Mon. Not. R. astr. Soc. (1974) **169**, 631–641.

THE TRANSFER FUNCTION, SIGNAL-TO-NOISE RATIO, AND LIMITING MAGNITUDE IN STELLAR SPECKLE INTERFEROMETRY

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(Communicated by Professor J. Ring)

(Received 1974 June 17)

SUMMARY

Recent theoretical work has shown that atmospheric seeing which would be regarded as poor for conventional imaging is desirable for attaining diffraction-limited resolution from an aberrated telescope using speckle interferometry. This result is discussed and illustrated with computational examples. An expression is derived for the overall signal-to-noise ratio in speckle interferometry. The signal-to-noise decreases as the seeing deteriorates; in practice therefore the optimum form of seeing for speckle interferometry depends in a complicated manner on the desired signal-to-noise ratio at a specified spatial frequency and on the telescope aberration. It is also shown that binary stars of magnitude $m_V \simeq 16$ should be detectable in reasonable observing times using realistic detection parameters.

1. INTRODUCTION

The technique of stellar speckle interferometry was first proposed and used by Labeyrie *et al.* (**1–4**) to obtain diffraction-limited resolution in the presence of atmospheric seeing. It has also been used for solar astronomy (**5**). A large number of short exposure ($t \simeq 10^{-2}$ s) photographs are taken through a narrow band colour filter ($\Delta\lambda \simeq 25$ nm). For each instantaneous record, the quasi-monochromatic incoherent imaging equation applies:

$$I(\xi, \eta) = O(\xi, \eta) \otimes t(\xi, \eta) \tag{1}$$

where $I(\xi, \eta)$ is the instantaneous image intensity,

$O(\xi, \eta)$ is the object intensity,

$t(\xi, \eta)$ is the instantaneous point spread function of the atmosphere/telescope system, and

$\otimes$ denotes convolution.

The analysis of this data may be carried out in either one of two equivalent ways. In the spatial domain, the ensemble average space autocorrelation is found, giving the resultant imaging equation,

$$\begin{aligned} C(\xi, \eta) &\equiv \langle I(\xi, \eta) \star I(\xi, \eta)\rangle \\ &= \{O(\xi, \eta) \star O(\xi, \eta)\} \otimes \{\langle t(\xi, \eta) \star t(\xi, \eta)\rangle\} \end{aligned} \tag{2}$$

where $\star$ denotes space autocorrelation.

In the spatial frequency domain the average squared modulus of the Fourier transform (i.e. the Wiener or power spectrum) of the image intensity is found,

$$W(u, v) \equiv \langle |i(u, v)|^2 \rangle$$
$$= |o(u, v)|^2 . \langle |T(u, v)|^2 \rangle \quad (3)$$

where $i(u, v)$ is the Fourier transform of the image intensity,

$o(u, v)$ is the Fourier transform of the object intensity, and

$T(u, v)$ is the instantaneous transfer function and equals the Fourier transform of $t(\xi, \eta)$.

The operations described by equations (2) and (3) are equivalent, $C(\xi, \eta)$ and $W(u, v)$ being Fourier transform pairs; when implementing the method, autocorrelation may be more suitable if digital techniques are used, whereas determination of the Wiener spectrum is more conveniently carried out using coherent optical methods.

When evaluating the potential resolution of speckle interferometry it is more convenient to use the spatial frequency domain expression given by equation (3). The quantity $\langle |T(u, v)|^2 \rangle$ is the transfer function of the speckle technique, and a number of authors have derived its possible forms using models of varying degrees of sophistication (**6–11**). It can be shown (see Section 2) that for very *poor* seeing the transfer function has a component proportional to the *diffraction-limited* transfer function of the telescope, regardless of telescope aberrations. However, this analysis also indicates that the constant of proportionality and hence the signal-to-noise ratio also decreases as the seeing deteriorates.

A detailed examination of the signal-to-noise ratio is more conveniently carried out in the space domain following equation (2), and this is considered in Section 3 for binary stars. The rms signal-to-noise ratio is inversely proportional to the square root of the average number of speckles per picture and therefore decreases as the seeing deteriorates. Thus in practice there is an optimum quality of seeing that depends on the interaction of the incident wavefront and telescope aberrations, and on the desired signal-to-noise ratio.

2. THE TRANSFER FUNCTION

The form of the transfer function $\langle |T(u, v)|^2 \rangle$ is found by calculating the Wiener spectrum of the instantaneous intensity from an unresolvable star (see Fig. 1). In order to do this, it is necessary to make some assumption about the form of the probability density function of the complex amplitude of the perturbed wavefront from the star, $Z(x, y)$. It is fairly well established (**12, 13**) that the log amplitude and phase each have a Gaussian probability density function; this model was used by Korff (**8**), but leads to considerable mathematical complexity. It can be shown that the phase of the incident wavefront is random in the interval $-\pi$ to π if the time-averaged image does not have a significant central core; this is the case for the usual seeing conditions. Because of this, we may use a model in which the complex amplitude is a complex Gaussian process (real and imaginary parts separately Gaussian) as a good approximation to the true situation, and this leads to a very much simpler analysis (**6, 7, 11**).

Assuming that the received wavefront is complex Gaussian, the transfer

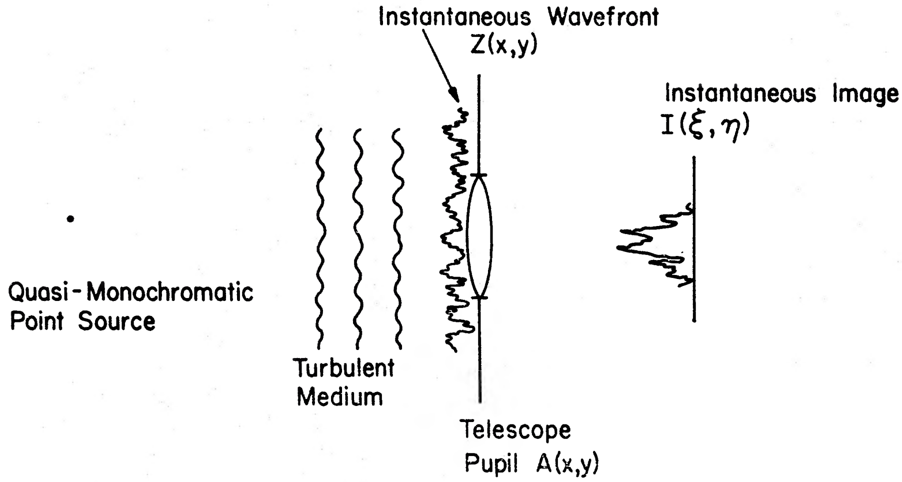

FIG. 1. *The formation of an instantaneous image of a point source through the atmosphere.*

function of the speckle technique is given by the expression (**7**),

$$\begin{aligned}\langle|T(u,v)|^2\rangle = {} & |C_z(x,y)|^2|T_0(u,v)|^2 \\ & +k\iiiint|C_z(x_2-x_1,y_2-y_1)|^2 A(x_1,y_1)\,A^*(x_2,y_2) \\ & \times A^*(x_1+x,y_1+y)\,A(x_2+x,y_2+y) \\ & \times dx_1\,dy_1\,dx_2\,dy_2 \end{aligned} \tag{4}$$

where, referring to Fig. 1,

$A(x, y)$ is the pupil function of the telescope,

$C_z(x, y)$ is the autocorrelation function of the complex amplitude of the received wavefront,

$T_0(u, v)$ is the optical transfer function of the telescope and depends on any aberrations that are present,

k is a constant,

and (x, y) and (u, v) are related by $x = \lambda fu$ and $y = \lambda fv$ where f is the focal length.

It can be seen from equation (4) that the overall transfer functions consists of two components. The normal time-average transfer function $\langle T(u, v)\rangle$ is given by (**7, 14, 15**),

$$\langle T(u,v)\rangle = C_z(x,y)\,T_0(u,v) \tag{5}$$

regardless of the first-order statistics of the random medium. Thus the first term of the transfer function of the speckle technique is simply the squared modulus of the time-average transfer function, and we may write,

$$\langle|T(u,v)|^2\rangle = |\langle T(u,v)\rangle|^2 + \text{a second term.}$$

The function $C_z(x, y)$ describes, in general terms, the correlation of the complex amplitude across the received wavefront and under average seeing conditions may take on fairly large values over a diameter d_0 of approximately 20 cm. For a *well-corrected* telescope of diameter large compared to d_0, the second term simplifies to the form of the diffraction-limited transfer function $T_A(u, v)$ associated with the telescope aperture, and therefore,

$$\langle|T(u,v)|^2\rangle = |\langle T(u,v)\rangle|^2 + kT_A(u,v) \tag{6}$$

where for an unshaded pupil

$$T_A(u, v) = |A(x, y)|^2 \star |A(x, y)|^2.$$

The first term in equation (6) arises from the average distribution of intensity in the instantaneous image, whereas the second term arises from the *fluctuation* of intensity in each image and carries the diffraction-limited information.

Typical forms of $|\langle T(u, v)\rangle|^2$ and $T_A(u, v)$ each normalized to 1·0 at zero spatial frequency are shown in Fig. 2(a) for the case of $R = d_0/D = 0{\cdot}1$ (i.e. telescope diameter 10 times the diameter of the seeing correlation area). However, in equation (6) the two functions are normalized such that their volumes are equal;

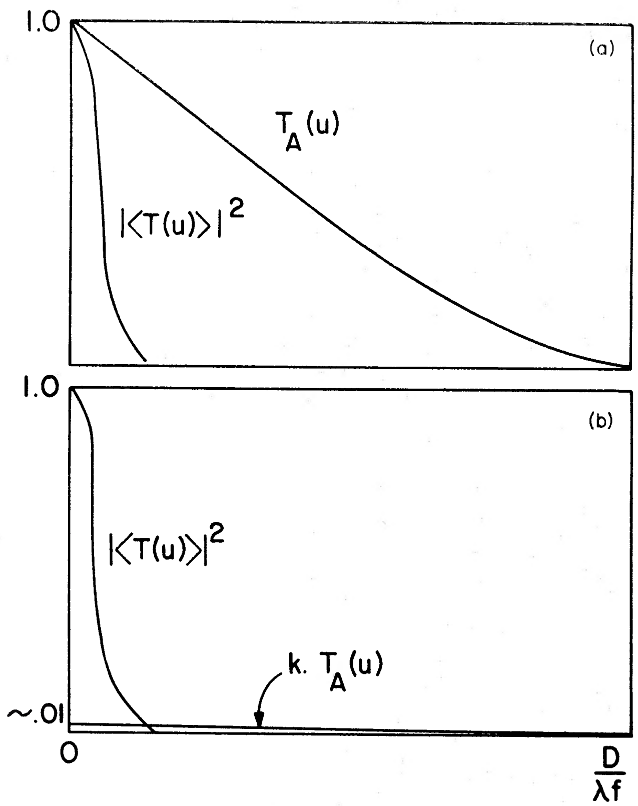

FIG. 2. (*a*) *The square of the time-averaged transfer function,* $|\langle T(u, v)\rangle|^2$, *and the diffraction-limited transfer function,* $T_A(u, v)$, *each normalized to* 1·0 *at the origin.* (*b*) *As* (*a*), *with* $T_A(u, v)$ *scaled by the constant k.*

this follows from the result that the variance of a speckle pattern equals the square of the mean intensity. If the first term stays normalized to 1·0 at the origin, then the factor k is approximately equal to R^2, and the two normalized terms of equation (6) are shown in Fig. 2(b). Clearly, the diffraction-limited information appears to have a low signal value when this type of normalization is applied. However, the question of normalization of the second term and of signal-to-noise ratio are more conveniently examined from another point of view (Section 3).

The second term in equation (4) only reduces to a form proportional to the diffraction-limited transfer function when the seeing is poor ($d_0 \ll D$) and the telescope is free of aberrations. In general, the second term depends on the form of the seeing correlation function $C_z(x, y)$ and the telescope aberrations, and this dependence is illustrated in Figs 3 and 4. In each case the form of $C_z(x, y)$ is taken

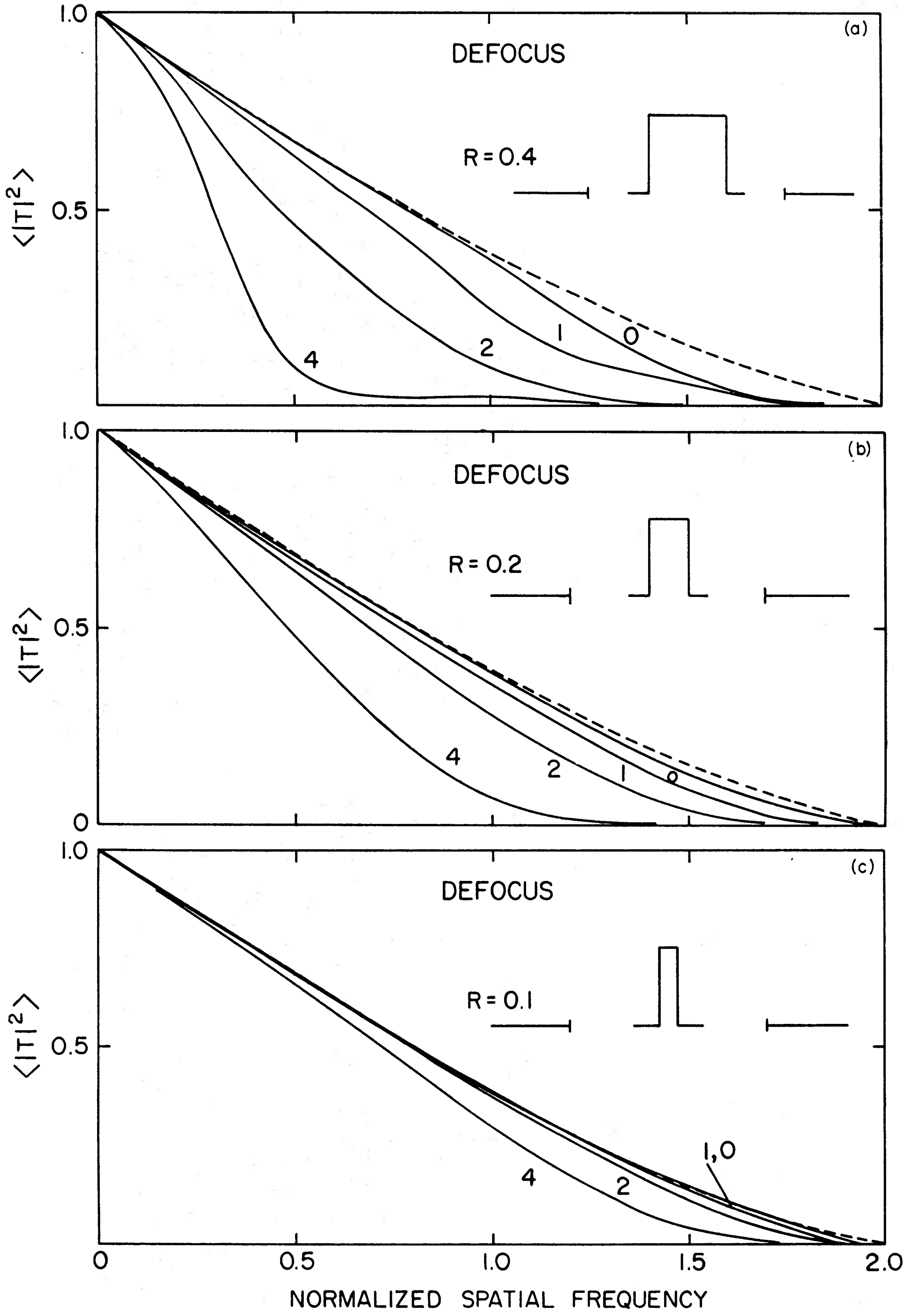

FIG. 3. *Speckle interferometry transfer functions for defocus values of* 0, $1/\pi$, $2/\pi$ *and* $4/\pi$ *wavelengths. The parameter* $R = d_0/D$ *is a measure of the quality of the seeing, for a given telescope diameter.* (*a*) $R = 0{\cdot}4$, (*b*) $R = 0{\cdot}2$, (*c*) $R = 0{\cdot}1$.

to be a 'top hat' function as shown in each figure in which the extent of the correlation area is shown relative to the telescope aperture; this form is chosen for computational convenience to illustrate the essential features of the dependence.

In Fig. 3 the speckle interferometry transfer functions (second term normalized at origin) for a defocused telescope are plotted for three cases of seeing, $R = 0{\cdot}4$,

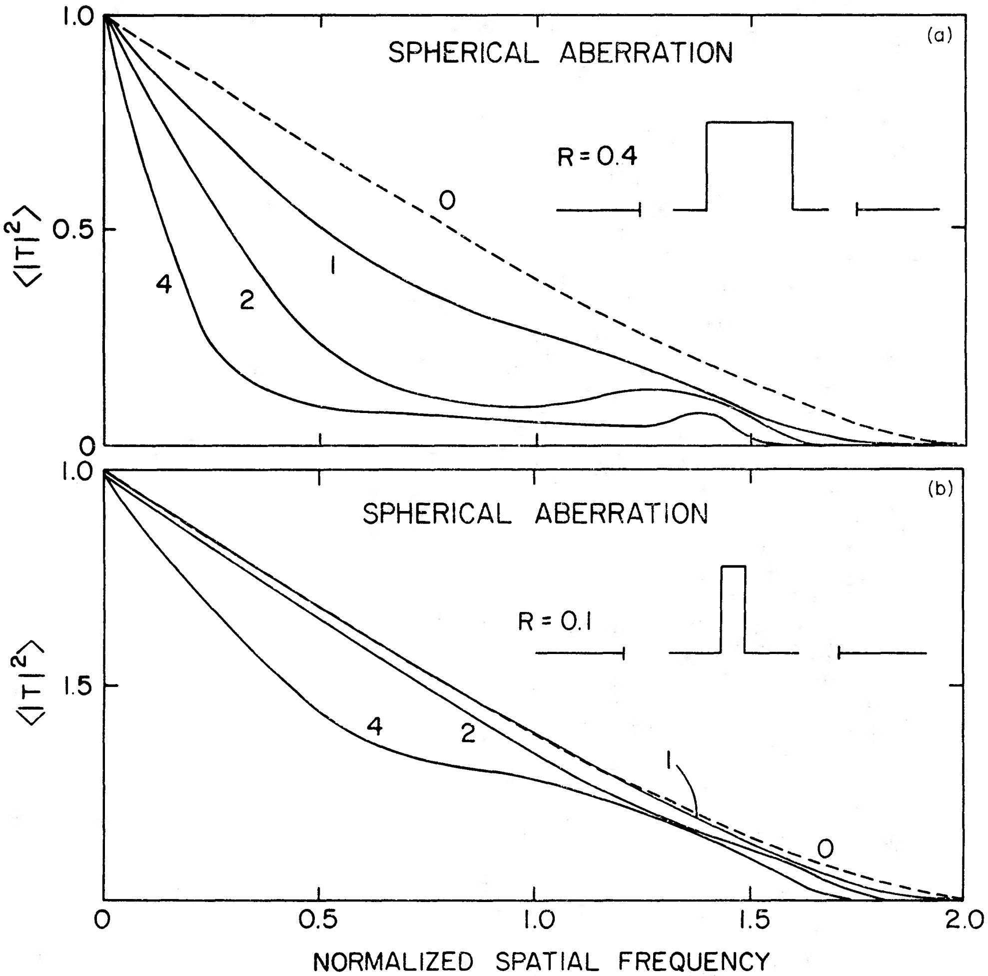

FIG. 4. *Speckle interferometry transfer functions for various values of spherical aberration optimally balanced by defocus: the values indicated on curves are the spherical aberration in wavelengths.* (*a*) $R = 0{\cdot}4$, (*b*) $R = 0{\cdot}1$.

0·2 and 0·1. For each seeing condition, transfer functions are shown for defocus values of 0, $1/\pi$, $2/\pi$ and $4/\pi$ wavelengths; the diffraction-limited transfer function of the telescope is also drawn. In Fig. 4 the speckle interferometry transfer functions for a telescope with spherical aberration optimally balanced by defocus are shown for seeing $R = 0{\cdot}4$ and 0·1. From Figs 3 and 4 it is clear that poor seeing (R small) is desirable if diffraction-limited resolution is to be achieved from an aberrated telescope.

Thus, providing that the seeing is sufficiently poor, the overall transfer function is independent of both telescope aberration and the exact form of the seeing, and is simply proportional to the diffraction-limited transfer function. If it is important that the transfer be a well-defined function, for example when there is no suitable reference star available, then poor seeing is desirable and could be achieved by introducing an *additional* scattering medium in the telescope optics (**7**, **11**). It is interesting to note that in their analysis of a laboratory method of measuring the spatial coherence which is identical to the Labeyrie method, Asakura *et al.* (**16**, **17**) come to the conclusion that the most desirable diffusers to use are ‘ white noise ’ diffusers (equivalent to very poor seeing).

3. SIGNAL-TO-NOISE RATIO

An instantaneous picture of a bright binary star whose components are equal in magnitude can be considered to consist of two superimposed identical speckle patterns, provided that both components lie within the isoplanatic region. For a faint binary, the two recorded patterns will tend to become dissimilar as a result of the fluctuation in the numbers of photons that are detected.

Consider two 'corresponding' speckles labelled 1 and 2. Let the average number of detected photons in each speckle be $\langle N \rangle$ and the actual numbers be N_1 and N_2. The correlation coefficient C between the numbers N_1 and N_2 is defined as

$$C \equiv \frac{\langle N_1 N_2 \rangle - \langle N \rangle^2}{\langle N^2 \rangle - \langle N \rangle^2} \tag{7}$$

where $\langle \; \rangle$ denotes the ensemble average. At very low light levels, N_1 and N_2 will be statistically independent, $\langle N_1 N_2 \rangle = \langle N_1 \rangle \langle N_2 \rangle = \langle N \rangle^2$, and hence $C = 0$. At very high light levels N_1 and N_2 are equal, $\langle N_1 N_2 \rangle = \langle N^2 \rangle$, and hence $C = 1$.

In speckle interferometry the autocorrelation of each picture is found (equation (2)). The average normalized value of the peak of the autocorrelation is simply the correlation coefficient; in other words, C is the signal of the speckle technique and depends only on the average number of detected photons per speckle $\langle N \rangle$. The accuracy with which the correlation coefficient can be measured depends on the total number of speckles used in a given measurement; this accuracy governs the noise of the speckle technique. An expression for the correlation coefficient is now found.

At low light levels the numbers of photons detected fluctuate due to (i) the random intensity of the speckle pattern, and (ii) the Poisson nature of photon detection in a 'uniform' field. The probability density function for the intensity of a speckle selected at random in a speckle pattern formed in polarized light is given by

$$p(I) = \frac{1}{\langle I \rangle} \exp\left(-I/\langle I \rangle\right). \tag{8}$$

This probability density function results from an application of the central limit theorem which assumes that a large number of seeing cells with random phase are present in the telescope aperture. When equation (8) is combined with the Poisson detection process, the overall probability of recording N photons for an average number of $\langle N \rangle$ is given by the Bose–Einstein probability density

$$P_N = \frac{\langle N \rangle^N}{(\langle N \rangle + 1)^{N+1}}. \tag{9}$$

The variance $\langle N^2 \rangle - \langle N \rangle^2$ of a Bose–Einstein distribution is equal to $\langle N \rangle(1 + \langle N \rangle)$, and the expression for the correlation coefficient reduces to

$$C = \frac{\langle N_1 N_2 \rangle - \langle N \rangle^2}{\langle N \rangle (1 + \langle N \rangle)}. \tag{10}$$

The average $\langle N_1 N_2 \rangle$ is defined as

$$\langle N_1 N_2 \rangle \equiv \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} P_{N_1 N_2} . N_1 N_2$$

where $P_{N_1 N_2}$ is the probability of detecting N_1 photons in 1, and simultaneously detecting N_2 photons in 2, where 1 and 2 are corresponding speckles which have the same average intensity, m. For a particular value of m the probability density is the product of two independent Poisson distributions

$$[P_{N_1N_2}]_m = \frac{1}{N_1!N_2!} \exp(-2m)\, m^{N_1+N_2}.$$

The value of m fluctuates according to the negative exponential distribution (equation (8)), and therefore for any pair of corresponding speckles selected at random,

$$P_{N_1N_2} = \frac{1}{\langle N\rangle\, N_1!N_2!} \int_0^\infty \exp(-2m)\, m^{N_1+N_2} \exp(-m/\langle N\rangle)\, dm. \quad (11)$$

The right-hand side of equation (11) can be written in the form of a Laplace transform which can be evaluated to give

$$P_{N_1N_2} = \frac{(N_1+N_2)!}{N_1!N_2!} \frac{\langle N\rangle^{N_1+N_2}}{(1+2\langle N\rangle)^{N_1+N_2+1}}.$$

The average $\langle N_1N_2\rangle$ is therefore given by

$$\langle N_1N_2\rangle = \frac{1}{1+2\langle N\rangle} \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \frac{(N_1+N_2)!}{N_1!N_2!} \left(\frac{\langle N\rangle}{1+2\langle N\rangle}\right)^{N_1+N_2} N_1N_2 \quad (12)$$

and the correlation coefficient is found by combining equations (10) and (12).

The correlation coefficient C is plotted as a function of the average number of detected photons per speckle $\langle N\rangle$ in Fig. 5, which has two notable features. First, the correlation coefficient is large for relatively low light levels; for example, $C \simeq 0.5$ with an average of only one detected photon per speckle. Second, it can be seen both from Fig. 5 and equations (10) and (12) that at very low light levels,

$$C \simeq \langle N\rangle, \qquad \langle N\rangle \ll 1.$$

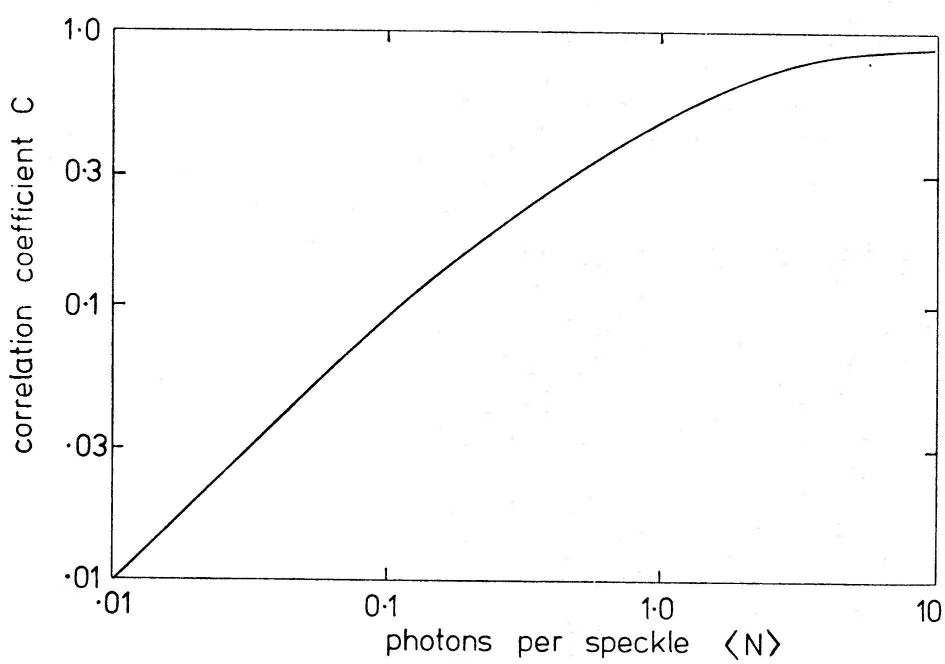

FIG. 5. *The correlation coefficient C as a function of the average number of photons per speckle* $\langle N\rangle$.

It can also be shown that if there is an additive uncorrelated Poisson noise term, due for example to sky background, then the correlation coefficient is given by,

$$C' = \frac{\langle N \rangle}{1+k}, \quad \langle N \rangle \ll 1$$

where k is the ratio of the average background level to the average signal level.

The absolute rms error of an estimate of the correlation coefficient is given by (**18**),

$$\sigma \simeq \frac{1-C^2}{\sqrt{M}}$$

where M is the total number of speckles which in this particular case is equal to the product of the total number of pictures N_p and the average number of speckle pairs per picture $\langle N_s \rangle$,

$$M = N_p \langle N_s \rangle.$$

The signal-to-noise ratio Q is defined as C'/σ, and for small $\langle N \rangle$ (and hence small C),

$$Q = \frac{\langle N \rangle \sqrt{N_p \langle N_s \rangle}}{1+k}. \tag{13}$$

It is also useful to write this expression in the form,

$$Q = \frac{\langle N_{ph} \rangle}{2(1+k)} \sqrt{\frac{N_p}{\langle N_s \rangle}} \tag{14}$$

where $\langle N_{ph} \rangle = 2\langle N \rangle \langle N_s \rangle$ is the average number of photons per picture. It should be noted that the rms signal-to-noise ratio Q is proportional to the number of photons per picture (for small $\langle N_{ph} \rangle$), and not to the square root of that number as in conventional imaging.

For given seeing conditions both $\langle N_{ph} \rangle$ and $\langle N_s \rangle$ increase as the square of the telescope diameter, and therefore the signal-to-noise ratio increases in proportion to the telescope diameter. This result is contrary to the transfer function analysis given in Section 2 which ignores the number and statistical fluctuation of detected photons. For a given telescope diameter, the signal-to-noise ratio increases in proportion to the average number of photons per picture; this is in contrast to the usual Poisson detection situation where the signal-to-noise ratio increases as the square root of the photon flux. It should also be noted that additive random noise does not have a particularly significant effect on the signal-to-noise ratio, although given sufficient observing time it will ultimately determine the limiting magnitude recordable.

The average number of speckle pairs $\langle N_s \rangle$ increases as the square of $1/R$, where R is ratio of the seeing patch 'diameter' to the telescope diameter. The signal-to-noise ratio Q is therefore proportional to R, that is, the overall signal-to-noise ratio decreases as the seeing deteriorates (small R).

The overall signal-to-noise ratio calculated above can be considered to be a scaling factor that should be applied to the speckle transfer function; the relative value of this scaling factor is simply R if the total number of pictures and photons per picture remain constant. When examining the effect of seeing on the transfer function, such as in Figs 3 and 4, this scaling factor should be taken into account. For example, Fig. 3(b) and (c) should be scaled by factors of $0{\cdot}2/0{\cdot}4 = 0{\cdot}5$ and

0·1/0·4 = 0·25 relative to Fig. 3(a). In Fig. 6(a) and (b) the results shown in Fig. 3 for defocus values of 0 and $4/\pi$ have been redrawn with this normalization.

In speckle interferometry we are generally interested in the high spatial frequency information, and it can be seen from Fig. 6 that the relative amplitude of this information depends in a complicated way on the aberrations and seeing. For

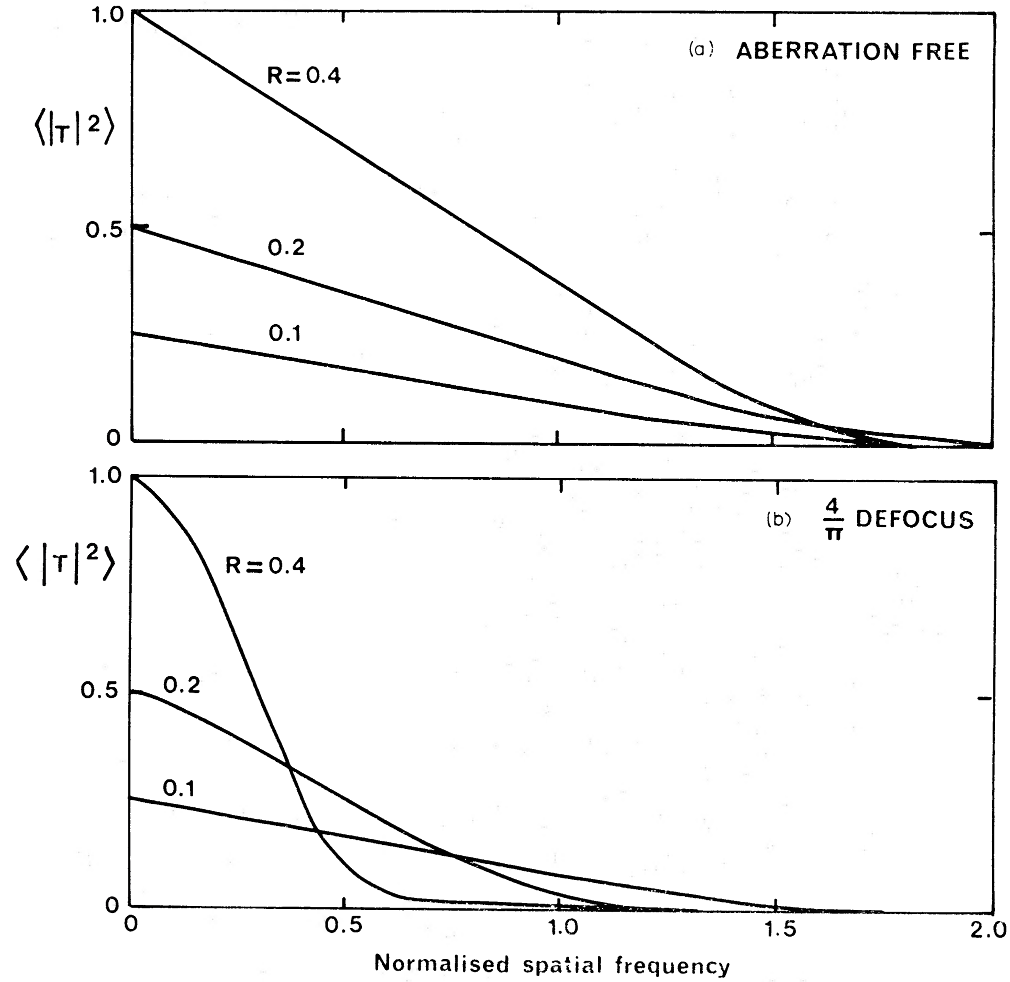

FIG. 6. *Speckle interferometry transfer functions scaled by the overall signal-to-noise ratio.* (*a*) *Aberration-free,* (*b*) *defocus of* $4/\pi$ *wavelengths.*

a well-corrected telescope, then we can in general say that the amplitude of the high frequency information decreases as the seeing deteriorates (Fig. 6(a)). But for a highly aberrated system (Fig. 6(b)), the amplitude of this information will increase as the seeing deteriorates up to some maximum value, after which it also decreases.

4. LIMITING MAGNITUDE

Suppose that an rms signal-to-noise ratio of 5 defines a just detectable result. Then assuming no additive noise ($k = 0$), 10^3 speckle pairs per picture on average, and 10^6 pixtures used in the superposition, we require an average of 0·3 detected polarized photons per picture (from equation (14)). Assuming 15 per cent detective quantum efficiency for the detector this corresponds to an average of 2 incident polarized photons or 4 incident unpolarized photons per picture. At an exposure time of 10^{-2} s this is equivalent to a flux rate of 400 photons s^{-1}. Using the value of $6{\cdot}5 \times 10^3$ photons $in^{-2}\ s^{-1}$ Å^{-1} for the flux from a zero magnitude object (**19**), and assuming a 200-in. telescope and a 250 Å bandpass, we have approximately 5×10^{10} photons s^{-1} for a zero magnitude star; therefore the limiting magnitude is

$$m_V \simeq \log_{2{\cdot}51}\left(\frac{5\times 10^{10}}{400}\right) \simeq 20.$$

In principle this limit is achieved in $10^6 . 10^{-2}$ s $= 2{\cdot}5$ hr of observing time, although in practice there is a certain dead time between records.

More realistic observing parameters are 10^4 pictures, a detector DQE of 10 per cent, and background noise equivalent to $k = 1$, giving a limiting magnitude of approximately $m_V = 16$ for a binary object. It should be emphasized that this value applies only to binary objects. The main purpose of this analysis of limiting magnitude is to indicate that speckle interferometry is capable of detecting much fainter objects than either intensity interferometry or Michelson interferometry.

ACKNOWLEDGMENT

A substantial proportion of the work described in this paper was undertaken whilst the author was a Guest Scientist at Kitt Peak National Observatory, and I am grateful to the staff there for many helpful discussions.

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