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THE DYNAMICAL PROPERTIES OF TWISTED ROPES OF MAGNETIC FIELD AND THE VIGOR OF NEW ACTIVE REGIONS ON THE SUN*

E. N. PARKER
Department of Physics and Department of Astronomy and Astrophysics, University of Chicago, Chicago, Illinois 60637
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ABSTRACT

The equilibrium configuration following expansion of all, or part, of a force-free magnetic rope is worked out formally from the hydromagnetic equilibrium equations. It is shown that expansion of a portion of the length of a long magnetic rope beyond a certain radius causes the expanded portion to be wrapped with an azimuthal field composed of lines of force from the surface layer of the unexpanded portion. This singular redistribution takes an extended period of time to achieve. Further, it automatically causes the expanded portion to become unstable to buckling.

We suggest that the restless behavior of the magnetic fields in new active regions in the solar photosphere is largely a result (*a*) of the buckling brought about by the expansion of the fields as they emerge through the photosphere, and (*b*) of the long-term redistribution of the surface lines of force onto the expanded portion of the rope.

Subject headings: hydromagnetics — magnetic fields, solar — solar activity

I. INTRODUCTION

The restless behavior of the magnetic fields in the photosphere is a conspicuous aspect of solar activity. The activity is complicated by such exotic nonlinear hydromagnetic effects as rapid reconnection and topological dissipation (Dungey 1953; Sweet 1969; Yeh and Axford 1970; Parker 1972, 1973*a*, *b*, and references therein). This paper points out another singular property of fields which plays a role in the complicated magnetic meteorology of the Sun. It concerns force-free ropes of magnetic field.

A priori one might expect any magnetic field $\boldsymbol{b}$ in the turbulent atmosphere of the Sun to be subject to torsion (i.e., the curl of $\boldsymbol{b}$—the electric current—to have a component parallel to $\boldsymbol{b}$) as a consequence of the vorticity in the fluid motions. The helical striations in eruptive prominences suggest fields subject to torsion. Alfvén (1961, 1963) has emphasized the appearance of field-aligned currents (torsional fields) in the solar corona. The torsional topology of sunspot fields is obvious in the Hα fine structure and the individual sunspots.

Relatively little is known of the general theoretical dynamical behavior of torsional and force-free fields because of the nonlinearity of the hydromagnetic equations, although extensive linear stability studies of special cases have been carried out (see, for instance, Lundquist 1951; Roberts 1955, 1956; Bernstein *et al.* 1958; Trehan 1958; Trehan and Reid 1958; Chandrasekhar 1961).

This paper examines the simple question of the expansion of an infinitely long, axisymmetric, force-free rope of flux in a tenuous, infinitely conducting medium. The problem is nonlinear but tractable, and the result of § 3*c* unanticipated.

The vector field in an infinitely long uniform rope of flux lying along the z-axis can be described in terms of a generating function $f(\varpi)$ of the distance $\varpi = (x^2 + y^2)^{1/2}$ from the z-axis,

$$b_z^2(\varpi) = f(\varpi) + \tfrac{1}{2}\varpi df/d\varpi\,, \qquad b_\phi^2(\varpi) = -\tfrac{1}{2}\varpi df/d\varpi \tag{1}$$

(Lüst and Schlüter 1954). The generating function must be positive and declining with increasing ϖ, but not faster than ϖ^{-2}, in order that b_z and b_ϕ be real. The square of the magnitude of the field is just $f(\varpi)$. Hence the magnitude of the field cannot decrease with ϖ faster than ϖ^{-1}. This limiting case is just the familiar magnetic field $b_\phi = 1/\varpi$ around a line current. It has no longitudinal component b_z. If b_z is not zero, then obviously $-\frac{1}{2}\varpi df/d\varpi$ must be smaller than f, and the field declines less rapidly than ϖ^{-1}.

So far as we are aware, there has been no general stability criterion for force-free ropes of field, apart from the problems treated in the references listed above. But it is obvious that instability must arise if the rope is twisted so tightly that the total tension is negative, i.e., if the rope is under longitudinal compression (Parker 1966). The total tension is

$$T = 2\pi\int_0^\infty d\varpi\varpi(b_z^2 - b_\phi^2)/8\pi = \tfrac{1}{4}\int_0^\infty d\varpi\varpi(f + \varpi df/d\varpi)$$

in the special case that the radius of the rope extends to infinity.

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If $f = Q/\varpi^{\alpha}$, with $0 < \alpha < 2$ in order that the field be real, then the tension is

$$T = \tfrac{1}{4}Q[(1 - \alpha)/(2 - \alpha)] ,$$

and is positive provided $\alpha < 1$. If $\alpha > 1$, the rope is under compression and unstable to buckling. We suspect, but cannot prove, that instabilities occur if $f + \varpi df/d\varpi$ is negative anywhere within the rope, even though the total integral tension may be positive. The limiting case $\alpha = 1$ yields

$$b_{\phi}^2 = b_z^2 = Q/2\varpi$$

and represents a 45° helix.

II. RADIAL DILATATION OF A FORCE-FREE ROPE

a) *Basic Equations*

Consider a force-free rope of flux with outer radius c described by the generating function $f(\varpi)$. At the outer surface $\varpi = c$ the pressure of the field $b^2/8\pi = f(c)/8\pi$ is balanced by an external gas pressure $p = f(c)/8\pi$ dyn cm^{-2}.

Now suppose that the external confining force retreats to a radial distance C. The field expands to fill the space, forming a new force-free rope of radius C. If the lines of force initially at the radius ϖ map into the radial position Π, so that the final field is described in terms of a new generating function $F(\Pi)$, then the field is

$$B_z^2(\Pi) = F(\Pi) + \tfrac{1}{2}\Pi dF/d\Pi, \qquad B_{\phi}^2(\Pi) = -\tfrac{1}{2}\Pi dF/d\Pi . \tag{2}$$

The mapping $\Pi = \Pi(\varpi)$ relates F to f, based on the conditions of conservation of magnetic flux. The longitudinal flux $2\pi\varpi d\varpi b_z(\varpi)$ through the annulus $(\varpi, \varpi + d\varpi)$ appears in the annulus $(\Pi, \Pi + d\Pi)$. Hence,

$$b_z(\varpi)\varpi d\varpi = B_z(\Pi)\Pi d\Pi$$

and

$$\varpi^2\left(f + \tfrac{1}{2}\varpi\frac{df}{d\varpi}\right) = \Pi^2\left(F + \tfrac{1}{2}\Pi\frac{dF}{d\Pi}\right)\left(\frac{d\Pi}{d\varpi}\right)^2 . \tag{3}$$

Similarly, the azimuthal flux per unit length (the number of turns per unit length) is conserved, $b_{\phi}(\varpi)d\varpi = B_{\phi}(\Pi)d\Pi$, so that

$$\varpi\frac{df}{d\varpi} = \Pi\frac{dF}{d\Pi}\left(\frac{d\Pi}{d\varpi}\right)^2 . \tag{4}$$

Simultaneous solution of these two equations yields $F(\Pi)$ and $\Pi(\varpi)$ for any given $f(\varpi)$.

Let $u \equiv \varpi^2/L^2$, $U = \Pi^2/L^2$, where L is a suitable scale length. Then (3) and (4) can be put in the concise form

$$\frac{dF}{df} = \frac{dUF}{duf} = \frac{du}{dU} . \tag{5}$$

To obtain solutions, rewrite equations (3) and (4) as

$$F + U\frac{dF}{dU} = \left(\frac{du}{dU}\right)^2\left(f + u\frac{df}{du}\right) , \tag{6}$$

$$\frac{dF}{dU} = \left(\frac{du}{dU}\right)^2\frac{df}{du} . \tag{7}$$

Then eliminate dF/dU between these two equations,

$$F = \left(\frac{du}{dU}\right)^2\left[f + (u - U)\frac{df}{du}\right] . \tag{8}$$

Differentiate this with respect to U and use equation (7) again to eliminate dF/dU. The result is the nonlinear differential equation

$$0 = \left\{\frac{d^2u}{dU^2}\left[f + (u - U)\frac{df}{du}\right] - \frac{df}{du}\frac{du}{dU} + \left(\frac{du}{dU}\right)^2\left[\frac{df}{du} + \tfrac{1}{2}(u - U)\frac{d^2f}{du^2}\right]\right\}\frac{du}{dU} \tag{9}$$

for the mapping $u = u(U)$ in terms of the initial generating function $f(u)$. The equation can be recast to give $U = U(u)$ if desired.

One typical case is sufficient to illustrate the physical principles. Let

$$f(\varpi) = 1 - \varpi^2/L^2 \tag{10}$$

for $\varpi \leq c$, and with $c \leq L/2^{1/2}$ so that the fields

$$b_z = (1 - 2\varpi^2/L^2)^{1/2}, \qquad b_\phi = \varpi/L,$$

are real. The field terminates at the surface $\Pi = C$ where the pressure of the field $F/8\pi$ is confined by the external gas pressure. Then equation (9) becomes

$$\frac{d^2u}{dU^2}(1 - 2u + U) + \frac{du}{dU} - \left(\frac{du}{dU}\right)^2 = 0. \tag{11}$$

To obtain the solution to this nonlinear second-order equation, let $\phi = 1 - 2u + U$, so that

$$2\phi\frac{d^2\phi}{dU^2} = 1 - \left(\frac{d\phi}{dU}\right)^2.$$

The first integral of this is

$$(d\phi/dU)^2 = 1 + R/\phi, \tag{12}$$

where R is a constant of the integration. Before going further with the integration, it is necessary to decide on the sign of R. Since $0 \leq u \leq \frac{1}{2}$ and $U \geq 0$, it follows that $\phi > 0$. Hence R is positive or negative depending upon whether $|d\phi/dU|$ is larger or smaller than unity. The former obtains when the rope is compressed and the latter when expanded.

To show this, note that from the definition of ϕ it follows that

$$d\phi/dU = 1 - 2du/dU. \tag{13}$$

Then suppose that the rope is *compressed*. Clearly $du/dU > 1$, so that $d\phi/dU < -1$, or $(d\phi/dU)^2 > +1$, and R must be positive as a consequence of equation (12). If the rope is expanded, then $0 < du/dU < 1$. Then $-1 < d\phi/dU < +1$ so that $R < 0$.

b) *Compression*

Suppose that the rope is compressed. Then

$$\pm dU = \frac{d\phi}{(1 + R/\phi)^{1/2}}.$$

Integration yields

$$\pm U = P + (1 - 2u + U)^{1/2}(1 - 2u + U + R)^{1/2} - R\sinh^{-1}\left(\frac{1 - 2u + U}{R}\right)^{1/2}, \tag{14}$$

where P is the second constant of integration. The two constants P and R are to be evaluated from the condition that $u = 0$ maps into $U = 0$, and u_1 maps into U_1. The former condition leads to

$$\pm U = (1 - 2u + U)^{1/2}(1 - 2u + U + R)^{1/2} - (1 + R)^{1/2} + R\sinh^{-1}\frac{1}{R^{1/2}} - R\sinh^{-1}\left(\frac{1 - 2u + U}{R}\right)^{1/2},$$

so that R is related to $u_1 \equiv c^2/L^2$ and $U_1 \equiv C^2/L^2$ by a transcendental algebraic equation. Consider the complete solution, all the way out to $u_1 = \frac{1}{2}$ $(c = L/2^{1/2})$, in which case

$$\pm U_1 = U_1^{1/2}(U_1 + R)^{1/2} - (1 + R)^{1/2} + R\sinh^{-1}\frac{1}{R^{1/2}} - R\sinh^{-1}\left(\frac{U_1}{R}\right)^{1/2}. \tag{15}$$

It is sufficient for the present purposes of illustration to consider extreme cases. So suppose that the rope is enormously compressed, $U_1 \ll 1$. Then R is very large and a straightforward expansion of (15) leads to the conclusion that the $\pm$ must be minus and

$$R^{1/2} \simeq \frac{2}{3U_1}(1 - U_1^{3/2}). \tag{16}$$

To this order it follows that[1]

$$1 - 2u + U = +[1 - (1 - U_1^{3/2})U/U_1]^{2/3} . \tag{17}$$

The generating function follows from equation (8):

$$F(U) = \tfrac{1}{9}\alpha^2\left[1 + \frac{3}{2\alpha}(1 - \alpha U)^{1/3}\right] \simeq \tfrac{1}{9}\alpha^2 , \tag{18}$$

where $\alpha \equiv (1 - U_1^{3/2})/U_1 = \frac{3}{2}R^{1/2}$, and

$$\frac{dF}{dU} \simeq -\frac{\alpha^2/9}{(1 - \alpha U)^{2/3}} . \tag{19}$$

Hence

$$B_z = \tfrac{1}{3}\alpha\left[1 - \frac{U}{(1 - \alpha U)^{2/3}}\right]^{1/2} , \qquad B_\phi = \tfrac{1}{3}\alpha\frac{U^{1/2}}{(1 - \alpha U)^{1/3}} . \tag{20}$$

Noting that $\alpha U_1 = 1 - U_1^{3/2} \simeq 1$ and $1 - \alpha U_1 = U_1^{3/2}$, it follows that B_ϕ increases from zero on the axis of the rope to $\frac{1}{3}\alpha$ at the surface, while B_z decreases from $\frac{1}{3}\alpha$ on the axis to zero at the surface. The outstanding feature is that the variation of B_z and B_ϕ is all in a small neighborhood of the outer surface U_1, essentially within a fraction $U_1^{3/2}$ of U_1. The rope is essentially an untwisted uniform field, with $B \simeq 0$ from $U = 0$ out to $U = U_1 - U_1^{5/2}$, where a thin layer of azimuthal field is wrapped on. The magnitude of the field $B_\phi^2 + B_z^2$ is uniform across the entire tube. Had the original rope stopped short of $u = \frac{1}{2}$, the layer of azimuthal field would not appear and, to the order considered, the field would be untwisted.

This is what one would expect on physical grounds. Radial compression of a twisted rope enhances B_z more than B_ϕ, by the factor ϖ/Π.

c) *Expansion*

Suppose that the rope is expanded. Then let $R \equiv -S$ so that $S > 0$. Then

$$\pm dU = \frac{d\phi}{(1 - S/\phi)^{1/2}} .$$

Integration yields

$$\pm U = Q + (1 - 2u + U)^{1/2}(1 - 2u + U - S)^{1/2} + S\cosh^{-1}\left(\frac{1 - 2u + U}{S}\right)^{1/2} ,$$

where Q is the second constant of integration. Since $u = 0$ where $U = 0$, we have

$$\pm U = (1 - 2u + U)^{1/2}(1 - 2u + U - S)^{1/2} - (1 + S)^{1/2} + S\cosh^{-1}\left(\frac{1 - 2u + U}{S}\right)^{1/2} - S\cosh^{-1}\frac{1}{S^{1/2}} . \tag{21}$$

Again let $u_1 = \frac{1}{2}$ and treat the extreme case that $U_1 \gg 1$. Then $S \ll 1$ and one finds that the $\pm$ must be plus, with

$$S = \frac{2}{\ln U_1}\left[1 - \frac{\frac{3}{2}}{\ln^2 U_1} + O\left(\frac{1}{\ln^3 U_1}\right)\right] . \tag{22}$$

To this order, then,

$$2u\left[1 + \frac{1 - \frac{3}{2}/\ln^2 U_1}{(1 + U_1)\ln U_1}\right] + O\left(\frac{1}{\ln^4 U_1}\right) = \frac{\ln(1 + U)}{\ln U_1}\left(1 - \frac{\frac{3}{2}}{\ln^2 U_1}\right) + \frac{3U}{2(1 + U)\ln^2 U_1} . \tag{23}$$

In lowest order[2]

$$u \simeq \frac{\ln(1 + U)}{2\ln U_1}\left[1 + O\left(\frac{1}{\ln^2 U_1}, \frac{1}{(1 + U)\ln U_1}\right)\right] , \tag{24}$$

[1] Alternatively one may return to (11) and note that $du/dU \gg 1$, while $U \ll 1 - 2u$ except in the outermost layer $u = \frac{1}{2}$. The equation is then immediately integrable to give $1 - 2u = \{1 - U/U_1[1 - (1 - 2u_1)^{3/2}]\}^{2/3}$, which agrees with (17) except in the outermost layer.

[2] Alternatively, one may return to (11) and note that $du/dU \ll 1$, while $2u \ll 1 + U$ everywhere in $0 < U < C^2/L^2$. The equation is then immediately integrable to give $u = u_1 \ln(1 + u)/\ln(1 + u_1)$, which agrees with equations (23) and (24) to the order considered.

Then equation (8) yields

$$F(U) = \frac{1}{4 \ln^2 U_1(1 + U)} \left[1 + O\left(\frac{1}{\ln U_1}\right)\right], \tag{25}$$

and

$$B_z = \frac{1}{2 \ln U_1(1 + U)}, \quad B_\phi = \frac{U^{1/2}}{2 \ln U_1(1 + U)}. \tag{26}$$

Thus except for a remnant of the original field ($b_z = (1 - 2u)^{1/2}$, $b_\phi = u^{1/2}$) in the core, $U \lesssim 1$, the field is essentially azimuthal, with $B_\phi \sim 1/\Pi$. The effect of expansion is the opposite of compression. The azimuthal field density declines only inversely with the logarithm of the expansion so that the azimuthal field becomes dominant. The expanded rope is much more tightly twisted, $B_\phi/B_z = U^{1/2} \gg 1$, then the initial rope, $b_\phi/b_z = u^{1/2}/(1 - 2u)^{1/2} \ll U^{1/2}$.

The important point is that the expansion enhances instability, by increasing the twisting of the rope and changing the initial tension

$$T = 2\pi \int_0^{L/2^{1/2}} d\varpi\varpi(b_z^2 - b_\phi^2)/8\pi = +\frac{L^2}{64}$$

into compression in the expanded tube,

$$T = -[\ln (1 + U_1) - 2]L^2/32 \ln^2 U_1, \tag{27}$$

for $U_1 \gg 1$. Hence, expansion of a stable force-free rope leads to buckling as the tension falls to zero and becomes compression. In the present example, with the asymptotic form (25), (26) the net tension falls to zero when $U_1 = 6.4$, i.e., when the tube has been expanded by a factor of $(U_1/u_1)^{1/2} \simeq 3.6$. This is only a crude estimate, of course, because the asymptotic form for the field was worked out assuming that $\ln U_1 \gg 1$. But it serves to show that a modest expansion of a force-free rope can put the rope into a fundamentally unstable state. When twisted ropes of field expand up through the photosphere to form an active region on the Sun, their expansion must cause buckling instability, leading to complicated contortions of the field.

III. RADIAL DILATION OF SEGMENT OF A ROPE

a) *Basic Equations*

Consider the equilibrium field configuration when a segment of length l of a force-free rope of total length s is expanded. It will be sufficient to give the results in the limit that $s \gg l$. The interested reader can easily write down the equations for any $s > l$. The behavior of the field in an expanded portion of the rope is singular and may perhaps contribute to the dissolution of sunspots.

Consider, then, an infinitely long twisted rope of magnetic flux, with its field described in terms of the generating function $f(u)$ by equation (1), where again $u = \varpi^2/L^2$. The outer radius of the rope is $\varpi = c$, with $u_1 \equiv c^2/L^2$ again. Expand a long section of the rope to the radius C, with the condition that the expanded section is uniform along its length $l \gg C$. The field in the expanded section is given by equation (2) in terms of the generating function $F(U)$, with $U = \Pi^2/L^2$ and $U_1 \equiv C_1^2/L^2$ again.

Conservation of magnetic flux along the rope leads again to equation (3). But now the equilibrium condition is that the torque in the shell (ϖ, $\varpi + d\varpi$) is uniform along the tube. Hence $b_z b_\phi \varpi^2 d\varpi = B_z B_\phi \Pi^2 d\Pi$. Divide this by $b_z \varpi d\varpi = B_z \Pi d\Pi$, for conservation of magnetic flux, and we have simply $b_\phi \varpi = B_\phi \Pi$. Hence

$$u^2 df/du = U^2 dF/dU \tag{28}$$

in place of equation (4). The problem is to solve equations (6) and (28) simultaneously. Use (28) to eliminate dF/dU from (6) to obtain an expression for F,

$$F(U) = \left(\frac{du}{dU}\right)^2 \frac{d}{du} uf - \frac{u^2}{U} \frac{df}{du}. \tag{29}$$

Differentiate this expression with respect to U and eliminate dF/dU from equation (28). The result is

$$0 = \left\{2 \frac{d^2u}{dU^2} \frac{d}{du} uf + \left(\frac{du}{dU}\right)^2 \frac{d^2uf}{du^2} - \frac{1}{U} \frac{d}{du} \left(u^2 \frac{df}{du}\right)\right\} \frac{du}{dU}. \tag{30}$$

Consider again the particular example (10). Then equation (30) becomes

$$0 = \left[(1 - 2u) \frac{d^2u}{dU^2} - \left(\frac{du}{dU}\right)^2 + \frac{u}{U}\right] \frac{du}{dU}. \tag{31}$$

This nonlinear differential equation is apparently not subject to exact integration by elementary methods.

b) *Compression*

Suppose that $C \ll c$, so that $U \ll u(U)$. Drop the linear term u/U from equation (31). The remainder of the equation can then be integrated to give

$$2u = 1 - \{1 - [1 - (1 - 2u_1)^{3/2}]U/U_1\}^{2/3}\,. \tag{32}$$

Again we treat the whole tube, $u_1 = \frac{1}{2}$, so that equation (32) reduces to

$$2u = 1 - (1 - U/U_1)^{2/3}\,. \tag{33}$$

The generating function $F(U)$ follows from (29) as

$$F(U) = \frac{1}{9U_1^2}[1 + O(U_1)]\,. \tag{34}$$

In this approximation, then

$$B_z^2 = \frac{1}{9U_1^2}\,, \qquad B_\phi^2 = 0\,, \tag{35}$$

and the twist has disappeared from the compressed section. The next-order terms can be computed if $F(U)$ is obtained from an integration of (28):

$$\frac{dF}{dU} = -\frac{u^2}{U^2} = -\frac{1}{4U^2}[1 - (1 - U/U_1)^{2/3}]^2\,. \tag{36}$$

Then

$$F(U) = \frac{1}{9U_1^2} + \frac{1}{U_1}\left(\frac{2\pi}{3^{3/2}} - 1\right) + [1 + (1 - U/U_1)^{1/3} - 2(1 - U/U_1)^{2/3}]\frac{1}{4U} + \frac{3(1 - U/U_1)^{1/3}}{4U_1} - \frac{2}{3^{1/2}U_1}\arctan\frac{1 + 2(1 - U/U_1)^{1/3}}{3^{1/2}}\,, \tag{37}$$

where the constant of integration has been chosen so that $F(0)$ reduces to (34). This expression for $F(U)$ omits constant terms $O(1/U_1)$, which are without physical interest. The slight decrease in $F(U)$, in passing from $U = 0$ to $U = U_1$, may be seen from the fact that

$$\frac{F(U_1) - F(0)}{F(0)} = \left(\frac{\pi}{3^{3/2}} - \frac{3}{4}\right)9U_1 \simeq -1.31U_1\,.$$

Thus the magnitude of the field declines by the factor $1 - 1.31U_1 \simeq 1$ from the axis to the surface, as a consequence of the small azimuthal field

$$B_\phi^2 = \frac{1}{4U}[1 - (1 - U/U_1)^{2/3}]^2\,, \tag{38}$$

which has the form

$$B_\phi \simeq U^{1/2}/3U_1 = L\Pi/3C^2$$

near the axis, and increases nearly linearly to the value $L/2C$ at the surface. The longitudinal field declines only very slightly toward the surface.

c) *Expansion*

Consider now the form of the field in a section of the magnetic rope that is expanded. The expansion leads to a singular redistribution of field. It is sufficient to consider the asymptotic form for large expansion ($C \gg c$) of a rope with but little initial twist ($u_1 \ll 1$, $b_z \simeq 1 - u$, $b_\phi = u^{1/2}$). Then $1 - 2u \simeq 1$ and $u/U \ll 1$, so that equation (31) reduces to

$$d^2u/dU^2 + u/U \simeq 0\,. \tag{39}$$

The solution which satisfies the present boundary conditions $u(0) = 0$, $u(U_1) = u_1$ is

$$u = u_1\frac{U^{1/2}J_1(2U^{1/2})}{U_1^{1/2}J_1(2U_1^{1/2})}\,. \tag{40}$$

The solution (40) exhibits the singular property of the equilibrium configuration, mentioned earlier. Recall that $u(U) = \varpi^2/L^2 \geq 0$ and must be a monotonically increasing function of U, i.e., $du/dU > 0$. But

$$\frac{du}{dU} = u_1 \frac{J_0(2U^{1/2})}{U_1^{1/2}J_1(2U_1^{1/2})} \tag{41}$$

and is positive only out to the first zero of J_0, at $U^{1/2} \equiv U_0^{1/2} = 1.2024$, $U_0 = 1.4458$. A physically acceptable solution to approximation (39) does not exist beyond this point. There is no problem, then, if $U_1 < U_0$. But we know on physical grounds that we are not limited in the amount of expansion that can be introduced. So what happens when $U_1 > U_0$? The dilemma cannot be attributed to the approximation of equation (31) by (39), neglecting u compared with 1 and $(du/dU)^2$ compared with u/U. That $u \leq u_1$ over the interval $(0, u_1)$ is obvious, since u is a monotonically increasing function of U with a maximum of u_1 at $U = U_1$. Hence $1 - 2u \simeq 1$ to any degree of accuracy by choosing u_1 sufficiently small. Compare $(du/dU)^2$ and u/U. It is evident from equations (40) and (41) that

$$\frac{U}{u}\frac{du}{dU} = \frac{U^{1/2}J_0(2U^{1/2})}{J_1(2U^{1/2})}\,.$$

This ratio declines monotonically from one at $U = 0$ to zero at $U = U_0$. Hence at the origin the neglected term $(du/dU)^2$ can be made arbitrarily small compared with u/U in the limit of small u_1, and becomes even smaller in proportion to u/U with increasing U. So there is no possibility for a *qualitative* error in the approximation. The effect is real. The solution that passes through the origin rises to a maximum and then declines with increasing U. The solution of equation (31) that is valid in the neighborhood of the origin extends out to the point $U = U_0$ at which $du/dU = 0$, and is physically unacceptable beyond.

But note that equation (31) can be satisfied either by setting equal to zero the quantity in braces, or by requiring that

$$du/dU = 0\,.$$

Hence beyond U_0 the solution is $u =$ *constant*. Altogether, then, if $U_1 < U_0 = 1.4458$, equation (40) is the solution. If $U_1 > U_0$, then equation (40) needs to be renormalized. In place of (40) write

$$u = u_1 \frac{U^{1/2}J_1(2U^{1/2})}{U_0^{1/2}J_1(2U_0^{1/2})} = u_1 1.6020 U^{1/2}J_1(2U^{1/2}) \tag{42}$$

for $0 < U < U_0$. Beyond U_0 the solution is

$$u = u_1\,, \tag{43}$$

and has the singular property that all the lines of force in the interval (U_0, U_1) map from the surface $u = u_1$ of the original force-free magnetic rope.

If $U_1 \leq U_0$, it follows from equations (29), (40), and (41) that

$$F(U) = u_1^2 \frac{J_0^2(2U^{1/2}) + J_1^2(2U^{1/2})}{U_1 J_1^2(2U_1^{1/2})}\,.$$

Hence

$$B_z = u_1 \frac{J_0(2U^{1/2})}{U_1^{1/2}J_1(2U_1^{1/2})}\,, \tag{44}$$

$$B_\phi = u_1 \frac{J_1(2U^{1/2})}{U_1^{1/2}J_1(2U_1^{1/2})}\,, \tag{45}$$

over the entire range $0 \leq U \leq U_1$. If, on the other hand, $U > U_0$, then use equation (42) in place of (40) in $U < U_0$. It follows that in $U < U_0$

$$B_z = u_1 \frac{J_0(2U^{1/2})}{U_0^{1/2}J_1(2U_0^{1/2})}\,, \tag{46}$$

$$B_\phi = u_1 \frac{J_1(2U^{1/2})}{U_0^{1/2}J_1(2U_0^{1/2})}\,, \tag{47}$$

with B_z falling to zero at $U = U_0$. In $U > U_0$, it follows from equations (29) and (43) that

$$F(U) = +u_1^2/U\,, \tag{48}$$

so that

$$B_z = 0\,, \qquad B_\phi = u_1/U^{1/2}\,. \tag{49}$$

The field in $U_0 \leq U \leq U_1$ is just the usual azimuthal field $1/\varpi$ associated with a line current.

The expansion enhances the azimuthal component, of course, with $b_\phi^2 \simeq u \ll b_z^2 \simeq 1$ before, and B_ϕ comparable to B_z after. The total tension in the rope after expansion is

$$T(U_1) = \frac{u_1^2 L^2 J_0(2U_1^{1/2})}{4U_1^{1/2} J_1(2U_1^{1/2})} \tag{50}$$

for $U_1 < U_0$. The tension is $+u_1^2/L^2/4U_1$ for $u_1 \ll U_1 \ll 1$. It decreases monotonically with increasing U_1, vanishing as $U_1 \to U_0$. Hence for $U_1 > U_0$ the tension is negative because of the purely azimuthal field in $U > U_0$. The net contribution from $0 \le U \le U_0$ is zero, so that

$$T(U_1) = -2\pi \int_{LU_0^{1/2}}^{LU_1^{1/2}} d\Pi\, \Pi \frac{B_\phi^2}{8\pi} = -\frac{u_1^2 L^2}{8} \ln \frac{U_1}{U_0}\,. \tag{51}$$

The rope is under compression and hence unstable if expanded beyond U_0.

IV. APPLICATION AND CONCLUSION

The calculations show that radial compression of a force-free rope of magnetic flux reduces the degree of twisting in the rope. The total number of turns is preserved, but the reduced radius means that the lines of force are more nearly parallel to the axis.

Radial expansion has the opposite effect, of course. The ratio B_ϕ/B_z on each line of force increases. We have pointed out elsewhere (Parker 1966) that a magnetic rope with even a modest twist has little, if any, net tension. For instance, the net tension along any annulus $(\varpi, \varpi + d\varpi)$ in a force-free rope vanishes when the lines of force reach an angle of $\pi/4$ with the axis of the rope ($b_\phi = b_z$).[3] Any further twisting, or any expansion, causes buckling.

The high-resolution Hα photographs of active regions in the solar photosphere (Smith 1971; Beckers 1971; Zirin 1971, 1972) show twisted ropes of flux emerging from the photosphere. It is evident that as the ropes rise and expand they move toward instability. Hence it may be the simple act of rising and expanding that causes much of the complicated and rapid variation in magnetic pattern in a newly forming active region. It may be the instability caused by expansion that triggers flare activity. In any case, magnetic fields in the form of stable ropes of flux at some depth below the photosphere automatically lapse into instability, and buckling and kinking, if they rise and expand to the surface.

But the story is more complicated than this. It may take a very long period of time for a rope of flux that rises and expands through the surface of the Sun to reach equilibrium. The rope spends a very long time shifting coils of field along its length from the unexpanded to the expanded portion. To describe the basic physics of this effect, consider an infinitely long uniform twisted rope of magnetic field of radius c immersed in a gas of uniform pressure. Imagine, then, that some finite segment of length l is expanded to a larger radius C. (Presumably this happens when a portion of the rope rises to the photosphere.) For simplicity suppose that the expansion takes place rather quickly, in a time small compared to the Alfvén transit time l/V_A along the segment. Then the number of turns in the segment l is conserved and the calculations of § IIc are appropriate. The expanded segment achieves a quasi-equilibrium (26) that may not be subject to buckling.[4]

Equation (26) does not represent a true equilibrium for the expanded segment because the torque in each cylindrical shell of field in the rope is not the same in the expanded and unexpanded portions of the rope. The field must redistribute itself into the configuration described by equations (44)–(49), requiring more turns in the expanded segment. And that takes a much longer time. Torsional Alfvén waves propagate in and out of the expanded segment, eventually bringing torsional equilibrium along each cylindrical shell of field.

Quantitatively, note that for $f(u) = 1 - u$ and $u \ll 1$, the number of turns per unit length in the unexpanded portion of the rope is

$$1/\lambda(\varpi) = b_\phi(\varpi)/2\pi\varpi b_z(\varpi) \simeq 1/2\pi L\,.$$

This is preserved in the sudden expansion. But torsional equilibrium requires the fields (44) and (45), or (46)–(49). If equations (44) and (45) are used, then the number of turns per unit length for torsional equilibrium is

$$1/\Lambda(\Pi) = B_\phi(\Pi)/2\pi\Pi B_z(\Pi) = \frac{J_1(2U^{1/2})}{2\pi L U^{1/2} J_0(2U^{1/2})}\,.$$

[3] There are a number of restrictions on field enhancement in force-free ropes that are not generally recognized (Parker 1966, 1974). For instance, it is readily shown from equation (1) that the mean square longitudinal field over a rope of radius c is just equal to the square of the field at the surface, $\langle b_z^2 \rangle = f(c) = b_z^2(c) + b_\phi^2(c)$, and cannot be larger. For the special case that $f = 1/\varpi^{2\alpha}$ so that $b_z = (1-\alpha)^{1/2}/\varpi^\alpha$, $b_\phi = \alpha^{1/2}/\varpi^\alpha$ and the pitch angle arc tan $(b_\phi/b_z) = [\alpha/(1+\alpha)]^{1/2}$ is uniform throughout the rope, it is readily shown that the mean longitudinal field is $\langle b_z \rangle = [2(1-\alpha)^{1/2}/(2-\alpha)]b(c) = [2/(2-\alpha)]b_z(c)$. Instability occurs for $\alpha > \frac{1}{2}$. For the critical value $\alpha = \frac{1}{2}$, the mean longitudinal field is $2^{3/2}/3 = 0.943$ times the total surface field. The mean longitudinal field *declines* in comparison with the surface field as the rope is twisted and becomes increasingly force free!

[4] For the present case that $u_1 \ll 1, U_1$ the net tension in the initial quasi-equilibrium is $[2U_1/(1+U_1) - \ln(1+U_1)]L^2/32 \ln^2 U_1$ in place of (27). Thus, for instance, if $U_1 = 2$, the tension is positive and the initial quasi-equilibrium is not subject to buckling.

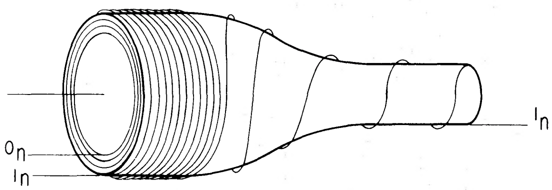

FIG. 1.—A schematic drawing of a line of force on the surface $\varpi = c$ of the unexpanded portion of the tube in the process of sliding its coils onto the expanded portion and filling the space between U_0 and U_1 with azimuthal field.

Near the axis of the rope ($U \ll 1$) this reduces to

$$1/\Lambda(\Pi) \simeq (1/2\pi L)[1 + O(\Pi^2/L^2)],$$

which is the same as in the original rope. But $1/\Lambda(\Pi)$ increases with Π and goes to infinity as $U \to U_0 = 1.4458$ at the first zero of J_0. The number of turns per unit length for torsional equilibrium is infinite in $U > U_0$ since, according to equation (49), $B_z = 0$ there.

It follows, therefore, that the lines of force coiled about the unexpanded portion slide along the tube onto the expanded portion where they coil increasingly tightly as time goes on. The topology of the lines does not change, of course, because we have assumed that the electrical conductivity is infinite. Figure 1 sketches the topology of the lines of force as they shift from the unexpanded to the expanded portion of the tube. The coils are propelled onto the expanded portion by the torque carried in the field. The energy comes from the slight unwinding of the field on the surface of the unexpanded portion. The surface layer of field forms a coil spring, which is weak (so far as torsion is concerned) on the expanded portion. Hence the unexpanded portion *unwinds*, and twists up the expanded portion. It is shown in the Appendix that the work done by the unwinding of the unexpanded portion of the coil spring exceeds the energy which goes into winding up the expanded portion.

If the expansion is fixed at some value $U_1 > U_0$, then the expanded portion must ultimately become unstable as more and more turns of field slip onto the expanded portion and tension gives way to compression, equation (51).

The curious feature is that the recoiling goes on only in the surface layers of the rope.[5] The entire field in the expanded portion beyond U_0 comes only from the surface $u = u_1$ of the unexpanded portion of the rope, as noted following equation (43). Hence when a magnetic rope emerges through the photosphere, the redistribution of coils goes on for periods *very large* compared with the Alfvén transit time along the surface of the expanded portion. The redistribution and re-coiling may go on for days, as a consequence of the singular mapping, (42) and (43), of lines from the expanded to the unexpanded portion of the rope. And if the rope is not unstable to buckling when it first emerges, it will become unstable when the redistribution of coils proceeds far enough.

The idea suggests itself that the slow development of buckling may be responsible for the final demise of individual sunspots, and perhaps individual active regions.

Altogether, then, the calculations show some of the complicated and time-consuming readjustments and instabilities in magnetic flux tubes that rise up from below into the solar photosphere. The restless, and sometimes explosive, behavior of the fields in newly forming active regions is presumably a direct consequence.

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APPENDIX

Consider the work done by the unwinding of the thin layer of field $(c - \Delta\varpi, c)$ at the surface of the unexpanded portion of the tube, and compare it with the energy in the azimuthal field beyond U_0 of the expanded portion. The magnetic field in the annulus $(c - \Delta\varpi, c)$ is $b_z = (1 - 2c^2/L^2)^{1/2}$ and $b_\phi = c/L$. Hence for $c/L \ll 1$ the magnetic flux in the annulus is

$$\Delta\Phi = 2\pi c\Delta\varpi b_z \simeq 2\pi c\Delta\varpi$$

to lowest order in c/L. Suppose that this same flux $\Delta\Phi$ is wrapped into the azimuthal field (49) $B_\phi = c^2/L\Pi$ in the annular ring of radius Π, thickness $\Delta\Pi$, and width ΔZ somewhere in $U > U_0$ of the expanded portion of the tube. Then

$$\Delta\Phi = \Delta\Pi\Delta Z B_\phi,$$

and

$$2\pi\Delta\varpi/c = \Delta\Pi\Delta Z/L\Pi$$

is the condition for conservation of flux.

In unwinding one revolution on the unexpanded portion of the tube, the work done is

$$W = 2\pi c\Delta\varpi \frac{b_z b_\phi}{4\pi} 2\pi c \simeq \pi c^3\Delta\varpi/L.$$

[5] We have been unable to find a means for escaping this behavior by introducing small amounts of diffusion.

The energy in the field after it is transferred to the expanded portion is

$$\mathscr{E} = 2\pi\Pi\Delta\Pi\Delta Z B_\phi^2/8\pi = \Delta\Pi\Delta Z c^4/4L^2\Pi\ .$$

This field energy is less than the work done by the unwinding, $W > \mathscr{E}$. For we have then

$$2\pi\Delta\varpi/c > \Delta\Pi\Delta Z/2L\Pi\ .$$

Employing the condition for flux conservation reduces this to $1 > \frac{1}{2}$. The work done is always twice as great as the final energy of the field once it is wrapped around the expanded portion of the tube. It is this extra work that shifts the coils of field from the unexpanded to the expanded portion of the tube, as illustrated in figure 1.

It is a simple matter to deduce the pitch of the field wound onto the expanded portion in terms of the thickness $\Delta\varpi$ of the surface layer on the unexpanded portion. If we suppose that the flux $\Delta\Phi$ from one turn of unwinding becomes coiled into the interval ΔZ on the expanded portion and fills the space from $\Pi_0 = LU_0^{1/2}$ to $\Pi_1 = LU_1^{1/2}$, then

$$\Delta\Phi = \Delta Z \int_{\pi_0}^{\pi_1} d\Pi B_\phi = \frac{c^2}{2L}\Delta Z \ln\frac{U_1}{U_0}\ ;$$

and, since $\Delta\Phi$ is also equal to $2\pi c\Delta\varpi$, it follows that

$$\Delta Z = \frac{4\pi L\Delta\varpi}{c \ln U_1/U_0}\ .$$

We note that ΔZ is the pitch of the winding, containing one revolution of field. We have $\Delta Z \to 0$, so that the field is precisely azimuthal, as the thickness $\Delta\varpi$ of the surface layer on the unexpanded portion approaches zero.

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E. N. Parker
Laboratory for Astrophysics and Space Research, 933 E. 56th Street, Chicago, Illinois 60637