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ANALYTIC PULSAR MODELS

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ABSTRACT

An analytic solution to Einstein's field equations is given. This analytic model of a uniform density slowly rotating neutron star can describe differentially as well as uniformly rotating stars.
Subject headings: neutron stars — pulsars — relativity — rotation

I. INTRODUCTION

Pulsars are now generally believed to be rotating neutron stars (Gold 1968; Oppenheimer and Volkoff 1939). According to generally accepted ideas, a typical pulsar has a radius of 10 km, a mass of about 1 $M_\odot$, and a rotational period of 30 ms to several seconds. Detailed discussions of some basic properties about which astrophysicists are reasonably confident can be found in e.g., Cohen and Cameron (1971), Harrison *et al.* (1965), Börner and Cohen (1973). In brief, the most massive neutron stars have crusts (about 200 m thick) composed of highly conducting crystalline material with a density ranging from 10^4 g cm^{-3} near the surface to $\sim 2 \times 10^{14}$ g cm^{-3}.

Deeper within the stellar interior the density increases and the material becomes predominantly a degenerate neutron sea in thermodynamic equilibrium with small admixtures of degenerate protons and electrons. In the central core where supernuclear densities ($> 10^{15}$ g cm^{-3}) can be encountered, hyperons are also present. One of the interesting features of the most recent high-mass models is that the density is practically constant throughout most of the star. (Baym, Pethick, and Sutherland 1972; Börner 1973; Börner and Cohen 1973, 1974; Cohen and Cameron 1971). Thus, the interior Schwarzschild solution (Adler, Bazin, and Schiffer 1965) should provide a good approximation to such a star.

The problem of solving the Einstein field equations for a rotating fluid body is quite complicated. It becomes tractable if the simplifying assumption of slow rotation is made (Brill and Cohen 1966; Cohen 1967*a*, *b*, 1970, 1971; Cohen and Brill 1968). In this case it is necessary to solve only one ordinary differential equation once the usual stellar-structure equations for nonrotating spherically symmetric stars are solved.

For the case of nonrotating incompressible matter, one obtains the Schwarzschild interior solution (Schwarzschild 1916). Unfortunately, the rotation equation is quite involved. Because of this, computer solutions were resorted to in the past (Cohen and Brill 1968). Here we circumvent this difficulty by treating a model which is approximately rigidly rotating. This allows an analytic solution to be obtained by choosing a reasonable form for a metric function and determining what stellar rotation rate can produce this result. Although the stellar rotation rate is a function of radius, it can be made arbitrarily close to rigid rotation. This approach can be used whenever one can find a spherically symmetric stellar model.

II. TREATMENT OF ROTATING BODIES

Since pulsars are generally believed to be rotating neutron stars (Gold 1968), they must be treated via general relativity. Even in the nonrotating case, the treatment is more complicated than the corresponding Newtonian one. In addition, there are qualitatively new effects such as the dragging of inertial frames by rotating bodies (Brill and Cohen 1966; Cohen 1967*a*).

The general relativistic equations describing a body rotating at an arbitrary rate are rather complicated (Brill and Cohen 1966; Cohen and Brill 1968; Cohen 1971). Even extensive use of axial symmetry and time independence allows only the reduction of the 10 Einstein equations to six. These six equations may be obtained from the metric describing an axially symmetric rotating body (Cohen and Brill 1968):

$$ds^2 = -A^2dt^2 + B^2dr^2 + C^2d\theta^2 + E^2(d\phi - \Omega dt)^2\,, \tag{1}$$

where A, B, C, E, and Ω are functions of r and θ.

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Fortunately, the problem simplifies greatly for slow rotation, i.e., work only to first order in the stellar rotation rate and related quantities such as Ω. The rotation is considered slow if the centrifugal force on each mass element is small compared with the gravitational force, and the velocity of each mass element is much less than that of light. Even for the most rapidly rotating pulsar, the Crab pulsar, for which $\omega \simeq 190\ \mathrm{s}^{-1}$, the rotation is slow by this criteria. Within this slow rotation approximation, the Einstein equations become (Brill and Cohen 1966; Cohen and Brill 1968; Cohen 1971; Cohen and Cohen 1968)

$$8\pi r^2\rho = [r(1 - B^{-2})]' , \tag{2a}$$

$$A^2B^2(1 + 8\pi r^2 p) = (rA^2)' , \tag{2b}$$

$$p' = -(\rho + p)A^{-1}A' , \tag{2c}$$

$$-16\pi(\rho + p)r^4BA^{-1}(\omega - \Omega) = (r^4\Omega' A^{-1}B^{-1})' . \tag{2d}$$

If we let $B^{-2} \equiv 1 - 2m(r)r^{-1}$ in equation (2a), it reduces to $m' = 4\pi r^2\rho$ which has the same form as the corresponding flat space equation.

Equation (2d) is a useful form of the rotation equation. By adding equations (2a) and (2b) and using the definition of $m(r)$ to obtain the quantities $A^{-1}A' + B^{-1}B'$ and B in terms of ρ, p, and $m(r)$ the rotation equation (2d) can be written in another form

$$[1 - 2m(r)r^{-1}](\Omega'' + 4\Omega' r^{-1}) = 4\pi(\rho + p)r[\Omega' + 4(\Omega - \omega)r^{-1}] . \tag{3}$$

For slow rotation, the problem can be divided into two parts (Brill and Cohen 1966; Adams *et al.* 1973): (1) the solutions to the usual stellar structure equations (2a)–(2c), and (2) the solution to the rotation equation (2d), or (3) either by numerical or analytic means. The function ω can be chosen arbitrarily, close to any desired function, e.g., uniform rotation.

We have concentrated on analytically tractable solutions in order to illustrate the fundamental physical effects and the role of relativity in the problem. Also, analytic solutions can be studied and applied by students without using computers. The application of the method to more realistic problems is straightforward (Cohen and Cameron 1971; Cohen 1971).

III. ROTATING INCOMPRESSIBLE FLUID

Detailed computer models (Cohen and Cameron 1971; Börner and Cohen 1973, 1974) of neutron stars near the mass peak share the property that the density changes very little with radius near the center and then drops rapidly towards zero near the surface. This behavior is very similar to an incompressible fluid model (ρ = constant) of Schwarzschild (1916) described in standard texts (Adler *et al.* 1965; Weinberg 1972)

$$m(r) = 4\pi\rho r^3/3 , \tag{4a}$$

$$p = -\rho + 2(1 + \sigma)\rho\{3(1 + \sigma) - (1 + 3\sigma)[1 - 2m(r)r^{-1}]^{1/2}\}^{-1} , \tag{4b}$$

and

$$8\pi\rho R^2/3 = 2m/R = 4\sigma(1 + 2\sigma)(1 + 3\sigma)^{-2} , \tag{4c}$$

where $\sigma = p_c/\rho$ the ratio of central pressure to density and R is the stellar radius. For a detailed discussion and new insights into this solution see Bludman (1973*a*, *b*).

Substitution of equations (4) into equation (3) results in a complicated differential equation for Ω for the case of rigid rotation (ω = constant). This has been solved heretofore only by numerical methods (Cohen and Brill 1968). Here we solve the problem analytically. We do this by inverting it. We choose a physically reasonable form for $\Omega(r)$ and solve equation (3) for the resulting $\omega(r)$:

$$\omega(r) = \Omega(r) + r\Omega'/4 - [1 - 2m(r)r^{-1}](\Omega'' + 4\Omega' r^{-1})[16\pi(\rho + p)]^{-1} . \tag{5}$$

To illustrate the method, we consider a simple function to represent Ω:

$$\Omega(r) = \Omega(0)[1 - b(r/R)^2 - b\tau(r/R)^4] , \tag{6}$$

where R is the stellar radius. This form contains the arbitrary constants $\Omega(0)$, b, and τ. The quantity $\Omega(0)$ may be viewed as a scaling factor while b can be determined in terms of τ by the requirement that Ω and its first derivative be continuous at the stellar surface. The exterior solution (Brill and Cohen 1966) is

$$\Omega = 2Jr^{-3} , \tag{7}$$

where J is the total angular momentum of the star (Cohen 1967*b*, 1970). The continuity requirement yields

$$b = 3/(5 + 7\tau) \tag{8a}$$

and

$$J = R^3\Omega(0)(1 + 2\tau)(5 + 7\tau)^{-1} . \tag{8b}$$

Only τ is a free parameter. Substituting equation (6) into equation (5) gives the angular velocity of the star

$$\omega(r) = \Omega(0)\left\{1 - \frac{9}{2(5+7\tau)}\left(\frac{r}{R}\right)^2 - \frac{6\tau}{(5+7\tau)}\left(\frac{r}{R}\right)^4 + \frac{9(1-8\pi\rho r^2/3)}{16\pi\rho R^2(5+7\tau)} \times \left[1 - \frac{(1+3\sigma)}{3(1+\sigma)}\left(1-\frac{8\pi\rho r^2}{3}\right)^{1/2}\right]\left[5 + 14\tau\left(\frac{r}{R}\right)^2\right]\right\}. \quad (9)$$

Thus, we have obtained an analytic solution representing a rotating neutron star. The angular momentum of this model is

$$J = 5^{-1}2mR^2\omega_c(1+\sigma)(1+3\sigma)^2(1+2\tau)[(1+10\sigma+21\sigma^2+8\sigma^3+5^{-1}28\tau\sigma(1+2\sigma)(1+\sigma)]^{-1}. \quad (10)$$

The case $\tau = 0$ is the simplest giving, in the Newtonian limit, completely rigid rotation. For more realistic values of σ, however, there is some variation in $\omega(r)$. In this case $\omega(r)$ is monotonically decreasing and its value at the surface is

$$\omega(R)/\omega_c = (1+\sigma)(5+12\sigma+9\sigma^2)/5(1+10\sigma+21\sigma^2+8\sigma^3). \quad (11)$$

The departure of this quantity from unity is a measure of how good the original choice in $\Omega(r)$ is.

The amount of differential rotation can be reduced (or increased if desired) by choosing a nonzero value for τ. For example, if τ is chosen in such a way that the surface angular velocity $\omega(R)$ equals the central angular velocity $\omega(0)$, the differential rotation is reduced considerably. This value of τ is

$$\tau_0 = \sigma(33+84\sigma+31\sigma^2)(1+\sigma)^{-1}(14+4\sigma-34\sigma^2)^{-1}. \quad (12)$$

Table 1 gives the maximum deviation of the angular velocity $(\omega/\omega_c)_{\max}$ from the central value at various densities $\sigma = (p/\rho)_{\text{central}}$ and for various choices of τ. For computer models of neutron stars using various equations of state (Bethe 1974; Bethe, Börner, and Sato 1971; Börner 1973; Cohen and Cameron 1971; Börner and Cohen 1973), the maximum value of σ is about 0.3 near the mass peak. It is interesting to note that even with the simple form of equation (6) for Ω, maximum differential rotation can be chosen to be less than 12 percent for all models.

In summary the metric corresponding to our model is

$$ds^2 = -A^2dt^2 + B^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta(d\phi - \Omega dt)^2, \quad (13)$$

$$A = \tfrac{3}{2}(1-8\pi\rho R^2/3)^{1/2} - \tfrac{1}{2}(1-8\pi\rho r^2/3)^{1/2}, \quad r \le R,$$

$$A^2 = [1 - 2m(R)/r], \quad r \ge R,$$

$$B^2 = (1-8\pi\rho r^2/3)^{-1}, \quad r \le R,$$

$$B^2 = (1-2m(R)/r)^{-1}, \quad r \ge R,$$

$$\Omega = \Omega(0)[1 - b(r/R)^2 - b\tau(r/R)^4], \quad r \le R,$$

$$\Omega = 2Jr^{-3}, \quad r \ge R.$$

TABLE 1

MAXIMUM DEVIATION OF ANGULAR VELOCITY FROM CENTRAL VALUE FOR VARIOUS $\sigma = p/\rho_{\text{central}}$

σ	τ/τ_0	$\lvert\omega/\omega_c\rvert_{\max}$
10^{-8}......	0	7×10^{-8}
10^{-8}......	1	2×10^{-11}
10^{-8}......	1	2×10^{-11}
0.1......	0	0.38
0.1......	0.9	0.04
0.1......	1	0.05
0.2......	0	0.52
0.2......	0.8	0.09
0.2......	1	0.14
0.3......	0	0.60
0.3......	0.7	0.12
0.3......	1	0.22

The free parameter τ is related to the degree of differential rotation and may be chosen to suit the physics of the problem; b is given by equation (8a), the angular momentum J by equation (8b), and the stellar rotation rate ω by equation (9).

More complicated forms (Pechenik 1973) for Ω can give even smaller amounts of differential rotation. Thus the method allows an analytic treatment of both differentially and uniformly rotating neutron stars.

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