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ANALYTIC STELLAR MODELS IN GENERAL RELATIVITY*

RICHARD C. ADAMS† AND JEFFREY M. COHEN‡
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ABSTRACT

In this paper we describe a method for deriving analytic stellar models in general relativity. The models are exact solutions to Einstein's equations. By inverting the usual procedure in astrophysics, we reduce Einstein's static field equations to one first-order linear differential equation. Such an equation can always be reduced to quadratures. Two new exact solutions are presented and the rotational properties explored.

Subject headings: collapsed stars — neutron stars — pulsars

I. INTRODUCTION

For a given equation of state the determination of the structure of a neutron star requires large complicated computer programs. Since one has neither perfect knowledge concerning a nuclear potential (Reid 1968; Bludman 1973; Börner 1973) and its applicability, nor complete confidence in the present state of many body theory at high densities, a less ambitious program is often sufficient. Also, it is desirable to have a quick, easy way to determine the relationships between the mass, radius, moment of inertia, stability, and other physical quantities without extensive computer analysis. For these reasons we present a method for calculating analytically the structure of a spherically symmetric star in general relativity. The method relies upon being able to do two definite integrals. The equation of state, i.e., the relationship between the pressure and the density for cold matter, is the end result of the calculation rather than the beginning. The solution will be taken as reasonable if both the pressure and density are positive and monotonically decreasing functions of r, if the pressure goes to zero at a finite radius, and if the mass, radius, and moment of inertia have reasonable values compared with computer-generated ones for the same central pressure and density. The idea behind this is similar to that used to solve the rotation equation for incompressible matter (Adams *et al.* 1974). The method used in that paper will be used to calculate the angular momentum as a function of the rotation rate.

II. THE METHOD

The metric for a static, spherically symmetric star is (Landau and Lifshitz 1971):

$$ds^2 = -A^2dt^2 + B^2dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{1}$$

where A and B are functions only of r. This metric leads to the familiar form of Einstein's field equations for a static body:

$$8\pi T^{00} = r^{-2} - (rB)^{-2} + 2B_r(rB^3)^{-1}\,, \tag{2}$$

$$8\pi T^{11} = 2A_r(rAB^2)^{-1} - r^{-2} + (rB)^{-2}\,, \tag{3}$$

$$8\pi T^{22} = 8\pi T^{33} = A_{rr}(AB^2)^{-1} - A_rB_r(AB^3)^{-1} + A_r(rAB^2)^{-1} - B_r(rB^3)^{-1}\,, \tag{4}$$

in which the subscript r denotes differentiation with respect to r. The usual assumption, that the matter is isotropic, implies

$$\rho = T^{00}\,, \tag{5}$$

$$p = T^{11} = T^{22} = T^{33}\,. \tag{6}$$

After subtracting equation (3) from equation (4), equation (6) leads to

$$0 = -rB_rB^{-3}(A + rA_r) + B^{-2}(r^2A_{rr} - rA_r - A) + A\,. \tag{7}$$

It is interesting to note that Buchdahl (1959; 1967) independently investigated a similar equation using different methods. He derived a second-order equation which must be solved to obtain A in terms of the average density $[\bar{\rho} = 3m(r)(4\pi r^3)^{-1}]$. He obtained a solution for A and $\bar{\rho}^{-1}$ in terms of a hypergeometric function and a linear

* Supported in part by the Naval Research Laboratory, Washington, D.C. 20375.
† Physics Department, University of Pennsylvania.
‡ Physics Department, University of Pennsylvania and Mathematical Sciences, Naval Research Laboratory.

function in r^2, respectively. Our approach has the advantage that we solve a first-order equation for B in terms of A rather than a second-order equation for A in terms of B. By letting $x = r^2$, denoting differentiation with respect to x by a prime, and defining f by

$$f = (1 - B^{-2})/x\,, \tag{8}$$

equation (7) can be written as

$$f' + 2f(A' + 2xA'')(A + 2xA')^{-1} = 4A''(A + 2xA')^{-1}\,. \tag{9}$$

Assuming that A is a known function of x, f can be found in terms of a function g:

$$g'(x) = 2(A' + 2xA'')(A + 2xA')^{-1} \tag{10}$$

via the relation

$$f(x) = e^{-g(x)}\left\{f_0 + 4\int_0^x dtA''(t)e^{g(t)}[A + 2tA'(t)]^{-1}\right\}, \tag{11}$$

where f_0 is a constant of integration. The method for finding analytic models is to assume a reasonable but definite form for A as a function of x and to perform the integrals in equations (10) and (11). Once f is known, the pressure and density can be found by expressing equations (2) and (3) as

$$8\pi\rho = 3f + 2xf'\,, \tag{12}$$

$$8\pi pA = 4A' - f(A + 4xA')\,. \tag{13}$$

The total mass, defined from the exterior Schwarzschild solution,

$$m(R) = R(1 - B^{-2})/2\,, \tag{14}$$

can be expressed as

$$m = R^3f(R^2)/2\,. \tag{15}$$

The integration constant f_0 can be found by the requirement that the pressure in equation (13) be zero at the outer boundary of the star, radius R. In general, if one chooses a certain form for A and does the integrals to find f and p, the density will not be zero at the radius R. The requirement that this occur restricts the possible forms of A.

For gaseous stars, both the pressure and density vanish at the outer boundary of the star. In the method presented in this paper all physical quantities are determined from A. Hence, it is useful to formulate this condition on the density and pressure in terms of A alone. By evaluating equations (9), (12), and (13) at the outer boundary, this restriction on A takes the form

$$0 = 3A(R^2)A'(R^2) + 2R^2[A'(R^2)]^2 + 2R^2A(R^2)A''(R^2)\,. \tag{16}$$

The procedure used here is an inversion of the usual problem solved in astrophysics. Instead of assuming that some equation of state is given and then solving the equations of structure, e.g., the Tolman-Oppenheimer-Volkoff (TOV) equations for the metric coefficients (Harrison *et al.* 1965; Börner and Cohen 1973; Cohen and Cameron 1971), we assume some form for the metric coefficients and find what equation of state induced this form. If we make a judicious choice of $A(x)$, then a simple but physically significant form for $p(\rho)$ will result.

III. ROTATIONAL PROPERTIES

Some of the material in this section has been dealt with elsewhere (Adams *et al.* 1974; Brill and Cohen 1966), but for completeness we will give here a brief review. The equations for a fluid body rotating about some axis are, in general, difficult to solve. The simplifying assumption of slow rotation, i.e., that the centrifugal force is small compared with the gravitational force, makes the problem tractable. Through first order in Ω (the metric coefficient which measures the dragging of inertial frames by the rotating body [Brill and Cohen 1966; Cohen and Brill 1968]), the equations to be solved are the three structure equations (2), (3), and (4), and a rotation equation. This rotation equation can be written as

$$-16\pi r^4B^2(\omega - \Omega)(\rho + p) = (r^4\Omega_r)_r - 4\pi r^5\Omega_rB^2(\rho + p)\,, \tag{17}$$

where ω is the rotation rate about the z-axis as measured by a distant observer. Even if one has solved the equations of structure analytically, the rotation equation is often too difficult to solve analytically, except for a star (with equation of state $p = \alpha\rho$) surrounded by a thin shell (Adams *et al.* 1973). In more complicated cases, information about the rotational properties can be obtained simply by inverting the problem. Instead of trying to solve the rotation equation for some assumed rotation rate ω (for rigid rotation ω is a constant independent of r), we assume some reasonable form for the dragging of inertial frames and (from eq. [17]) find what form of ω produced this result. The advantages of this procedure are that the equations can be simply and analytically solved, and that one readily knows how close to the physically desired solution one is.

In order to match smoothly to the exterior solution (Cohen and Brill 1968)

$$\Omega(r) = 2Jr^{-3}, \tag{18}$$

in which J is the total angular momentum of the body, Ω must satisfy the boundary condition at the stellar radius R that Ω and Ω_r be continuous.

It was found (Adams *et al.* 1974) that a form for Ω which produced substantial variation in ω was

$$\Omega(r) = \Omega_0[1 - 0.6(r/R)^2]. \tag{19}$$

Subsequent to this work (Adams 1974), it was found that a form for Ω which gave less differential rotation for the relativistic incompressible stars ($\sigma = p_c/\rho_c \sim \frac{1}{3}$) at the expense of Newtonian stars ($\sigma \ll 1$) was

$$\Omega(r) = \Omega_0\{1 - 3r^2[R^2(2\alpha + 3)]^{-1}\}^{\alpha} \tag{20}$$

for $\alpha \sim \frac{1}{2}$. This form for Ω with $\alpha = \frac{1}{2}$ will be used for one of the examples in the next section. If α is a function of σ, then the differential rotation is small over a wide range of σ (Adams 1974).

In terms of $x = r^2$, $\Omega(x)$, and $f(x)$, $\omega(x)$ can be written from equation (17) as

$$\omega(x) = \Omega(x) + x\Omega'(x)/2 - (5\Omega' + 2x\Omega'')(1 - xf)[8\pi(\rho + p)]^{-1}. \tag{21}$$

IV. EXAMPLES

In this section we present a number of new analytic solutions using the method described above.

Case 1. Assume that $A''(x) = 0$ which implies

$$A = a_0 + a_1x. \tag{22}$$

From equation (10) we obtain

$$g(x) = \tfrac{2}{3}\ln(a_0 + 3a_1x), \tag{23}$$

which (from eq. [11]) leads to

$$f(x) = f_0(a_0 + 3a_1x)^{-2/3}. \tag{24}$$

From equation (13) and the requirement that the pressure be zero at the outer radius, f_0 is found to be

$$f_0 = 4a_1(a_0 + 3a_1R^2)^{2/3}(a_0 + 5a_1R^2)^{-1}. \tag{25}$$

From equations (12) and (13) we obtain expressions for the density and pressure:

$$8\pi\rho = 4a_1(a_0 + 3a_1R^2)^{2/3}(3a_0 + 5a_1x)(a_0 + 3a_1x)^{-5/3}(a_0 + 5a_1R^2)^{-1}, \tag{26}$$

$$8\pi p = 4a_1[1 - (a_0 + 3a_1R^2)^{2/3}(a_0 + 5a_1x)(a_0 + 3a_1x)^{-2/3}(a_0 + 5a_1R^2)^{-1}](a_0 + a_1x)^{-1}. \tag{27}$$

The ratio of central pressure to density ($\sigma = p_c/\rho_c$) can be expressed in terms of the parameter $y = a_1R^2/a_0$. By evaluating equations (26) and (27) at $x = 0$, we see that σ can be expressed as

$$3\sigma = (1 + 5y)(1 + 3y)^{-2/3} - 1. \tag{28}$$

Table 1 gives σ and $dy/d\sigma$ as a function of y. This table will be used later to compute stellar models. In terms of y and ρ_c the radius and mass are

$$R^2 = 3y(2\pi\rho_c(1 + 3\sigma))^{-1}, \tag{29}$$

$$m = 2yR(1 + 5y)^{-1}. \tag{30}$$

The parameter a_0 can be determined from the requirement that this solution match smoothly to the exterior Schwarzschild solution, i.e.,

$$A(R) = (1 - 2mR^{-1})^{1/2}, \tag{31}$$

TABLE 1

σ AND $dy/d\sigma$ VERSUS y

y	σ	$dy/d\sigma$	y	σ	$dy/d\sigma$
0.0..........	0.0	1.00	0.72..........	0.379	3.09
0.03..........	0.0286	1.10	0.75..........	0.388	3.17
0.06..........	0.0547	1.20	0.78..........	0.398	3.25
0.09..........	0.0788	1.30	0.81..........	0.407	3.32
0.12..........	0.101	1.39	0.84..........	0.416	3.39
0.15..........	0.122	1.49	0.87..........	0.425	3.47
0.18..........	0.142	1.58	0.90..........	0.433	3.54
0.21..........	0.160	1.67	0.93..........	0.441	3.61
0.24..........	0.178	1.76	0.96..........	0.450	3.69
0.27..........	0.194	1.85	0.99..........	0.458	3.76
0.30..........	0.210	1.94	1.02..........	0.466	3.83
0.33..........	0.225	2.03	1.05..........	0.473	3.90
0.36..........	0.240	2.12	1.08..........	0.481	3.97
0.39..........	0.253	2.20	1.11..........	0.489	4.04
0.42..........	0.267	2.29	1.14..........	0.496	4.11
0.45..........	0.280	2.37	1.17..........	0.503	4.17
0.48..........	0.292	2.46	1.20..........	0.510	4.24
0.51..........	0.304	2.54	1.23..........	0.517	4.31
0.54..........	0.316	2.62	1.26..........	0.524	4.38
0.57..........	0.327	2.70	1.29..........	0.531	4.44
0.60..........	0.338	2.78	1.32..........	0.538	4.51
0.63..........	0.349	2.86	1.35..........	0.544	4.57
0.66..........	0.359	2.94	1.38..........	0.551	4.64
0.69..........	0.369	3.02	1.41..........	0.557	4.70

which implies from equation (22)

$$a_0^{-2} = (1 + 5y)(1 + y)\,. \tag{32}$$

As for the rotational properties, we assume that

$$\Omega(x) = \Omega_0(1 - 3x(2R)^{-2})^{1/2}\,. \tag{33}$$

By evaluating equation (21) at the origin [$\omega_c = \omega(0)$], we obtain the relation between Ω_0 and ω_c:

$$\Omega_0 = 32y(1 + \sigma)\omega_c[5(1 + 3\sigma) + 32y(1 + \sigma)]^{-1}\,. \tag{34}$$

The angular momentum of this body can be found by evaluating equations (18) and (33) at the outer radius and by using equation (34) to express Ω_0 in terms of ω_c:

$$J = 8yR^3(1 + \sigma)\omega_c(5(1 + 3\sigma) + 32y(1 + \sigma))^{-1}\,. \tag{35}$$

Since from equation (30), $R = m(1 + 5y)(2y)^{-1}$, equation (35) can be rewritten as

$$J = 4MR^2\omega_c(1 + \sigma)(1 + 5y)[5(1 + 3\sigma) + 32y(1 + \sigma)]^{-1}\,. \tag{36}$$

Although the concept of a moment of inertia is not useful if the rotation is not rigid, as long as $\omega(x)$ does not change sign for x less than R^2 the ratio of J to ω_c is a measure of how well the body can resist an external torque. Hence, for want of a better term, we will call the ratio of J to ω_c the "moment of inertia." As a measure of the uniformity of the rotation we will use the ratio $\omega(R^2)/\omega_c$. From equation (21) this is

$$\omega(R) = 4(1 + \sigma)\omega_c(12 + 51y + 41y^2)(3 + 5y)^{-1}[5(1 + 3\sigma) + 32y(1 + \sigma)]^{-1}\,. \tag{37}$$

Case 2. Assume that

$$A(x) = b_0 \exp(b_1 x)\,. \tag{38}$$

Retracing all the steps from equations (23) to (32), we have for $y = 2b_1R^2/b_0$

$$g(x) = 2b_1 x\,, \tag{39}$$

$$f(x) = [f_0 + 2b_1 J(2b_1 x)] \exp(-2b_1 x)\,, \tag{40}$$

where

$$J(q) = e^{-1}[\mathrm{Ei}(1 + q) - 1.89512] \tag{41}$$

TABLE 2

BETHE JOHNSON EQUATION OF STATE AND NEUTRON STAR PARAMETERS AS CALCULATED BY BORNER AND COHEN (1973)

ρ_{15}	P_{35}	σ	R_5	M_{33}	Z_0	I_{45}
3.16...........	12.5	0.439	9.78	3.72	2.08	1.54
2.51...........	8.23	0.364	10.25	3.69	1.64	1.64
2.00...........	5.34	0.296	10.73	3.59	1.28	1.68
1.58...........	3.37	0.237	11.19	3.39	0.98	1.66
1.26...........	2.11	0.186	11.59	3.10	0.74	1.55
1.00...........	1.27	1.141	11.91	2.71	0.55	1.35

NOTE.—$\rho_{15} = \rho_0/10^{15}$ g cm^{-3}, $P_{35} = P_0/10^{35}$ dyn cm^{-2}, $\sigma = P_0/\rho_0$, $R_5 = R/10^5$ cm, $M_{33} = M/10^{33}$ g; $Z_0 = A(0)^{-1} - 1$; $I_{45} = I/10^{45}$ gm-cm^2.

and $\text{Ei}(1 + q)$ is the exponential integral (Abramowitz and Stegun 1964). The integration constant f_0 is given by

$$f_0 = 2b_1[2e^y(1+2y)^{-1} - J(y)]\,, \tag{42}$$

and ρ and p by

$$8\pi\rho = 8b_1{}^2x(1 + 2b_1x)^{-1} + 2b_1(3 - 4b_1x)[J(2b_1x) - J(y) + 2e^y(1 + 2y)^{-1}]\exp(-2b_1x)\,, \tag{43}$$

$$8\pi p = 4b_1[1 - (1 + 4b_1x)(1 + 2y)^{-1}\exp(y - 2b_1x) + (1 + 4b_1x)[J(y) - J(2b_1x)]\exp(-2b_1x)/2]\,. \tag{44}$$

In terms of y, the ratio of P_c to ρ_c is

$$3\sigma = [2 + J(y) - 2e^y(1 + 2y)^{-1}][2e^y(1 + 2y)^{-1} - J(y)]^{-1}\,. \tag{45}$$

The total mass is

$$m = yR(1 + 2y)^{-1}\,, \tag{46}$$

where R is given by

$$R^2 = 3y[2e^y(1 + 2y)^{-1} - J(y)](8\pi\rho_c)^{-1}\,. \tag{47}$$

Finally, the expression for b_0 is

$$b_0 = [e^y(1 + 2y)]^{-1/2}\,. \tag{48}$$

From equation (46) it can be seen that as $y \to \infty$, $2mR^{-1} \to 1$. In this same limit, however, $\sigma \to -\frac{1}{3}$, so this limit is not physically achievable. For nonnegative σ, the maximum y is about 2.

Case 3. The form $A(x) = e^{-bx}(1 + 2bx)$ gave a very simple result for $f(x)$ but resulted in negative pressure for all values of b.

V. APPLICATIONS

In this section we use the results of the first example (case 1) to calculate the mass, radius, redshift, and "moment of inertia" of a fluid body whose equation of state at the center is the same as that calculated by Bethe and Johnson (Börner and Cohen 1973). This, of course, does not mean that we are using the Bethe-Johnson equation of state, since ours would differ from theirs for all x greater than zero. We are using their pressure versus density curve to set the scale and to show that a simple analytic model can give reasonably accurate answers in certain limited density ranges. This model cannot be expected to give reliable answers for $\sigma < 0.2$, since the density becomes more and more like incompressible matter as σ goes to zero. This approach has many points in common with that of Bludman (1973) who used polytropes.

TABLE 3

NEUTRON STAR PARAMETERS AS CALCULATED FROM ANALYTIC FORMULAE

ρ_{15}	σ	y	R_5	M_{33}	Z_0	I_{45}
3.16.........	0.439	0.921	9.00(8)	4.00(7.5)	2.28(9.6)	1.94(26)
2.51.........	0.364	0.675	9.10(11)	3.79(2.7)	1.71(4.1)	1.87(14)
2.00.........	0.296	0.490	9.13(13)	3.49(2.8)	1.26(1.6)	1.76(4.8)
1.58.........	0.237	0.355	9.20(17)	3.16(6.8)	0.94(4.1)	1.74(4.8)
1.26.........	0.186	0.256	9.15(21)	2.77(11)	0.69(6.8)	1.44(7.1)
1.00.........	0.141	0.179	9.04(24)	2.32(14)	0.49(11)	1.20(9.6)

NOTE.—Percentage errors are in parentheses.

Table 2 gives the results from the Bethe-Johnson equation of state as calculated by Börner and Cohen (1973). Table 3 gives the results of the analytic model as calculated by a slide rule. The numbers in parentheses are the errors of our results relative to the computer-generated ones. The procedure used was that for each density, σ was calculated from Table 2, and y was calculated by interpolation from Table 1. From equations (29), (30), (32), and (36) the radius, mass, redshift at the center ($z_0 = a_0^{-1} - 1$), and moment of inertia were calculated.

VI. CONCLUSIONS

From Table 3 it can be seen that the gross features of sophisticated computer models can be duplicated within 15 percent by a simple analytic model. It is to be hoped that there will soon be discovered a simple form for $A(x)$ for which all the integrals can be done analytically, which will give even better numerical agreement. If these integrals can be done, we can obtain additional solutions besides the two presented in the paper.

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RICHARD C. ADAMS and JEFFREY M. COHEN: Physics Department, University of Pennsylvania, Philadelphia, PA 19174