bcc lattice and Wigner-Seitz spheres ($\delta u_l \approx 10^{-1} kT/AH$). At $\log \rho_c = 7.53$, a difference of this size would shift the values of Γ and $\log T_c$ at crystallization from 157.2 to 113.9 ($\delta \log \Gamma \approx 0.14$) and from 6.85 to 6.99 ($\delta \log T_c \approx 0.14$), respectively (recall Fig. 1 and Table 1), which corresponds to an uncertainty in the luminosity at the onset of crystallization of $\log (L/L_\odot) = -2.79 \pm 0.40$.

c) Convection

We restrict ourselves here to a few brief comments on convection. Comprehensive and detailed discussions of the role of convection in white dwarfs may be found in Fontaine and Van Horn (1975); Fontaine *et al.* (1974); and in the references quoted therein.

The marked difference in the ionization potential of the K and L shell electrons of carbon enables the envelopes of pure carbon white dwarfs to develop two separate convection zones. Depending on the luminosity and effective temperature of the star, one or both of these convection zones may be present, or, as occurs at low luminosities, the two zones may merge to form one large convective region. This general pattern is present throughout the evolution of our $1 M_\odot$ pure ^{12}C star and is illustrated in Figure 4.

During the early pre-white dwarf phase of gravitational contraction, the luminosity and effective temperature of the star rise. This has the effect of driving the two convection zones which are present outward through the envelope. Near $\log (L/L_\odot) = 3.00$, $\log T_e = 4.76$ the outer zone reaches the base of the atmosphere and then disappears. As the luminosity and effective temperature continue to increase, the inner zone also retreats outward until, near maximum luminosity, it too reaches the base of the atmosphere. At this point, the entire convection zone lies within a mass fraction 10^{-12} of the surface, in a region which is optically thin. As a result, convective transport of energy is unimportant.

As the luminosity and effective temperature begin to fall the convection zone returns to greater depths. Near $\log (L/L_\odot) = -1.18$, $\log T_e = 4.52$ (values which are somewhat sensitive to the envelope equation of state), the outer convection zone again reappears. At $\log (L/L_\odot) = -2.90$, $\log T_e = 4.08$ the two convection zones merge, and near $\log (L/L_\odot) = -3.20$, $\log T_e = 4.05$ the convective region reaches the degenerate core ($\psi = 0$). Nonideal effects are important in the envelope already at $\log (L/L_\odot) = -2$ ($\Gamma_e \approx 5$; see Table 3) and require accurate treatment in the equation of state (Fontaine *et al.* 1974). At $\log (L/L_\odot) = -3.40$, $\log T_e = 3.97$ the convective region reaches its greatest depth, extending through a mass fraction 5.5×10^{-8} . The increasing degeneracy of the envelope subsequently drives the convective region outward again as the corresponding increase in the efficiency of electron conduction flattens the temperature gradient at the base of the envelope. At the lowest luminosity for which calculations were carried out, $\log (L/L_\odot) = -5.60$, the entire convective region is degenerate, beginning at optical depth $\tau_0 = 0.275$. The physical approximations employed in the opacity and equation of state calculations have become sufficiently insecure in this extreme case, however, that the results for this model can be regarded only as indicative of the true physical conditions.

d) Structure of the Crystallizing Star

We have singled out model 12 listed in Tables 3 and 4 to exemplify the structure of a $1 M_{\odot}$ carbon white dwarf undergoing crystallization. Figures 7 and 8 illustrate the structure of the model. The time step between the previous calculated model and model 12 is $\delta\tau = 2.2 \times 10^8$ years, corresponding to the changes $\delta \log T_c = -0.028$ and $\delta \log (L/L_{\odot}) = -0.051$. During this interval, the crystallization boundary has moved outward from $q = 0.0648$ to $q = 0.727$ (its position in model 12). Figure 7 shows, as a function of the mass fraction q , the temperature and density and values of the dimensionless variables ψ , Γ , and θ , which specify the thermal properties of the ions at any point within the star. Since the values of these dimensionless variables are mainly determined by the density distribution, which itself depends primarily upon the electronic equation of state, all of the variables change quite smoothly across the crystallization boundary [a discontinuity of $O(\psi^{-1})$ does occur in each, but this is too small to be shown in Fig. 7]. In contrast, the quantities plotted in Figure 8 depend sensitively on the thermal properties of the ions, and all show large changes at the crystallization boundary. The discontinuity in entropy at the crystallization boundary is a measure of the latent heat, while the thermal energy supplied by each point in the star, $T\delta s$, reflects the contribution from the heat capacity C_v/Nk as well as that from latent heat over the time interval $\delta\tau$. The size of $\delta(L/L_{\odot})$ across the crystallizing region indicates that the release of latent heat supplies ~ 30 percent of the luminosity of model 12 in agreement with our earlier estimate $\delta\tau/\tau \approx 0.30$. We note that the fact that C_v/Nk has fallen to a value of 1.57 at the center of the star shows that the effects of Debye cooling are already significant.

e) Luminosity and Discovery Functions

To compare theory and observation, it is useful to construct the luminosity function $\Phi(L/L_{\odot})$ and the "discovery function" $P(L/L_{\odot})$. The quantities $\Phi(L/L_{\odot}) d \log (L/L_{\odot})$ and $P(L/L_{\odot}) d \log (L/L_{\odot})$ represent the relative space density of nearby white dwarfs and the probability of observing a nearby white dwarf, respectively, in the luminosity interval $d \log (L/L_{\odot})$. If the birth rate of white dwarfs has been uniform in space and constant over the lifetime of the galaxy, the luminosity function for white dwarfs of a particular composition and mass is just $\Phi(L/L_{\odot}) \propto [d \log (L/L_{\odot})/d\tau]^{-1}$. If we also assume an observational survey complete down to a given apparent magnitude,

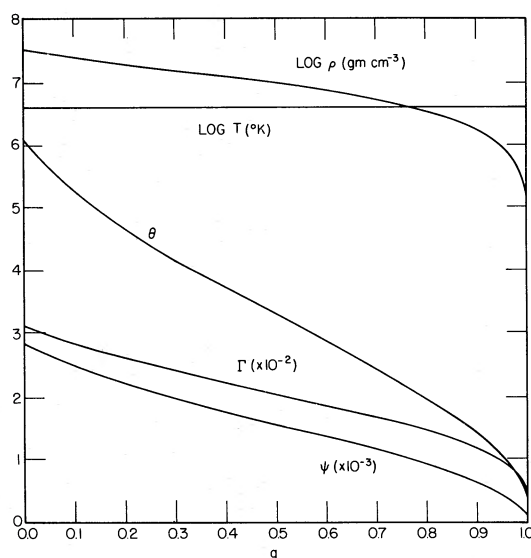


FIG. 7

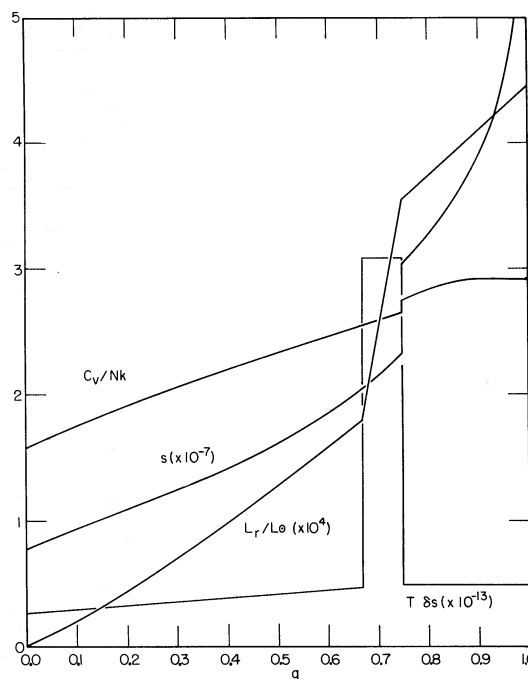


FIG. 8

FIG. 7.—The density ρ , temperature T , and dimensionless variables ψ , Γ , θ as a function of mass fraction q for model 12 listed in Tables 3 and 4. The crystallization boundary lies at $q = 0.727$. Discontinuities of fractional size $O(10^{-4})$ in the density and in ψ , Γ , and θ are too small to be shown.

FIG. 8.—The heat capacity per ion, C_v/Nk (in dimensionless units); the specific entropy s (in $\text{ergs g}^{-1} \text{K}^{-1}$); and the luminosity at radius r , L_r (in solar units); as a function of mass fraction q for model 12 listed in Tables 3 and 4. The crystallization boundary lies at $q = 0.727$. The quantity δs is the change in entropy between the previous calculated model (in which the crystallization boundary lay at $q = 0.648$) and model 12; the quantity $T\delta s$ (in ergs g^{-1}) shows the thermal energy supplied by each point in the star.

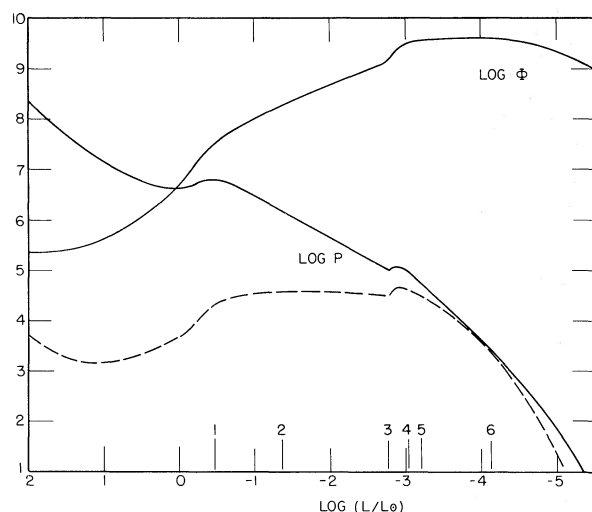


FIG. 9

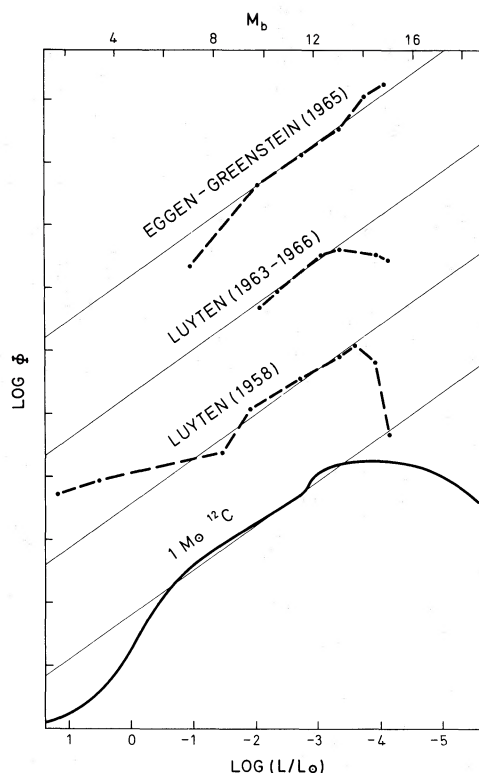


FIG. 10

FIG. 9.—The luminosity function Φ and the discovery function P . The dashed curve shows the discovery function after taking into account a bolometric correction appropriate for a black body. The epochs numbered 1–6 correspond to those listed in Table 5.

FIG. 10.—Theoretical luminosity function for a $1 M_{\odot}$ pure ^{12}C star (heavy solid line) and the observational functions derived by Weidemann (1967) from the observations of Luyten and of Eggen and Greenstein (dashed lines) as a function of luminosity (or bolometric magnitude M_b). The behavior of the luminosity function predicted by Mestel's (1952) original cooling theory [$\log \Phi(L/L_{\odot}) \propto -5 \log (L/L_{\odot})/7$] is also shown (light solid lines).

the corresponding discovery function is $P(L/L_{\odot}) \propto \Phi(L/L_{\odot})(L/L_{\odot})^{3/2}$. Figure 9 shows curves for $\log \Phi$ and $\log P$ calculated numerically from models spanning the cooling phase of the $1 M_{\odot}$ star.

Because of longer cooling time scales at lower luminosities, the luminosity function increases with decreasing luminosity. Release of latent heat during crystallization does not produce a peak in the luminosity function (however, see Fig. 10), although it does produce a small peak in the discovery function. This peak may be somewhat larger in lower-mass stars in which Debye cooling begins at lower luminosities and release of latent heat supplements a relatively constant, as opposed to a rapidly falling, ionic heat capacity. The rapid decrease in luminosity caused by Debye cooling also produces the broad maximum in the luminosity function below $\log L/L_{\odot} \approx -3$, with a width of about $\delta \log (L/L_{\odot}) \approx 2$. It is interesting that the second small peak in the discovery function corresponds to the epoch at which $L_v = L$.

IV. DISCUSSION

a) Theoretical Work

It is of interest to compare our results, based on detailed evolutionary calculations, with the earlier results obtained by Van Horn (1968) and by Koester (1972) using approximate treatments of crystallization. Van Horn's calculations combined Chandrasekhar models of white dwarf structure with a Debye model for the ion heat capacity and a latent heat contribution based on equation (16). The resulting cooling function for an $M = 1.081 M_{\odot}$ pure ^{12}C white dwarf is superposed on the cooling function for our $1 M_{\odot}$ white dwarf in Figure 6. The comparison shows that the earlier result follows the qualitative form of our cooling function quite closely. The principal difference lies in the much higher luminosity at which crystallization begins in the earlier model. This is a direct result of the value $\Gamma = 52$ at crystallization used in the earlier work. Following the onset of convection and Debye cooling, the two cooling functions are qualitatively similar but show quantitative differences whose source is at present not clear.

We can also compare our discovery function with that of Van Horn. Close inspection of his Figure 4 shows that for ^{12}C the peak in the discovery function is narrow and relatively weak, in qualitative agreement with our results

shown in Figure 6. If account is taken of the difference in Γ at crystallization, the results appear to be consistent. Van Horn also found that the discovery function increased at high luminosities [$-1.5 \leq \log(L/L_\odot) \leq -2.5$] and suggested that inclusion of neutrino energy losses might alter this. Our present results however, show that for a $1 M_\odot$ white dwarf neutrino losses are unimportant below $\log(L/L_\odot) \approx -0.5$.

Koester (1972) combined detailed models of the outer convective envelope with Chandrasekhar models of the interior, and his treatment of crystallization generally followed that of Van Horn (1968). For the ionic heat capacity, however, Koester used the value $C_v/Nk = 3/2$ in the dense liquid phase, changing discontinuously to the high-temperature crystal value $C_v/Nk = 3$ at crystallization. As we saw in § IIb, Coulomb interactions are already sufficiently important in the dense liquid phase to produce an ionic heat capacity close to the high-temperature crystal value of $C_v/Nk = 3$. Consequently there is in fact only a relatively small discontinuity across the phase transition boundary (see also Mestel and Ruderman 1967). The effect of Koester's incorrect physical approximation can be seen clearly in his mean stellar heat capacity (his Fig. 6) which shows a sharp rise as the crystal phase begins to contribute. *This masks, to a large extent, the contribution from latent heat.*

Koester plotted luminosity functions for $M = 0.57 M_\odot$ white dwarfs of ^{12}C and ^{24}Mg composition (his Fig. 8). He found no feature corresponding to the release of latent heat, but—in addition to the masking effect caused by his treatment of C_v —his bin size was sufficiently large that a small effect of the size we have found would not have been visible. In addition, the luminosity functions do not show a decline due to Debye and convective cooling since these become important for the intermediate-mass white dwarfs considered only at luminosities below those plotted. There thus exist substantial differences in the treatment of crystallization and in the display of results between our work and Koester's.

Another point of interest is the relation between the central temperature and luminosity for white dwarfs: Mestel (1952) derived the relation $d \log(L/L_\odot)/d \log T_c = 3.5$, assuming a Kramers law opacity in the envelope. Van Horn (1968) and Ostriker and Axel (1969) found values of 2.7 and 2.3, respectively, by fitting to existing model calculations, and Koester (1972) in the work described above found a value of 2.5. The curve of central temperature versus luminosity for our $1 M_\odot$ pure ^{12}C star included in Figure 6 shows that $d \log(L/L_\odot)/d \log T_c$ varies almost continuously with luminosity, making it difficult to compare our results with the earlier values. We therefore quote results separately for three luminosity intervals of possible interest, using the endpoints to determine the slope. Over the intervals $\log(L/L_\odot) = -1$ to the onset of crystallization and from the latter point to $\log(L/L_\odot) = -4.60$ we find the values 2.6 and 1.8, respectively. For the entire interval $-1 \leq \log(L/L_\odot) \leq -4$, we find a value of 2.3.

Finally, it is of interest to compare the central density and radius of our coolest model with the values for $1 M_\odot$ ^{12}C zero-temperature Chandrasekhar (1939) and Hamada and Salpeter (1961) models. The results, given in Table 6, provide a good measure of the accuracy of our calculations.

b) Observation

The sample of known white dwarfs is strongly concentrated in the luminosity interval $-2 \leq \log(L/L_\odot) \leq -4$ (cf. Weidemann 1968; Van Horn 1968). Weidemann (1967) has derived observational luminosity functions for roughly this interval from the surveys of Luyten (1958, 1966) and of Eggen and Greenstein (1965a) by correcting for observational selection effects and employing a bolometric correction derived from the luminosity-color and two-color diagrams of Eggen and Greenstein (1965a). Figure 10 compares the theoretical luminosity function for the $1 M_\odot$ pure ^{12}C star with these observational functions. The behavior of the luminosity function predicted by Mestel's (1952) original cooling theory [$\log \Phi(L/L_\odot) \propto -\frac{2}{3} \log(L/L_\odot)$] is also shown. The $1 M_\odot$ theoretical luminosity function deviates strongly from that of Mestel's cooling theory above $\log(L/L_\odot) \approx -0.5$ due to neutrino energy losses and below $\log(L/L_\odot) \approx -3.5$ due to Debye cooling. Agreement between the theoretical luminosity functions and the observational functions derived by Weidemann is generally good, and it seems quite likely that the luminosity function for a stellar mass closer to the average for white dwarfs ($M \approx 0.8 M_\odot$ [Greenstein and Trimble 1967; Wehrse 1975]) would improve the agreement even more.

Important results of our calculations are theoretical luminosity and discovery functions extending from above to well below the luminosity interval containing most known white dwarfs. These functions provide a framework for further comparison with observation. For example, in contrast to the number of known white dwarfs, the

TABLE 6
COMPARISON WITH $1 M_\odot$ ZERO-TEMPERATURE MODELS

Quantity	Chandrasekhar*	Hamada-Salpeter*	This Work†
$\log \rho_c$	7.445 ± 0.006	7.517 ± 0.006	7.529
$\log R$	8.758 ± 0.002	8.734 ± 0.002	8.733

* Calculated by four point Aitken-Lagrange interpolation in Tables 1A and 1B of Hamada and Salpeter (1961).

† These values are taken from model 16 in Tables 3 and 4.

theoretical discovery function shown in Figure 9 continues to increase with increasing luminosity. At high luminosities the bolometric correction becomes very large, and this should be taken into account in the discovery function. However, although the changing weight factor flattens the discovery function it does not produce a cutoff at $\log(L/L_\odot) \approx -2$ (see Fig. 9). At very high luminosities the spatial distribution of the white dwarfs can no longer be presumed uniform, and the influence of the Galactic disk must be considered. If we assume the scale height of the white dwarf distribution in the Galaxy to be 300 pc (Allen 1963) and take the observational survey to be complete down to an apparent visual magnitude $m_V \approx 16$ (Eggen and Greenstein 1965*a, b*, 1967), the disk character begins to be important at $\log(L/L_\odot) \gtrsim +1.5$.

The lack of correspondence at high luminosities between the discovery function and the distribution of observed white dwarfs may arise partly from other observational selection effects. For example, selection of white dwarf candidates on the basis of proper motions (Luyten 1958, 1963) discriminates against high-luminosity objects because of their greater average distance. In any case, the difference between observation and current theory is significant, and it seems likely that the resolution of the problem may provide new information concerning the distribution and origin of the white dwarfs. Our results emphasize that interpretation of the small number of known high-luminosity white dwarfs as evidence for the existence of the universal Fermi weak interaction (Chin, Chiu, and Stothers 1966) must be approached with considerable caution.

At low luminosities, Debye cooling produces a rapid decrease in the luminosity and discovery functions. However, our results show that in a $1 M_\odot$ pure ^{12}C star the effect of Debye cooling is delayed until $\log(L/L_\odot) \approx -4$ because this cooling and the release of latent heat by continuing crystallization overlap. Debye cooling should begin at even lower luminosities in lower-mass white dwarfs. Even so, we note that at low luminosities the small volume of space searched may begin to play a role. For example, if we again consider a survey complete down to $m_V \approx 16$, at a luminosity of $\log(L/L_\odot) \approx -4$, such a survey will sample out to less than 20 pc. At even lower luminosities, the distance sampled by such a survey may become less than the distances to the nearest white dwarfs.

Greenstein (1969*a, b*, 1971, 1974) has frequently suggested that a deficiency of faint white dwarfs does exist. The luminosity functions based on the Luyten surveys do show a drop at low luminosities, but this may be due to decreasing completeness and accuracy of the proper-motion survey material near the faint magnitude limit (Weidemann 1967). The function derived from the spectroscopic survey of Eggen and Greenstein (1965*a*) shows no deficiency, but the survey may include stars that are not white dwarfs (Greenstein 1969*a*; Eggen 1969). Most recently a search for faint white dwarfs has been carried out with some success (Hintzen and Strittmatter 1974). This search is ongoing and may provide important evidence concerning the possible deficiency of faint white dwarfs.

There is a great need at present to determine whether observational evidence of white dwarf crystallization exists. Figures 3 and 10 indicate that the effect produced by the release of latent heat during crystallization is too small to provide such evidence at present. Debye cooling might offer such a possibility, however, because its effect is quite dramatic; unfortunately this may not be conclusive because high-density quantum liquids are expected to show a similar effect. The above theoretical and observational results indicate that an observational luminosity function accurate to below $\log(L/L_\odot) \approx -5$ may be required to provide strong evidence of Debye cooling and hence, indirectly, of crystallization.

We note finally that white dwarf ages determined from our cooling curve are consistent with the ages of white dwarfs in clusters. For example, the approximate ranges of luminosities of white dwarfs in the Hyades and Praesepe clusters are $-1.0 \leq \log(L/L_\odot) \leq -2.4$ and $-2.3 \leq \log(L/L_\odot) \leq -2.8$ respectively (Shipman 1972). If we take the ages given by the cooling curve for the $1 M_\odot$ pure ^{12}C star as representative, we find white dwarf ages of 4×10^7 – 5×10^8 years and 4×10^8 – 9×10^8 years, respectively. These compare with the cluster ages of $(4\text{--}5) \times 10^8$ years found by Kraft (1967) by comparison of color-magnitude diagrams and theoretical evolutionary tracks, and of 9×10^8 years suggested by van den Heuvel (1969) on the basis of revised effective temperature-color index relations near the main-sequence turnoff for the Hyades cluster.

V. CONCLUDING REMARKS

Perhaps the most significant result of the present, detailed evolutionary calculation is the finding of good qualitative agreement with the earlier work of Mestel and Ruderman (1967) and of Van Horn (1968), apart from the specific value of the parameter Γ at crystallization. Our work has used the new, accurate, dense-plasma equation of state developed by Lamb (1974), in which an exact thermodynamic treatment of the phase transition yields $\Gamma = 160 \pm 15$. Our pre-white dwarf models are also the first to include the neutrino bremsstrahlung emission process, although we find that this mechanism never contributes more than 10 percent of the total neutrino luminosity in those phases where neutrino losses dominate.

Our calculations have also served to provide several new goals for future work. Specifically, in view of the significant increase in the accuracy with which we treat the physics, it is now of interest to compute rather detailed grids of cooling white dwarf models. Such grids are essential if meaningful interpretations of the observed statistics are to be obtained, and this alone can materially improve upon the brief discussion of the luminosity function given here. Similarly, further study of ion quantum effects and mixtures in the dense plasma interior is required while much more detailed work is needed on theories of opacities and equations of state at the low temperatures and high densities characteristic of the envelopes of very cool white dwarfs ($T_{\text{eff}} \lesssim 6000$ K). Our present knowledge of

the theory is inadequate to permit us to determine whether the apparent deficiency of red degenerates (Greenstein 1969a) is consistent with current models for the hotter white dwarfs; there are some indications that it is not. Finally, both observational and theoretical work is needed to clarify the situation in the case of the intrinsically bright white dwarfs: Are the observed numbers of the hotter stars too small in comparison with theory? What is the effect of the hot subdwarfs upon this distribution? What can the distribution of these stars in the Galaxy tell us about their structure and origin?

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