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# STATISTICAL ANALYSIS OF CATALOGS OF EXTRAGALACTIC OBJECTS. I. THEORY*

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## ABSTRACT

This paper is the first in a series on a systematic analysis of a number of catalogs of extragalactic objects. The method is based on the "analysis of power spectra," as elucidated by Tukey. As workers in other fields have abundantly shown, this method, which combines the virtues of the power spectrum with the utility of its transform, the covariance function, can be a powerful and discriminating statistical test. Presented here are a number of theoretical results necessary for the analysis of data distributed on a sphere. Some results are close analogs of corresponding formulae for a distribution in flat space. Others are directed to the problem of relating the observed two-dimensional distribution to the three-dimensional spatial distribution.
*Subject headings:* galaxies — galaxies, clusters of

## I. INTRODUCTION

### *a*) *The Choice of Statistical Measure*

This paper describes a general method of statistical analysis of the catalogs of extragalactic objects—galaxies, clusters of galaxies, radio sources. A goal is to derive quantitative statistical statements on the nature of the distribution of matter, in the sky and in space. An equally important goal is to subject a variety of catalogs to like analysis. Any regularities that emerge, and any manifest differences among distributions of different objects, surely would be of prime interest.

There are different philosophies on how the catalogs should be analyzed. Neyman and Scott observe that the catalogs contain such a wealth of information on galaxies that one should be able to make quite detailed statements on the manner of distribution (cf. Scott 1962; Neyman 1962; and earlier references therein). Indeed, with the use of fast digital computers, one can very well think of deriving many different and useful statistical measures. In contrast, Limber (1953, 1954) was content to derive from the Shane-Wirtanen data only two numbers: an effective coherence length and a measure of the amplitude of a covariance function. It is impossible to say which philosophy is "best," and of course one hopes that they may complement each other. The present work is directed to *simple statistical measures*, of the sort discussed by Limber, even though *this means abandoning information available in the catalogs*. The choice is dictated in part by the desire to have systematic analyses of a variety of catalogs, some of which may be suitable for more detailed analysis, others not. Equally important, one observes that analysis in extragalactic astronomy almost always is beclouded with difficulties of interpretation. It is hoped that by concentrating on a statistical measure that does admit of direct and simple interpretation one may be able to deal more directly with the ambiguities of the phenomena.

The approach is based on two statistical measures—the covariance function and the power spectrum—both of which already have been applied in analyses of catalogs of

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extragalactic objects. The covariance function in effect is the average of the product of the density of objects at any point with the density of objects some fixed (angular or linear) distance away in a randomly chosen direction. This function, or a variant, or some analogous measure, has figured in many discussions. In some cases, it is an auxiliary measure in a more elaborate statistical model (cf. Scott 1962; Neyman 1962; Karachentsev 1966, 1971; Kiang 1967), while in other cases a main goal simply is to estimate the function (Limber 1953; Rubin 1954; Gunn 1965; Kiang and Saslaw 1969; de Vaucouleurs 1971, fig. 2; Bogart and Wagoner 1973). The covariance function alone does not measure some very interesting things, like the distribution function for the numbers of galaxies in clusters of a given linear scale (apart from the indirect measure provided by statistical irregularities in the apparent covariance function). It does measure the average degree of irregularity as a function of characteristic length scale. This is interesting in its own right, and as a quantity that naturally arises in the theory of evolution of irregularities in an expanding Universe.

The general problem with the covariance function as a statistical measure has been vividly described by Tukey (1967): *the apparent value of the function at any chosen argument depends on effects in the data that may have many different characteristic length scales*. In particular, in any catalog of extragalactic objects the apparent density almost invariably varies across the sky because of the variable effects of galactic obscuration and confusion. This "coherence" of large angular scale is to be expected, and, if observed, must be treated with caution unless one can make a reliable correction for it. But large-scale coherence affects the apparent covariance function on small scales as well as large. Limber (1953) and Bogart and Wagoner (1973) proposed that one may avoid the problem by exploiting the expected symmetry in galactic longitude. This should reduce the effect, but it would require careful discussion before one could be sure that it had been eliminated.

This problem of mixing of effects applies to many statistical measures (Yu and Peebles 1969, hereafter referred to as YP; Fullerton and Hoover 1972). A common technique has been to divide the sky into cells and then use some measure of the frequency distribution of counts of objects in the cells as a measure of clustering. This is described in detail and very complete references given by de Vaucouleurs (1971). The cell size for which the deviation from random is greatest according to the measure is taken as an indicator of the characteristic clustering scale. But any large-scale gradient of the density is guaranteed to make the frequency distribution in smaller cells deviate from Poisson, whether there is small-scale coherence or not. Generally, it is a difficult problem to untangle this mixing effect. In one case, for the measure of deviation from random introduced by Mowbray (1938) and by Zwicky (1942, 1952), it was shown by Neyman, Scott, and Shane (1954) that the behavior of the measure is quite different from what was intuitively expected. The significance of the "maximum deviation from random" is uncertain also because measures of deviation from random for different cell sizes surely are statistically related. On both counts, it is difficult to give a meaningful assessment of significance of results.

The power spectrum appears to be a most convenient way to get at both problems (Blackman and Tukey 1959, hereafter referred to as BT). This function is the square of the Fourier transform of the density for a distribution in flat space, the square of the spherical-harmonic transform of the density for a distribution on a sphere (YP). It has the following attractive features:

1) Because the basis functions are orthogonal (or close to it) and have characteristic scale that varies systematically with wavenumber $k$ or index $l$, the spectrum gives an immediate display of irregularity as a function of characteristic scale.

2) There are well-established means for estimating the degree of contamination of the apparent power spectrum by irregularities in different parts of the "*true*" spectrum.

3) It is easy to make models for the expected variance in the spectrum, and to estimate the correlation of different terms (BT).

4) The power spectrum is just the Fourier transform (or the analog for a distribution on a sphere) of the covariance function. By studying both functions, according to the general precepts of BT, we can hope to judge the effects of contamination of one scale by another, to judge the statistical significance of any result, and to arrive finally at a more believable estimate of the true covariance function.

5) We are interested ultimately in the spatial distribution. By examining the spherical-harmonic transform we are in effect making a spherical wave decomposition of the three-dimensional Fourier components. Thus it is not surprising that there is a fairly direct connection between the angular power spectrum and its spatial counterpart.

### *b*) *Outline of the Paper*

Although the statistical questions are simple, the mathematical analysis of them is somewhat lengthy. In an effort to minimize confusion, I have listed the main symbols and results in part *a* of the next section. It is hoped that this part will provide an adequate guide for people not interested in details to the subsequent papers describing the applications.

The main basis for the calculation is the homogeneity assumption by which one gives meaning to the ensemble averages or expectation values of quantities. This subject is discussed in § II*b*. Section II*c* deals with subsidiary assumptions like isotropy and geometry.

The basic relations among the covariance function and power spectrum in three dimensions and their counterparts on a sphere are derived in § III. Each of these four functions is related to each of the others by an integral equation or an infinite series. This means that there are many routes by which the analysis can proceed from one function to another. The relation between power spectrum and covariance function is discussed in § III*a* for a three-dimensional distribution, and in § III*b* for a distribution on a sphere. In §§ III*c*, *d*, the expression for the "power spectrum" on the sphere is derived from the spatial power spectrum. Finally, as a demonstration of consistency, the relation between corresponding covariance functions (Limber's formula) is derived in § III*e*. This latter is the most direct path to the desired result, equation (6) below, but the approach adopted here seems more natural and convenient.

The adjustments necessary to take account of a survey over a part of the sky are discussed in § IV. Section V deals with the interpretation of the fundamental relations in terms of a clustering picture, and with the expected scaling of the measured functions with the depth of the survey. The expected statistical properties (frequency distribution) of the power spectrum components requires some additional assumptions. This is discussed in § VI. Appendix A lists some needed properties and conventions for the spherical harmonics.

The book by Blackman and Tukey is not only an important guide to how this analysis was fashioned, but also an excellent introduction to the basic concepts. It will be assumed that the reader is acquainted with the elementary results presented in BT.

## II. DEFINITIONS AND ASSUMPTIONS

### *a*) *Definition of the Main Symbols*

The main terms and the symbols and equations relating them will be listed here for more convenient reference. The members of a catalog, whether galaxies, clusters of galaxies, or radio sources, are called *objects*. It will be supposed that each object has a well-defined pointlike position. The probability that an object is found within volume element $\delta v$ at distance $r$ from a randomly chosen object is

$$\delta P = n(1 + \xi(r))\delta v\,, \tag{1}$$

where $n$ is the mean number density of objects. The function $\xi(r)$ may be considered the covariance for the density $\rho(x)/n - 1$, and $\xi$ will be called the *spatial covariance function*. The probability that an object is seen in the sky (listed in a catalog) within solid angle $\delta\Omega$ at angular distance $\theta$ from a randomly chosen object is

$$\delta P = \mathcal{N}[1 + w(\theta)]\delta\Omega\,. \tag{2}$$

The quantities $\mathcal{N}$ and $w$ are the results of an average over an ensemble of catalogs obtained from randomly chosen positions (not objects). $\mathcal{N}$ is the *ensemble average number of objects per steradian*, $\delta\Omega$ is the measure of solid angle, in steradians, and $w$ will be called the *angular covariance function*. It will be assumed that $\mathcal{N}$ and $w$ are not affected by obscuration.

The analog of a power spectrum for a distribution on a sphere is

$$Z_l = \langle|a_l^m|^2\rangle\,, \qquad a_l^m = \sum_i Y_l^m(i)\,. \tag{3}$$

The argument of the spherical harmonic is the angular position of the $i$th object in the catalog (assumed to cover the whole sphere), and the square brackets mean an ensemble average. The *angular power spectrum* $u_l$ is a measure of the departure of $Z_l$ from the value expected for a random distribution,

$$u_l = Z_l/\mathcal{N} - 1\,. \tag{4}$$

The angular power spectrum is related to the angular covariance function by the fundamental equation (§ III*b*)

$$u_l = 2\pi\mathcal{N}\int_{-1}^{+1} w(\theta)P_l(\cos\theta)d\cos\theta\,. \tag{5}$$

This is the analog of the standard result that the power spectrum is the Fourier transform of the covariance function. It shows, among other things, that $u_l$ is equal to the mean number of objects in excess of random within angular distance $\sim 2/l$ radians from a randomly chosen object (§ V*c*).

A second fundamental equation relates the angular power spectrum to the spatial covariance function (§§ III*c*, *d*),

$$u_l = \frac{12\pi\mathcal{N}}{D^{*3}}\int r^2 dr\,\xi(r)\Lambda_l(r/D^*)\,. \tag{6}$$

This shows how $u_l$ is related to the number of objects within a chosen spatial distance $r$ of a randomly chosen object. The weight function $\Lambda$ (eq. [34] below) depends on the probability $P(r)$ that an object at distance $r$ is included in the catalog. Under the assumptions of the next section, this probability function may be written as

$$P(r) = \phi(r/D^*)\,, \qquad \phi(x) = \int_{-\infty}^{\infty}\frac{d\Theta}{dM}dM\,\Phi(M - M^* + 5\log x)\,. \tag{7}$$

The quantity $d\Theta/dM$ is the differential luminosity function, $\Phi(m - m_0)$ is the probability that an object with apparent magnitude $m$ would be included in the catalog,

$M^*$ is some convenient measure of the typical absolute magnitude of objects selected by apparent magnitude, and, finally, an object with absolute magnitude $M^*$ at distance $D^*$ would appear at the nominal limiting apparent magnitude $m_0$ of the survey.

When the survey covers the whole sphere, the quantities $|a_l^m|^2$ (eq. [3]) for different $m$-values are estimates of the desired function $Z_l$, these estimates having an exponential frequency distribution (eq. [89]) under suitable assumptions. When the survey covers only a portion of the sky, the spherical-harmonic transform derived from the observed portion is written as $b_l^m$. To get an estimate of $Z_l$, one must subtract a mean from $b_l^m$, square, and then divide by a normalizing factor (eq. [50] below). The resulting quantities $Z_l^m$ again are estimates of the "real spectrum" $Z_l$, with exponential distribution under suitable assumptions, but here the mean values of the $Z_l^m$ are correlated, and are running averages of the true spectrum.

### *b*) *Fair-Sample Hypothesis*

The basic assumption is that, in Hubble's phrase, the space surveyed in the catalogs is a *fair sample* of the Universe. That is, it is assumed that if catalogs were obtained in the same fashion from different parts of space, they would yield the same statistical measures, apart from statistical fluctuations. This is a bold extrapolation, as has been emphasized perhaps more strongly by some writers (e.g., Dingle 1933; de Vaucouleurs 1971) than others. One should also bear in mind, however, that it is subject to empirical test. The hypothesis clearly is invalid if the statistical measures derived from different parts of the sky are significantly different, or if measures derived from surveys to different depth do not scale properly with depth, or if the maximum coherence length (clustering size) derived from the analysis is comparable to the depth or angular extent times depth for the survey. To the extent that these things do not happen, that the results for different parts of the sky and different depths are numerically consistent, we have direct and empirical evidence that the hypothesis is a valid and useful working approximation.

The hypothesis may be stated as follows:

1) The distribution of objects (galaxies, clusters of galaxies, radio sources) is described by a random stationary process.

2) The ensemble-average values defined by the process are good approximations to the values obtained from our observed sample of the Universe.

The first assumption has been discussed by Neyman (1962). One can adopt the view that we have only one Universe, that we can see only part of it, and that the analysis ought to be based on that part alone. One can accordingly construct a purely descriptive "covariance function" by counting the frequency distribution of the separations of all observed pairs of points, $\theta_{ij}$ or $r_{ij}$. This distribution function is related to the square of the Fourier transform of the distribution by relations like equation (6) (cf. eq. [28] below). If these descriptive functions, when smoothed, give similar results when obtained from the data in different parts of the observable space, then one may be tempted to take the next step, to suppose that the observed distribution is one realization of a universal quasi-stochastic process by which galaxies form. On this assumption, one can formulate the descriptive statistical measures so that, on the average, they are expected to coincide with the "true process." We have then the set of positions $\boldsymbol{x}_i$ of all objects in some large region $V$ of space. The catalog is derived from some position $\boldsymbol{x}_0$ in $V$. Whatever physical process makes galaxies and other extragalactic objects may be used to generate a statistical ensemble of possible distributions of objects in $V$. It is assumed that, for any statistical measure derived from the catalog, the average across the ensemble exists, and that this average is independent of the position of observation. Equation (6) is understood in the sense of this ensemble average (e.g., Rosenblatt 1962).

The second assumption is just that the coherence length in the distribution of galaxies is much less than the depth surveyed in the catalog, so that it makes sense to estimate statistical measures like $\xi(r)$ from the catalog.

### c) *Other Assumptions*

The cosmological redshift and the expected deviation from flat space in relativistic cosmologies are only minor effects in the available catalogs of galaxies and clusters of galaxies. Accordingly, relativistic effects are ignored, and the space within $V$ taken to be flat with the objects at fixed Cartesian coordinates $\boldsymbol{x}_i$. This may be a bad approximation for the radio sources, but the analysis of angular correlation still is valid and useful. The approach fails only in the application of the equations relating angular and spatial measures.

The next assumption is *the isotropy of the ensemble averages*. This was already indicated by writing the spatial covariance function (eq. [1]) as a function only of the magnitude of the displacement $r$. This assumption has some empirical basis from the radiation background and the galaxy counts, but again the main justification in any application may be the degree of internal consistency.

For the catalogs of galaxies, we have information on apparent magnitude, not distance, so it is necessary to make some assumption about the *luminosity function*. The mathematically convenient assumption is that there is a universal luminosity function, so that (*a*) positions $\boldsymbol{x}_i$ and luminosities $M_i$ are not correlated; and (*b*) the probability that a randomly chosen object is brighter than absolute magnitude $M$ is

$$P(<M) = \Theta(M)\,, \tag{8}$$

where $\Theta$ is a known and given function.

This assumption was adopted by Neyman and Scott and by Limber. It appears that the luminosity functions of galaxies in rich clusters and the general field, so far as they are known, are remarkably similar, but of course at some level this assumption must break down. For the present, however, other uncertainties already are so great that it does not seem advisable to broaden the discussion.

Shane and Wirtanen found that their counts of galaxies are reproducible only in a statistical sense, that the probability that a galaxy with apparent magnitude $m$ is included in any given plate count is given by a selection function $\Phi$. This function approximates a step function, with step at the nominal magnitude limit $m_0$ of the survey, but the width of the "step" might be appreciable. As Neyman (1962) and Scott (1962) remark, this can be a serious effect. For the present discussion, all that is needed is the probability $\phi$ that an object at distance $r$ is included in the catalog. Given the probability $\Phi(r, M)$ that an object with absolute magnitude $M$ at distance $r$ is included, $\phi$ may be expressed as an integral over this probability multiplied by the differential luminosity function (cf. eq. [7]).

To discuss how the analysis scales with the limiting magnitude $m_0$, one must make an assumption on how the selection function depends on $m$. A convenient and possibly realistic assumption is that $\Phi$ depends only on $m - m_0$, $m$ being the apparent magnitude of the object and $m_0$ the nominal limiting magnitude,

$$\Phi = \Phi(m - m_0)\,, \qquad m_0 = M^* + 5\log(D^*/10\ \mathrm{pc})\,. \tag{9}$$

This with equation (8) yields equation (7). As in the discussion of the luminosity function, one can observe that this simple model must be inadequate at some level, but that, for the present rather crude analysis, it should give a fairly good representation of the situation.

## III. THE BASIC EQUATIONS

### *a*) *The Spatial Covariance Function and Power Spectrum*

It will be convenient to imagine that $V$ is rectangular, and, to avoid problems with boundaries, to suppose that the universe is periodic with period fixed by this rectangular volume. Under the hypothesis this is only a mathematical artifice, not a physical assumption, for the dimensions of $V$ may be large compared to the depth of the survey, so the boundaries make no difference.

The distribution of $N$ points $x_i$ in $V$ may be represented by the sum of delta functions

$$\rho(\boldsymbol{x}) = \sum_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) . \tag{10}$$

The Fourier transform of this distribution is defined as

$$\delta_k = N^{-1/2} \sum_i \exp(i\boldsymbol{k}\cdot\boldsymbol{x}_i) ,$$

$$\rho(\boldsymbol{x}) = N^{-1/2} V^{-1} \sum_k \delta_k \exp(-i\boldsymbol{k}\cdot\boldsymbol{x}) , \tag{11}$$

where the sum is over plane waves periodic in $V$. One can define a covariance function as

$$C(r) = N^{-1} \int \rho(\boldsymbol{x})\rho(\boldsymbol{x} + \boldsymbol{r}) d^3x$$

$$= V^{-1} \sum_k |\delta_k|^2 \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) . \tag{12}$$

The second line follows from equation (11). This is the standard result that the covariance function is the Fourier transform of the power spectrum.

For a random distribution of points, the expected value of $|\delta_k|^2$ for $k \neq 0$ is unity. The departure from unity may be expressed by rewriting equation (12) as

$$C(\boldsymbol{r}) = \delta(\boldsymbol{r}) + \frac{N-1}{V} + \frac{1}{V} \sum_{k \neq 0} (|\delta_k|^2 - 1) \exp(i\boldsymbol{k}\cdot\boldsymbol{r}) . \tag{13}$$

The second term on the right-hand side of this equation is the mean density of points, $n$ (assuming $N \gg 1$). The third term is a measure of the spatial correlation function $\xi(r)$ (eq. [1]). The delta function only says that there certainly is a point at zero displacement.

Equation (13) is general but not useful as it stands. The function $\rho(\boldsymbol{r})$ is a sum of delta functions, so $C(\boldsymbol{r}) - \delta(\boldsymbol{r})$ is not smooth but a very dense sum of delta functions, and, correspondingly, $|\delta_k|^2 - 1$ oscillates (with unit amplitude for a random distribution of points), as is apparent from equation (11), which gives

$$|\delta_k|^2 - 1 = N^{-1} \sum_{i \neq j} \exp[i\boldsymbol{k}\cdot(\boldsymbol{x}_i - \boldsymbol{x}_j)] . \tag{14}$$

The detailed fluctuations in $|\delta_k|^2$ and the individual delta functions in $C(\boldsymbol{r})$ are of no interest. The easiest way to proceed is to introduce the average over an ensemble of universes (§ II*b*),

$$\langle C(\boldsymbol{r}) - \delta(\boldsymbol{r})\rangle = n[1 + \xi(r)] ,$$

$$\langle |\delta_k|^2 \rangle = \mathscr{P}(k) + 1 , \tag{15}$$

where by isotropy the power spectrum $\mathscr{P}$ is a function of the magnitude of $\boldsymbol{k}$. Then equation (13) gives

$$\xi(r) = (nV)^{-1} \sum_{k \neq 0} \mathscr{P}(k) \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) ,$$

$$\mathscr{P}(k) = 4\pi n \int_0^\infty r^2 dr \xi(r) j_0(kr) . \tag{16}$$

The second equation is the result of transforming the Fourier series and then evaluating the angular integral, with

$$j_0(x) = \sin(x)/x . \tag{17}$$

In line with § II, it is assumed that $\xi$ differs from zero over a range small in comparison with the dimensions of the box.

### b) *The Relation Between $u_l$ and $w(\theta)$*

The purpose of this section is to discuss the relation analogous to equation (12) for the angular functions. Let $\sigma(\Omega)$ be some function of angular position $\Omega = (\theta, \phi)$ on the sphere. In analogy to equation (12), an angular correlation function for $\sigma$ may be defined as

$$K(\theta) = \int \sigma(\Omega_1)\sigma(\Omega_2)\delta(\cos\theta_{12} - \cos\theta) \frac{d\Omega_1 d\Omega_2}{8\pi^2} , \tag{18}$$

where $\theta_{12}$ is the angle between the directions $\Omega_1$ and $\Omega_2$. The delta function in the integral guarantees that the product of the functions is evaluated at angular separation $\theta$, the angular integration averages this product over position and orientation on the sphere, and the denominator normalizes the integral so that if $\sigma \equiv 1$ then $K = 1$. To understand the argument of the delta function, imagine that the polar axis for the integration over $\Omega_2$ is placed along $\Omega_1$. Then the integral over $\Omega_2$ becomes an integral of $\sigma_1\sigma_2$ over the azimuthal angle $\phi_2$ at fixed polar angle $\theta$, which is just what is wanted.

The spherical harmonic transform of $\sigma$ is

$$a_l^m = \int \sigma Y_l^m d\Omega . \tag{19}$$

The analog of equation (12) is

$$\sum_m |a_l^m|^2/(2l + 1) = 2\pi \int K(\theta) P_l(\cos\theta) d\cos\theta . \tag{20}$$

To check this equation, one has only to substitute equation (19) into the left-hand side and use the addition theorem (eq. [A3]), and note that the result is equal to the result of substituting equation (18) into the right-hand side of equation (20). Equation (20) apparently first was derived by Obukhov (1947).

Equation (20) is valid in the limit where $\sigma$ approaches a sum of two-dimensional delta functions, such as would represent the angular positions of objects in a catalog, and it remains valid if $K$ and $|a_l^m|^2$ are replaced with their ensemble average values. The expected value of $K(\theta)$ is (cf. Appendix D)

$$\langle K(\theta) \rangle = \mathscr{N}[\delta(0, \Omega) + \mathscr{N}(1 + w(\theta))] . \tag{21}$$

This equation applies in the sense of an average over many catalogs. $\mathscr{N}$ is the number of objects per steradian, averaged over the ensemble. Equation (21) says that

$\mathcal{N}(1 + w)\delta\Omega$ is the probability for finding an object at angular distance $\theta$ in the same catalog as an object picked at random from the ensemble. Strictly speaking, this means that the average is weighted in proportion to the number of objects in each catalog, but since the number always is large this is not an important point.

If $l > 0$, equations (20) and (21) yield

$$Z_l = \mathcal{N} + 2\pi\mathcal{N}^2 \int w(\theta)P_l(\cos\theta)d\cos\theta\,, \tag{22}$$

where $Z_l$ is defined by equation (3). Note that the expectation value in equation (3) does not depend on the index $m$, for by rotating coordinates we can always change $m$, but by isotropy the expectation value cannot change (cf. Appendix B). Equation (22) with equation (4) yields the fundamental result expressed in equation (5).

For $l = 0$, equations (20) and (21) give

$$Z_0 = \mathcal{N} + 4\pi\mathcal{N}^2\,, \tag{23}$$

if $\int w d\Omega = 0$. This is the expected result for Poisson statistics, for $a_0{}^0$ is equal to the number of objects in the catalog divided by $(4\pi)^{1/2}$.

### *c) The Integral Equation-Survey to Fixed Distance*

It will be assumed here that the catalog lists all angular positions of objects within a spherical volume of radius $D$ around $\boldsymbol{r}_0$. Then the spherical-harmonic transform of the distribution is (eq. [3])

$$a_l{}^m = \int r^2 dr d\Omega_r Y_l{}^m(\Omega_r)\rho(\boldsymbol{r} - \boldsymbol{r}_0)\,, \tag{24}$$

where $\rho$ is the sum of delta functions (eq. [10]). Now the basic idea is that $\rho$ may be represented as a sum of plane waves (eq. [11]), and each plane wave may be represented as an expansion in spherical harmonics (eq. [A5]).

It is convenient to proceed as follows. (1) Express $\rho$ as a sum over plane waves. (2) Average the expression for $|a|^2$ over randomly chosen position $\boldsymbol{r}_0$ of observation in $V$. This eliminates the cross terms in the square of the sum, to give

$$\langle |a_l{}^m|^2 \rangle = NV^{-2} \sum |\delta_k|^2 \left| \int_0^D d^3r\, Y_l{}^m(\Omega_r) \exp(-i\boldsymbol{k}\cdot\boldsymbol{r}) \right|^2 . \tag{25}$$

(3) Write $|\delta|^2 = 1 + (|\delta|^2 - 1)$. The second term is a measure of the deviation from random (eqs. [14], [15]). (4) For the first term, observe that the sum over $\boldsymbol{k}$ can be evaluated, using

$$\sum_k \exp[i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}')] = V\delta(\boldsymbol{x} - \boldsymbol{x}')\,, \tag{26}$$

so the integrals over position may be directly evaluated to give the first contribution to $\langle |a|^2 \rangle$,

$$\tfrac{1}{3}nD^3 = \mathcal{N}\,. \tag{27}$$

This is the expected result for a random distribution. (5) Use equation (A5) to evaluate the angular integrals in equation (25). This gives

$$\langle |a_l{}^m|^2 \rangle = \mathcal{N} + \frac{16\pi^2 N}{V^2} \sum (|\delta_k|^2 - 1) \left| Y_l{}^m(\Omega_k) \int_0^D r^2 dr j_l(kr) \right|^2 . \tag{28}$$

(6) Average this expression across an ensemble of universes generated by the process that makes galaxies. This replaces $|\delta|^2 - 1$ with the smooth function $\mathscr{P}(k)$ (eq. [15]). (7) Replace the sum over $\boldsymbol{k}$ with an integral. (8) Use equation (16) to replace $\mathscr{P}(k)$ with the spatial covariance function $\xi(r)$. The result with equation (4) is

$$u_l = 4\pi \int r^2 dr n \xi(r) \Delta_l(r/D) , \tag{29}$$

where the weight function is

$$\Delta_l(y) = \frac{6}{\pi} \int_0^1 r_1^2 r_2^2 dr_1 dr_2 I_l(y, r_1, r_2) ,$$

$$I_l(y, r_1, r_2) = \int_0^\infty k^2 dk j_0(ky) j_l(kr_1) j_l(kr_2) . \tag{30}$$

Some properties of $\Delta$ are described in § V.

For any given distribution of objects in $V$, one can regard the average of $|a_l^m|^2$ over position $\boldsymbol{r}_0$ in $V$ as a purely descriptive statistical measure of the distribution, just as $|\delta_k|^2$ may be regarded as a descriptive measure of the given distribution. These two measures are related by equation (28), with no assumption (like homogeneity) needed. To get to equation (29), one must introduce the assumption or approximation that $|\delta_k|^2 - 1$ may be replaced with the smoothed truncated and isotropic function $\mathscr{P}_k$.

### *d*) *Effect of the Luminosity and Selection Function*

To take account of the luminosity function, and the variable completeness of the catalog as a function of apparent magnitude, we can write (cf. eq. [24])

$$a_l^m = \int_0^\infty r^2 dr d\Omega_r Y_l^m(\Omega_r) \rho(\boldsymbol{r} - \boldsymbol{r}_0) H(\boldsymbol{r}) . \tag{31}$$

$H$ is a stochastic process with two values, 0 and 1, that determines whether an object at $\boldsymbol{r} + \boldsymbol{r}_0$ is included in the catalog. In accordance with the general assumptions (§ II*c*), values of $H$ at different positions are not correlated, and $H$ has expectation values

$$\langle H \rangle = \langle H^2 \rangle \equiv \phi(r/D^*) , \tag{32}$$

where $\phi$ is the probability that an object at distance $r$ from the observer is included in the catalog (cf. eq. [7]).

The calculation leading up to equation (29) may be repeated with little change. In step (4), one must average $H^2$ with the help of equation (32) to obtain the first contribution to $\langle |a|^2 \rangle$,

$$\mathscr{N} = D^{*3} n E , \qquad E = \int_0^\infty x^2 dx \phi(x) . \tag{33}$$

Again, as expected, if $\mathscr{P}(k) = 0$, the expectation value of $|a|^2$ is just the mean number of points per steradian. The rest of the calculation proceeds as before, yielding

$$u_l = \frac{12\pi\mathscr{N}}{D^{*3}} \int r^2 dr \xi(r) \Lambda_l(r/D^*) ,$$

$$\Lambda_l(y) = \frac{2}{3\pi E^2} \int x_1^2 x_2^2 dx_1 dx_2 \phi(x_1) \phi(x_2) I_l(y, x_1, x_2) . \tag{34}$$

Note that the normalization of $\phi$ does not affect the value of $\Lambda$. Equations (30) are recovered from equations (34) in the limit $\phi$ = constant, $x < 1$; $\phi = 0, x > 1, D = D^*$.

### *e*) *Relation to Limber's Formula*

Equation (5) relates $u_l$ to the angular covariance function $w(\theta)$, so equation (34) is equivalent to an equation for $w$ in terms of $\xi$. Limber (1953) wrote down the general equation relating $w$ and $\xi$ under assumptions equivalent to the present ones but with the extra assumption that the density of objects could be approximated as a continuous function. This is reasonable for the problem he was considering, the analysis of the Shane-Wirtanen catalog of galaxies, but should be modified for other catalogs. As it happens, all that is required in the discussion of the angular covariance function is the reinterpretation of symbols. For the power spectrum, the transition to the point picture adds the term $\mathcal{N}$ to $Z_l$.

In this section, Limber's formula is rederived in the point picture, and the equivalence to the integral equation (34) is demonstrated.

Suppose an object is chosen at random from the ensemble of catalogs. The probability that the object is at distance in the range $s$ to $s + ds$ is (eqs. [32], [33])

$$dP = \phi(s/D^*)s^2ds/(D^{*3}E) . \tag{35}$$

Then by equation (1), the probability that an object is found at angular distance $\theta$ from the chosen object, in the solid angle $\delta\Omega$, is

$$\delta P = \int \frac{\phi(s/D^*)s^2ds}{D^{*3}E} \int t^2dt\delta\Omega n[1 + \xi(r)]\phi(t/D^*) . \tag{36}$$

In the numerator, $r$ is the distance between the two objects,

$$r^2 = s^2 + t^2 - 2st \cos \theta . \tag{37}$$

The integrand farthest to the right is the probability of finding an object in the volume element $t^2dt\delta\Omega$ multiplied by the probability $\phi$ that the object is counted in the catalog. Then Limber's formula follows with equations (2) and (33),

$$w(\theta) = \int s^2t^2dsdt\phi(s/D^*)\phi(t/D^*)\xi(r)/D^{*6}E^2 . \tag{38}$$

With equation (5), Limber's formula gives for the angular spectrum

$$u_l = 2\pi\mathcal{N} \int d\mu dsdts^2t^2P_l(\mu)\phi(s/D^*)\phi(t/D^*)\xi(r)/D^{*6}E^2 , \tag{39}$$

where $\mu = \cos \theta$. But $P_l$ may be written in terms of $I_l$ (eq. [C9]),

$$P_l(\mu) = \frac{4rst}{\pi} I_l(r, s, t) . \tag{40}$$

Here $r, s, t$, and $\mu = \cos \theta$ are related by equation (37), and $I_l$ vanishes when $|\mu| > 1$. On substituting this formula into equation (39), and changing the variable of integration from $\mu$ to $r$ (eq. [37]), one obtains equations (34).

## IV. SURVEY OF A FRACTION OF THE SKY

When the survey covers only a portion of the sky, as is almost inevitably the case, it is useful to subtract a mean value from the density of objects in the catalog (BT), and to renormalize the apparent spectrum. Also, the method of averaging the apparent spectrum over the azimuthal quantum number $m$ must be considered more carefully.

### a) *Convolution Theorem*

The angular positions of points in the catalog may be represented as

$$\sigma_1(\Omega) = W(\Omega)\sigma(\Omega)\,, \tag{41}$$

where $\sigma$ is a sum of two-dimensional Dirac functions representing the distribution over the whole sky. The window function $W$ vanishes outside the region of the survey, and is unity inside the region.

The spherical-harmonic transform of equation (41) is

$$b_l^m = \int \sigma_1 Y_l^m d\Omega = \sum_{l'm'} a_{l'}^{m'} W_{ll'}^{mm'}\,,$$

$$W_{ll'}^{mm'} = \int d\Omega\, Y_l^m Y_{l'}^{-m'} W(\Omega)\,. \tag{42}$$

The second equation follows from the completeness relation (eq. [A6]). The quantity $a_l^m$ is the sum of spherical harmonics evaluated at the angular positions of all objects, over all the sphere, while in $b_l^m$ the sum is restricted to the portion of the sphere covered by the survey.

Equation (42) is the analog of the convolution theorem for a Fourier transform. The sum over $l'$ and $m'$ makes the apparent spectrum $b$ a running average over the true spectrum $a$. This gives rise to a possible problem (BT): if the true spectrum has large differences in power at different values of $l$, then the convolution may seriously alter the shape of the apparent spectrum. Fortunately, knowing the shape of $W_{ll'}^{mm'}$ and of the apparent spectrum, one can hope to judge whether this is a serious problem. The convolution in the apparent power spectrum is discussed further in part $b$ of this section and in § VI*d*.

It is possible to make analytic estimates of the shape of $W_{ll'}^{mm'}$ for a chosen survey region, but it seems simpler to use direct numerical results. This point is therefore deferred to the discussion of analyses of catalogs.

### b) *Renormalization*

By equations (11), (31), (32), and (33),

$$\langle a_l^m \rangle = (4\pi)^{1/2} \mathcal{N} \delta_{l,0}\,. \tag{43}$$

The running average (eq. [42]) introduces a systematic deviation in the apparent spectrum,

$$\langle b_l^m \rangle = \mathcal{N} I_l^m\,, \qquad I_l^m = \int_{\Delta\Omega} d\Omega\, Y_l^m\,, \tag{44}$$

where the integral is over the area $\Delta\Omega$ of the survey. To eliminate this effect, one should subtract the expected mean. The apparent density of objects is

$$\mathcal{N}_a = \frac{N_a}{\Delta\Omega} = \frac{(4\pi)^{1/2} b_0^0}{\Delta\Omega}\,, \tag{45}$$

where $N_a$ is the number of objects in the particular catalog, giving the "corrected spectrum"

$$c_l^m = b_l^m - \mathcal{N}_a I_l^m = \sum_{l'm'} a_{l'}^{m'} K_{ll'}^{mm'} ,$$

$$K_{ll'}^{mm'} = W_{ll'}^{mm'} - I_l^m I_{l'}^{-m'}/\Delta\Omega . \tag{46}$$

To estimate the expected value of $|c|^2$, we need the result

$$\langle a_l^m a_{l'}^{-m'} \rangle = \delta_{l,l'} \delta_{m,m'} Z_l . \tag{47}$$

This equation follows from isotropy plus the fact that $a_l^m$ transforms under rotation of the coordinates in the same way as does $Y_l^m$ (eq. [B2]).

Equations (46) and (47) yield

$$\langle |c_l^m|^2 \rangle = \sum_{l'} Z_{l'} R_{ll'}^m J_l^m ,$$

$$R_{ll'}^m \equiv \sum_{m'} R_{ll'}^{mm'} , \qquad R_{ll'}^{mm'} \equiv |K_{ll'}^{mm'}|^2 / J_l^m ,$$

$$J_l^m \equiv \int_{\Delta\Omega} |Y_l^m|^2 d\Omega , \tag{48}$$

where the weight function satisfies (eq. [A6])

$$R_{l0}^m = 0 ,$$

$$\sum_{l'} R_{ll'}^m = 1 - |I_l^m|^2/(\Delta\Omega J_l^m) . \tag{49}$$

If the survey covers an appreciable part of the sky, then for $l > 0$ this sum is very close to unity. Thus we are led to define

$$Z_l^m \equiv |b_l^m - \mathcal{N}_a I_l^m|^2 / J_l^m , \tag{50}$$

and we see from equation (48) that the expected value of this apparent spectrum is a running average of the true spectrum,

$$\langle Z_l^m \rangle = \sum_{l'} R_{ll'}^m Z_{l'} . \tag{51}$$

Because $R_{l0}^m = 0$, the large term $Z_0$ does not affect this mean value. If, for $l > 0$, $Z_{l'}$ varies only slowly with $l'$, then it may be taken outside the sum, to get

$$\langle Z_l^m \rangle \simeq Z_l . \tag{52}$$

In this approximation, $Z_l^m$ is the desired apparent spectrum corrected for the mean density and suitably renormalized. Equation (50) is equivalent to the equation derived in YP.

The width of the spectral window function in equation (51) is inversely proportional to the angular extent $\Theta$ of the survey region. For structure in the distribution of objects with angular scale $\theta$, $Z_l$ changes appreciably over $\Delta l \sim \pi/\theta$. Thus if $\theta \ll \Theta$, it is a good approximation to take $Z_{l'}$ out of the sum. However, for structure on the scale of $\Theta$ the approximation is not good. Thus the apparent spectrum as estimated from $Z_l^m$ may be in error at small $l$-values.

### c) *Averaging Over m*

#### i) *Averaging Formula*

Since the expectation value of $Z_l^m$ is (approximately) $Z_l$, it is reasonable to average $Z_l^m$ over $m$. The decision on how to weight different $m$-values in the average might be based on an estimate of expected variance, but it seems preferable to base it on the following considerations. Generally, one expects that the catalog is complete above some fixed latitude. When $l \gg 1$ and $m \sim l$, the $Y_l^m$ are very small at high latitude, and increase going toward lower latitude, so that they give maximum weight to the objects at low latitude. The small values of the $Y_l^m$ are compensated by the small $J_l^m$ in equation (50), but because the weight is concentrated at lowest latitude it seems unwise to give the $Z_l^m$ equal weight with those belonging to smaller $m$-values. An averaging formula that reflects this thought is

$$(Z_l^m)_m \equiv \sum_m |b_l^m - \mathcal{N}_a I_l^m|^2 \Big/ \sum_m J_l^m . \tag{53}$$

This weights the $Z_l^m$ in proportion to $J_l^m$. By the addition theorem (eq. [A3]), the denominator of this expression is

$$\sum_m J_l^m = (2l + 1)\Delta\Omega/4\pi . \tag{54}$$

#### ii) *Model for Galactic Obscuration in the "Null Hypothesis"*

The averaging formula (53) has some attractive features in the following model for galactic obscuration. Suppose objects are distributed at random, with true mean density $\mathcal{N}_0$ per steradian, according to Poisson statistics. To describe the effects of galactic obscuration, it will be supposed that objects are included in the catalog with a probability $f(\Omega)$ that is a smooth function of position in the sky.

The mean number of objects in the catalog is

$$\mathcal{N}\Delta\Omega \equiv N \equiv \langle N_a \rangle = \mathcal{N}_0 \int f d\Omega . \tag{55}$$

In this model, expectation values are readily converted to integrals—for example,

$$\begin{aligned} \langle |b_l^m|^2 \rangle &= \left\langle \sum_i | Y_l^m(i)|^2 \right\rangle + \left\langle \sum_{j \neq i} Y_l^m(i) Y_l^{-m}(j) \right\rangle \\ &= \mathcal{N}_0 \int f | Y_l^m|^2 d\Omega + \mathcal{N}_0^2 \left| \int f Y_l^m d\Omega \right|^2 . \end{aligned} \tag{56}$$

One finds in this way that ensemble average of $|c_l^m|^2$ (eq. [46]) is

$$\begin{aligned} \langle |c_l^m|^2 \rangle &= \mathcal{N}_0 \int f \left| Y_l^m \right|^2 d\Omega + \mathcal{N}_0^2 \left| \int f Y_l^m d\Omega \right|^2 \\ &\quad - \frac{2(N + 1)\mathcal{N}_0}{\Delta\Omega} \mathrm{Re} \left( I_l^{-m} \int f Y_l^m d\Omega \right) + \frac{N(N + 1)}{\Delta\Omega^2} |I_l^m|^2 . \end{aligned} \tag{57}$$

For a given smooth function $f$ and large enough $l$, the integrals over $Y_l^m$ are much less than the integral over $| Y_l^m|^2$. Thus, for sufficiently large $l$, it is a good approximation to write

$$\langle |c_l^m|^2 \rangle \simeq \mathcal{N}_0 \int f | Y_l^m|^2 d\Omega , \tag{58}$$

which with equations (53) and (A3) gives

$$\langle (Z_l^m)_m \rangle \simeq \mathcal{N} = \langle N_a/\Delta\Omega \rangle . \tag{59}$$

In this approximation, *equation (53) automatically renormalizes the apparent spectrum to the apparent density of objects.* If we had averaged the $Z_l^m$ over $m$ with uniform weight, we would have had a systematic error in equation (59) in the present model for local obscuration. It might be noted that this does not violate equation (52), for the systematic gradient $f(\Omega)$ due to obscuration is a fixed quantity, not subject to the ensemble average assumed in that equation.

When the extra terms in equation (57) may not be ignored, we can take $N \gg 1$. Then equations (53), (57), and (A2) give

$$\begin{aligned} u'_l &\equiv \langle (Z_l^m)_m - N_a/\Delta\Omega \rangle / \mathcal{N} \\ &\simeq \frac{4\pi}{(2l+1)N} \sum_m |\mathcal{N}_0 \int f Y_l^m d\Omega - \mathcal{N} I_l^m|^2 . \end{aligned} \tag{60}$$

This equation is obtained also if one considers $b_l^m$ as the spherical harmonic transform of the *continuous* surface density function $\sigma(\Omega) = \mathcal{N}_0 f(\Omega)$.

Equation (60) illustrates how the shape of the obscuration function $f$ affects the apparent spectrum. If $f$ is quite smooth, the major contribution is at small $l$. If $f$ has irregularities on small angular scale, this shows up at larger $l$ values. The apparent spectrum satisfies the sum rule (eq. [A6])

$$\sum (2l+1)u'_l = 4\pi\mathcal{N}\left[\frac{\Delta\Omega \int f^2 d\Omega}{(\int f d\Omega)^2} - 1\right] . \tag{61}$$

The term in brackets is the mean square deviation of $f$ from the mean.

## V. PROPERTIES OF THE WEIGHT FUNCTION, SCALING, AND CLUSTERING

### *a) Scaling Law*

An important check of interpretation of the results of analysis is to establish whether $u_l$ and $w$ vary with the depth of the survey in the manner expected according to the fair-sample hypothesis. In the case of practical interest, $\theta$ is small and $l$ large, for one supposes that the structure under study is much smaller than the depth of the survey. Here the expected variation of $u$ and $w$ may be stated as simple scaling laws.

With the change of variables

$$x_1 = u + v/2 , \qquad x_2 = u - v/2 , \tag{62}$$

and equation (C9), equation (34) becomes

$$\begin{aligned} \Lambda_l(y) = (6yE^2)^{-1} \int dudv(u^2 - v^2/4) \\ \times \phi(u + v/2)\phi(u - v/2)P_l[1 - (y^2 - v^2)/(2u^2 - v^2/2)] , \end{aligned} \tag{63}$$

where the limits of integration are

$$|v| \leq y \leq 2u . \tag{64}$$

If the luminosity function has a "reasonable" form, so we can say that the typical member of the catalog is at a distance on the order of $D^*$, and if the coherence length [maximum value of $r$ for which $\xi(r)$ is significantly different from zero] is much less than $D^*$, then we only need to evaluate $\Lambda$ for $y \ll u$. In this case $|v| \ll u$, so we can simplify equation (63). Also, we observe that the argument of $P_l$ is close to unity, so we can use equation (A8). The result is

$$\Lambda_l(y) \simeq \frac{2}{3\pi y E^2} \int_0^\infty du \int_0^v dv \int_0^{\pi/2} d\psi u^2 \phi(u)^2 \cos\left(l \cos\psi (y^2 - v^2)^{1/2}/u\right). \quad (65)$$

With the variables change $lv = x$ we find

$$\Lambda_l(y) \simeq \Lambda(yl). \quad (66)$$

The accuracy of this law is illustrated in the next section for the case that the survey is to a fixed limiting distance. Equations (33), (34), and (66) yield the first of the desired scaling laws,

$$u_l \simeq u(l/D^*). \quad (67)$$

If in equation (5) the largest contribution to the integral is at $\theta \ll 1$, then with equation (A8) we can approximate the integral as

$$u_l \simeq \mathscr{N} \int \theta d\theta \int_0^{2\pi} d\psi w(\theta) \cos(l\theta \cos\psi). \quad (68)$$

In this small-angle approximation, $u_l$ is just the Fourier transform of the covariance function integrated along a strip. Equations (33), (67), and (68) give the second scaling law,

$$w(\theta, D^*) = D^{*-1} f(\theta D^*). \quad (69)$$

The main assumption behind equations (67) and (69) is that the dominant contribution to the apparent covariance function or spectrum at small $\theta$ or large $l$ is from structure much smaller than $D^*$. If the selection function (§ II*c*) has appreciable width, then it must be assumed also that $\Phi$ scales as indicated in equation (9).

### *b*) *Discussion of the Weight Function*

#### i) *Limiting Value*

In the limit $y \to 0$, equations (34) and (C10) give

$$\Lambda_l(0) = \frac{1}{3} \int x^2 dx \phi(x)^2 \bigg/ \left[\int x^2 dx \phi(x)\right]^2. \quad (70)$$

If the survey is to a fixed limiting distance $x = 1$, this equation reduces to

$$\Delta_l(0) = 1. \quad (71)$$

The normalization was chosen to yield this convenient value in this important limiting case. If the luminosity function has an appreciable spread, and if $D^*$ is chosen so that about half the points are at distance greater than $D^*$, then large values of $x$ are more significant in the denominator than numerator of equation (70), so generally $\Lambda_l(0)$ is less than unity. In effect, the spread in luminosity function reduces the apparent clustering in the sky.

ii) *Numerical Evaluation*

Equations (65) and (66) may be rewritten as

$$\Lambda(y) = \frac{1}{9E^2}\int_0^\infty d(s^3)\phi(s)^2\Upsilon(y/s)\,,$$

$$\Upsilon(y) = \frac{2}{\pi}\int_0^{\pi/2} d\psi \int_0^1 dt\cos\left(y(1-t^2)^{1/2}\cos\psi\right). \tag{72}$$

This is readily evaluated by Monte Carlo integration. The results are plotted in figure 1 for the case of a survey to fixed limiting distance ($\phi = 0$, $s > 1$; $\phi =$ const., $s < 1$). For comparison, $\Delta_l(y)$ (eqs. [30], [C9]) is plotted as a function of $ly$ (eq. [66]) for $l = 1, 2, 3$. Apparently, equation (66) is not a band approximation even for small $l$-values.

Detailed discussion of the shape of $\Lambda$ for a spread luminosity function is reserved for the discussions of individual catalogs, but one can make a general observation. The result of Monte Carlo integration of $\Upsilon$ (eq. [72]) is shown in figure 2. It is apparent that if $s^3\phi(s)^2$ has peak value near $s \sim 1$, then $\Lambda(y)$ has a shape similar to that shown in figure 1, with characteristic width $y \sim 1$, but with peak value $\lesssim 1$. Other choices for $\phi$ could yield more complicated forms, like a relatively narrow peak plus a broader shoulder in $\Lambda$.

### c) *Meaning of the Integral Equation: Cluster Model*

If the objects are distributed in the sky in randomly placed clusters of $n_c$ objects each, and if the clusters size is much less than $\pi D/l$, so that $Y_l^m$ does not "resolve" the cluster, then

$$Z_l \sim \mathcal{N} n_c\,, \qquad u_l \sim n_c - 1\,. \tag{73}$$

This is because there are in effect $4\pi\mathcal{N}/n_c$ independent points, each with weight $n_c$ in $a_l^m$ or $n_c^2$ in $Z_l$ (YP). The integral equation (5) is a more detailed statement of this same result. The Legendre polynomial $P_l(\cos\theta)$ is unity at $\theta = 0$, reaches its first zero at angle (in radians)

$$\theta_1 \simeq 2.40/(l + \tfrac{1}{2})\,, \tag{74}$$

and for larger angles oscillates with roughly constant amplitude $\sim l^{-1/2}$ and period $\Delta\theta \sim 2\pi/l$ (Jahnke and Emde 1945, pp. 116, 117). If the cluster size is much less than $\theta_1$, then we can take $P_l = 1$ in equation (5), to get

$$u_l \simeq \int \mathcal{N} w(\theta) 2\pi d\cos\theta\,. \tag{75}$$

But $\mathcal{N} w d\Omega$ is the probability above random for finding an object in the element $d\Omega$ of solid angle at distance $\theta$ from a randomly chosen object (eq. [2]). Thus equation (75) says that $u_l$ *is the mean number of objects in excess of random around a randomly chosen object in the sky*. This agrees with equation (73). If the clusters have angular extent greater than $\theta_1$, then we can interpret $u_l$ as the number of objects in excess of random and within angular distance $\sim\theta_1$ of the randomly chosen object.

A similar interpretation holds for the spatial covariance function. Suppose the survey is to a fixed limiting distance. Then the weight function has the form shown in figure 1, and the effect of the weight function in equation (6) is to cut the integral off at $r \sim D/l$, so that equation (6) may be approximated as

$$u_l \simeq \int_0^{D/l} 4\pi r^2 dr n\xi(r)\,. \tag{76}$$

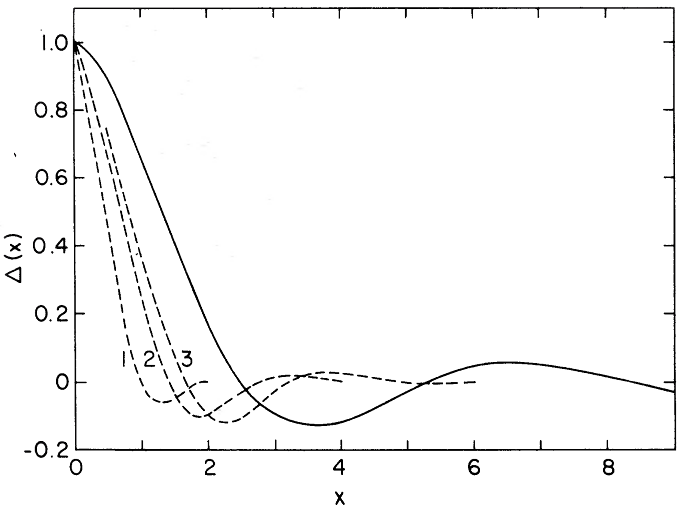


FIG. 1.—The weight function for a survey to a fixed limiting distance (eqs. [30], [66]). The dashed lines refer to harmonic index $l = 1, 2, 3$; and the solid line to the limit of large $l$ (eq. [72]).

As before, this is the average number of objects in excess of random within distance $D/l$ of a randomly chosen object. If $\xi$ goes to zero sufficiently rapidly at large $r$, then we have for small $l$ or large $D$,

$$n_c \equiv 1 + \int_0^\infty 4\pi r^2 n\xi(r) dr \simeq u_l + 1 \,, \tag{77}$$

as a measure of the "number of objects per cluster." One should bear in mind, however, that this interpretation holds in the sense of an average only. The power spectrum does not distinguish between a complete clustering model, as described here, and a model where a fraction $1 - f$ of the objects is randomly distributed, the remaining fraction in randomly distributed clusters of $(n_c - 1)/f + 1$ objects each.

If the survey is to fixed limiting apparent magnitude, and if there is an appreciable

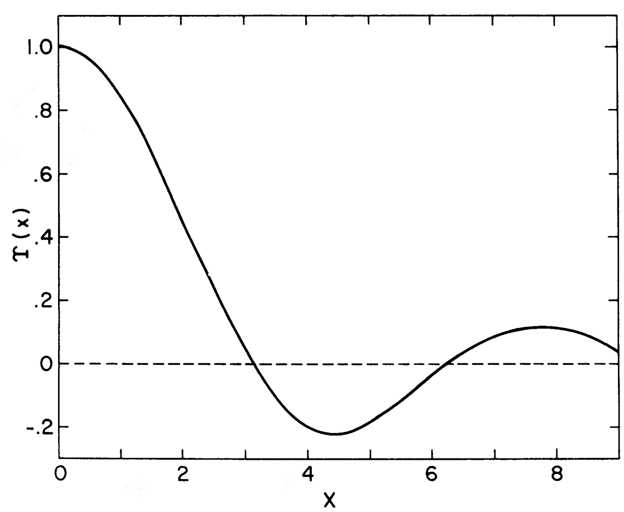


FIG. 2.—The second weight function (eq. [72])

spread in absolute magnitude, then there is the additional complication that similar clusters at different distances display different numbers of cluster members in the catalog. The interpretation of equation (75) as the mean apparent number of objects per cluster still applies, and the value $\Lambda_l(0)$ gives a rough estimate of the ratio of this apparent number to the true number (eq. [6]),

$$u_l \simeq \frac{3\mathcal{N}}{nD^{*3}}(n_c - 1)\Lambda_l(0)\,, \tag{78}$$

where this equation applies for $l$ small enough that the spherical harmonics do not resolve the clusters.

One can check equation (78) by directly computing $Z_l$ under the assumption that each cluster contains just $n_c$ members. For a cluster at distance $r$, the mean value for the number of members bright enough to be in the catalog is $\langle n_0 \rangle = n_c\phi(r/D^*)$, and the mean of the square of this number is $\langle n_0{}^2 \rangle = (n_c{}^2 - n_c)\phi^2 + n_c\phi$. The clusters are supposed to be distributed at random, the spatial density of clusters being $n/n_c$. Since each cluster acts like an independent point with average weight $\langle n_0{}^2 \rangle$, one finds

$$\begin{aligned} \langle |a_l{}^m|^2 \rangle &= \int r^2 dr (n/n_c)\{\phi^2(n_c{}^2 - n_c) + \phi n_c\} \\ &= \mathcal{N} + n(n_c - 1)\int r^2 dr \phi(r/D)^2\,. \end{aligned} \tag{79}$$

This result agrees with equations (4), (70), and (78).

## VI. ESTIMATES OF DISTRIBUTIONS AND STANDARD DEVIATIONS

The expected values of estimates of the power spectra and covariance functions have been discussed under fairly general assumptions. To estimate the expected statistical uncertainties in these estimates, one must introduce some new assumptions or more specific models for the distribution of objects. The simplest model is the hypothesis that the objects are distributed according to a random Poisson process. Slightly more general is the assumption that the objects are in many randomly placed clusters. To the desired approximation, this is consistent with the assumption that the objects are distributed according to a random Gaussian process.

### *a) Standard Deviation of $Z_l{}^m$ for a Random Poisson Process*

The computation goes along the lines indicated in equation (56), but with $f = 1$ in the solid angle $\Delta\Omega$ of the survey. The result for the "corrected spectrum" (eq. [46]), with the indices $l$ and $m$ dropped to shorten the expression, is

$$\langle |c|^2 \rangle = \mathcal{N}(J - |I|^2/\Delta\Omega)\,,$$

$$\begin{aligned} \langle |c|^4 \rangle - \langle |c|^2 \rangle^2 = \mathcal{N}^2 \bigg\{ & J^2 + \left| \int Y^2 d\Omega \right|^2 \\ & - \frac{2}{\Delta\Omega}\,\mathrm{Re}\left(I^{*2}\int Y^2 d\Omega\right) - \frac{2|I|^2 J}{\Delta\Omega} + \frac{2|I|^4}{\Delta\Omega^2}\bigg\} \\ + \mathcal{N}\bigg\{ & \int d\Omega |Y|^4 - \frac{4}{\Delta\Omega}\,\mathrm{Re}\left(I^*\int Y^* Y^2 d\Omega\right) \\ & + \frac{2}{\Delta\Omega^2}\,\mathrm{Re}\left(I^{*2}\int Y^2 d\Omega\right) + \frac{4|I|^2 J}{\Delta\Omega^2} - \frac{3|I|^4}{\Delta\Omega^3}\bigg\}\,. \end{aligned} \tag{80}$$

The first of these equations also follows from equations (48) and (49) with $Z_l = \mathcal{N}$. Both equations are greatly simplified if the survey covers an appreciable part of the sky, for then the integrals over odd powers of $Y$ are very small, and the integral over $Y^2$ is small unless $m = 0$ because of the oscillating factor $\exp(2im\phi)$. If, in addition, $\mathcal{N}$ is large, as generally would be the case, then the second of equations (80) may be approximated as

$$(\delta Z_l^m)^2 \equiv \langle (Z_l^m)^2 \rangle - \langle Z_l^m \rangle^2 \simeq \mathcal{N}^2(1 + \delta_{m,0}) . \tag{81}$$

In this model, the standard deviation of $Z_l^m$ (eq. [50]) is equal to the expected value of $Z_l^m$ if $m \neq 0$, $2^{1/2}$ times this value if $m = 0$.

### *b*) *Frequency Distribution of $Z_l^m$ in a Cluster Model*

If the objects in the catalog are distributed in a *large number* of *independently placed clusters*, then $c_l^m$ (eq. [46]) is the sum of a large number of statistically independent terms (representing the different independent clusters), each with zero ensemble average value. It follows (YP) that if $m \neq 0$ the real and imaginary parts of $c_l^m$ have Gaussian frequency distributions, and hence that $|c_l^m|^2$ has an exponential frequency distribution. The width of the distribution in $Z_l^m$ (eq. [50]) is fixed by equation (52). Thus the probability that a chosen $Z_l^m$ has value greater than $x$ is, for $m \neq 0$ (YP),

$$P(Z_l^m > x) \simeq \exp(-x/Z_l) . \tag{82}$$

This agrees with equation (81) for $m \neq 0$ and $Z_l = \mathcal{N}$. The standard deviation of $Z_l^0$ is larger by the factor $2^{1/2}$ because $c_l^0$ has no imaginary part.

A second consequence of the present model is that the spectral components $|a_l^m|^2$ are statistically independent to a good approximation. The desired quantity is

$$\langle |a_l^m|^2 |a_{l'}^{m'}|^2 \rangle = \sum \langle Y_l^m(\Omega_i) Y_l^{-m}(\Omega_j) Y_{l'}^{m'}(\Omega_k) Y_{l'}^{-m'}(\Omega_l) \rangle ,$$
$$(l, m) \neq (l', m') ; \qquad m, m' \geq 0 . \tag{83}$$

When objects $i$ and $j$ are in clusters different from the clusters of $k$ and $l$, the first two factors in the right-hand side of this equation are by assumption independent of the second two. The correlation between $|a|^2$ and $|a'|^2$ thus depends on the number of terms where objects in the second pair of factors are in the same cluster as objects in the first pair. By orthogonality, the only such terms with nonzero average are those with points $i$, $j$, $k$, and $l$ all in the same cluster. On taking $1 + u_l$ as a measure of the number of points per cluster within angular spread $\sim \pi/l \sim$ the "resolution" of the spherical harmonics, we have the correlation value

$$\frac{\langle |a_l^m|^2 |a_{l'}^{m'}|^2 \rangle}{\langle |a_l^m|^2 \rangle \langle |a_{l'}^{m'}|^2 \rangle} - 1 \simeq \frac{(1 + u_l)}{\mathcal{N}} \left[ \int |Y_l^m|^2 |Y_{l'}^{m'}|^2 d\Omega - 1/4\pi \right] \ll 1 ,$$
$$(l, m) \neq (l', m') ; \qquad m, m' \geq 0 . \tag{84}$$

As indicated, if $1 + u_l \lesssim n_c \ll \mathcal{N}$, as would be the case if there were many clusters, then the right-hand side is small, and we have the desired small correlation between $|a|^2$ and $|a'|^2$.

If the survey covers only a portion of the sky, then the apparent spectrum $b$ is a running average of $a$ (eq. [42]), so the $Z_l^m$ are not statistically independent. However, if the survey area covers a spherical cap in the sky (or any axisymmetric region), then $W_{ll'}^{mm'} \propto \delta_{mm'}$, and the $Z_l^m$ for different $m$-values but the same $l$ are to a good approximation statistically independent (in this model).

### *c*) *Random-Phase Assumption*

A statistical picture approximately equivalent to the preceding one may be based on the assumption that the large irregularities seen now developed from small density irregularities in the very early universe. It would seem natural to assume that these early density irregularities could be described as a random stationary process. If the process simply added a random contribution to the density at each point, the result would have been white noise, a flat power spectrum with randomly assigned phases. If the process displaced each element of matter by a small distance in a randomly chosen direction, the result would have been a power spectrum $\propto k^2$, but again with random phases. Neither of these processes is local in the sense that the effect could have been achieved by causal processes (operating within the Hubble length $ct$), but one does have the impression that at least neither is a particularly contrived or special situation. Extending this, the thought is that, whatever the (unknown) process that fixed the initial power spectrum, the least special assumption in the face of ignorance is the random-phase assumption (RPA). Thus, in a representation of the density irregularities in the form of equation (11), each Fourier component has randomly assigned phase. In the linear perturbation approximation, the expansion of the universe preserves the random-phase character. It might be a reasonable guess, therefore, to suppose that the RPA is valid now at sufficiently large wavelength.

Under the RPA, the spherical-harmonic transform of the observed distribution of objects may be written as (eqs. [11], [31], [A5])

$$a_l^m = \sum_k A_l^m(k)\, Y_l^m(\Omega_k) \exp{(i\phi_k)}\,, \tag{85}$$

where the sum is over the contribution of each Fourier component, $\phi_k$ being the phase. This is a sum of statistically independent terms, each with zero mean, leading as before to a Gaussian distribution of real and imaginary parts and an exponential distribution of $|a_l^m|^2$ if $m \neq 0$. The latter steps still apply if $a$ is convolved over the transform of a window function to obtain $b$.

In the RPA, the expectation value $\langle |a|^2 |a'|^2 \rangle$ involves a sum over four $\boldsymbol{k}$, but in the ensemble average the only nonzero terms are those in which pairs of $\boldsymbol{k}$ are equal so that the phases cancel. The orthogonality of the spherical harmonics eliminates two of the sets of pairs of $\boldsymbol{k}$, yielding equation (84).

The basic results (82) and (84) apply under "appropriate conditions"—if the objects are distributed at random (and $\mathscr{N} \gg 1$), or in many randomly placed groups, or if the RPA applies. The following sections give some consequences of these basic results.

### *d*) *Standard Deviation of the Covariance Function*

It will be assumed here that the survey covers the whole sky. $w(\theta)$ is estimated by counting the total number $\delta N$ of pairs of objects with angular separation in the range $\theta_a \leq \theta_{ij} \leq \theta_b$. This number, with each pair counted twice, may be written as (eq. [18])

$$\delta N = 8\pi^2 \int_{\theta_a}^{\theta_b} K(\theta) d\cos\theta\,, \tag{86}$$

and an estimate of $w$ is (eqs. [2], [45])

$$w_a = \delta N/(N_a \mathscr{N}_a \delta\Omega) - 1\,, \qquad \delta\Omega = 2\pi(\cos\theta_b - \cos\theta_a)\,. \tag{87}$$

$K$ is related to the spectrum by equation (20), which with equation (A7) gives

$$w_a = \frac{2\pi}{N_a \mathscr{N}_a \delta\Omega} \sum_{l>0} |a_l^m|^2 \int_{\theta_a}^{\theta_b} P_l d\cos\theta\,. \tag{88}$$

Consider first the expected value of $w_a$. If $N_a \gg 1$, it is reasonable to take $N_a \mathscr{N}_a$ as a fixed number. Then equations (3), (4), and (A7) give

$$\langle w_a \rangle = \frac{2\pi}{N\delta\Omega} \sum_{l>0} (2l + 1)u_l \int_{\theta_a}^{\theta_b} P_l d\cos\theta - \frac{1}{N} \cdot \tag{89}$$

The last term arises because $w$ is normalized to $N_a/4\pi$ objects per steradian; but as one object always is isolated as the starting point, there really are available only $N_a - 1$ objects. Ignoring this small zero shift, we obtain from equations (5) and (89) the reasonable result

$$\langle w_a \rangle \simeq \frac{2\pi}{\delta\Omega} \int_{\theta_a}^{\theta_b} w(\theta) d\cos\theta \,. \tag{90}$$

The expected standard deviation of $w_a$ can be estimated in the cluster model (§ VI*b*). Recalling that negative $m$-values should not be counted separately, we obtain

$$\begin{aligned}(\delta w_a)^2 &\equiv \langle w_a{}^2 \rangle - \langle w_a \rangle^2 \\ &= \left(\frac{4\pi}{N\delta\Omega}\right)^2 \sum_{l>0} (1 + u_l)^2 (l + \tfrac{1}{2}) \int_{\theta_a}^{\theta_b} P_l(\mu) P_l(\mu') d\mu d\mu' \,.\end{aligned} \tag{91}$$

If $u_l$ is constant, and $\theta_b - \theta_a \ll \theta_a \ll 1$, the largest contribution to the sum comes at

$$l' \sim \pi/(\theta_b - \theta_a) \,. \tag{92}$$

If $u_l$ does not vary too rapidly with $l$, it may be a good approximation to take it out of the sum, replacing it with $u_{l'}$. Then we can use equation (A7) to obtain

$$(\delta w_a)^2 \simeq \frac{2(1 + u_{l'})^2}{N\mathscr{N}\delta\Omega} \simeq \frac{(1 + u_{l'})^2 (1 + w_a)}{(\delta N/2)} \,, \tag{93}$$

where $\delta N/2$ is the number of different pairs of objects (each pair counted once) that enter the estimate of $w_a$ (eq. [87]).

If the objects are distributed at random, $u_l = 0$, then each different pair with angular separation between $\theta_a$ and $\theta_b$ in effect is statistically independent, the expected standard deviation in the count of pairs being the square root of the count. If the points are correlated, $1 + u_{l'}$ is a measure of the mean number of points per cluster (§ V*c*). In agreement with this, the effective number of independent pairs in equation (93) is reduced by the square of $1 + u_{l'}$.

If the survey covers only a portion of the sky, then we have at our disposal only the fraction of pairs that fall within the survey area. With fewer pairs, the standard deviation in $w_a$ is larger, but it is reasonable to expect that we can still use equation (93) if we replace $\delta N/2$ with the number of different pairs actually used in the estimate of $w_a$.

### *e*) *Correlation in the Apparent Spectrum*

When the survey covers only part of the sky, the apparent spectrum is a running average of $a_l^m$, so neighboring values of $Z_l^m$ are correlated. In the RPA, the spectrum $a_l^m$ derived from the whole sky satisfies

$$\begin{aligned}\langle a_l^m a_{l'}^{m'} a_\lambda^\mu a_{\lambda'}^{\mu'} \rangle = {} & \delta_{l,l'} \delta_{m,-m'} \delta_{\lambda,\lambda'} \delta_{\mu,-\mu'} Z_l Z_\lambda \\ & + \delta_{l,\lambda} \delta_{m,-\mu} \delta_{l',\lambda'} \delta_{m',-\mu'} Z_l Z_{l'} \\ & + \delta_{l,\lambda'} \delta_{m,-\mu'} \delta_{l',\lambda} \delta_{m',-\mu} Z_l Z_{l'} \,.\end{aligned} \tag{94}$$

This equation is obtained by using equations (11) and (31) to represent the product on the left-hand side as a sum over Fourier components, averaging over $H$ (eq. [32]), and then evaluating the angular integrals. In the RPA, the only nonvanishing terms in the sum over Fourier components are those in which the $\boldsymbol{k}$ indices are equal in pairs, resulting in equation (94).

Equations (46) and (94) give

$$\langle|c_l^m|^2|c_{l'}^{m'}|^2\rangle - \langle|c_l^m|^2\rangle\langle|c_{l'}^{m'}|^2\rangle = \left|\sum_{\lambda\mu} Z_\lambda K_{l\lambda}^{m\mu} K^*{}_{l'\lambda}^{m'\mu}\right|^2 + \left|\sum_{\lambda\mu} Z_\lambda K_{l\lambda}^{m\mu} K^*{}_{l'\lambda}^{-m'\mu}\right|^2 . \tag{95}$$

If the survey covers an appreciable part of the sky, the weight functions in this expression are small unless $l \sim l' \sim \lambda$. If in addition $Z_l$ is a smooth function of $l$, we can replace $Z_\lambda$ with $Z_l$ in the expression. Then equations (48), (50), and (A6) give

$$\langle Z_l^m Z_{l'}^{m'}\rangle - \langle Z_l^m\rangle\langle Z_{l'}^{m'}\rangle \simeq Z_l^2(R_{ll'}^{mm'} + R_{ll'}^{m,-m'})/J_{l'}^{m'} . \tag{96}$$

This is the expected covariance of the apparent spectrum in the RPA.

If $Z_l = \mathcal{N}$, equation (96) agrees with the term proportional to $\mathcal{N}^2$ in equation (80). This is expected, because the two pictures are equivalent in the limit of large numbers of objects.

It is interesting to consider how the expected statistical uncertainty in the apparent spectrum scales with the depth of the survey. If the goal is to study $\xi(r)$ at characteristic length $r$, then the relevant index in the spectrum is $l \sim D^*/r$. The apparent spectrum may be averaged over $m$ and over some fixed fraction of $l$. Since equation (96) fixes the correlation among different components of $Z_l^m$, the effective number of independent terms in this average varies as $\sim l^2 \propto D^{*2}$. Thus, the average spectrum is uncertain by an amount that varies as

$$\delta Z \propto D^{*-1} \propto \mathcal{N}^{-1/3} . \tag{97}$$

This varies with the number of objects more slowly than the familiar $\mathcal{N}^{-1/2}$, because the projection in the sky of groups at different depths reduces the apparent clustering.

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## APPENDIX A

## CONVENTIONS AND FORMULAE FOR THE SPHERICAL HARMONICS

The spherical harmonics are products of functions of the polar and azimuthal angles,

$$Y_l^m(\theta, \phi) = P'{}_l^m(\cos\theta)e^{im\phi} . \tag{A1}$$

When $l \gg 1$, $P'{}_l^m$ is very small for $\theta \lesssim \theta_0 = \sin^{-1} m/l$, and for $\theta > \theta_0$ has approximately regularly spaced zeros at intervals $\Delta\theta \sim \pi/l$ radians.

The normalization used here is the conventional one,

$$\int Y_l^m Y_{l'}^{-m'} d\Omega = \delta_{ll'}\delta_{mm'}\,, \tag{A2}$$

$\delta$ representing the Kronecker delta function. With this normalization, the *addition theorem* is (Edmonds 1957, p. 63)

$$\sum_{m=-l,l} Y_l^m(\Omega) Y_l^{-m}(\Omega') = \frac{2l+1}{4\pi} P_l(\cos\theta)\,. \tag{A3}$$

Here $P_l$ has the conventional normalization

$$P_l(1) = 1\,, \qquad \int_{-1}^{+1} P_l^2(\mu) d\mu = 2/(2l+1)\,, \tag{A4}$$

and $\theta$ is the angle between the directions $\Omega$ and $\Omega'$.

The *expansion of a plane wave in spherical waves* is (Edmonds 1957, p. 80)

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = 4\pi \sum_{l,m} i^l j_l(kr) Y_l^m(\Omega_r) Y_l^{-m}(\Omega_k)\,. \tag{A5}$$

$\Omega_r$ and $\Omega_k$ point in the direction of $\boldsymbol{r}$ and $\boldsymbol{k}$. The expansion valid when $\boldsymbol{k}$ points along the polar axis is standard, and may be transformed to the form of equation (A5) with the help of the addition theorem. The function $j_l$ is the spherical Bessel function (Edmonds 1957).

A (suitably well-behaved) function of angles may be expanded as a sum of spherical harmonics. This plus the normalization choice (eq. [A2]) may be expressed as the *completeness relation*:

$$\sum_{l,m} Y_l^m(\Omega) Y_l^{-m}(\Omega') = \delta(\Omega, \Omega')\,. \tag{A6}$$

The sum is over all $l$ and $m$, and $\delta$ is the two-dimensional Dirac delta function on a sphere. The corresponding formula for the Legendre polynomials is

$$\sum \frac{2l+1}{2} P_l(\mu_1) P_l(\mu_2) = \delta(\mu_1 - \mu_2)\,. \tag{A7}$$

A convenient approximation to $P_l$ when $\theta \ll 1$ is (Jahnke and Emde 1945, p. 116)

$$P_l(1 - \theta^2/2) \simeq \frac{2}{\pi}\int_0^{\pi/2} d\psi \cos\left[(l + \tfrac{1}{2})\theta \cos\psi\right]. \tag{A8}$$

## APPENDIX B

## ISOTROPY AND ROTATIONS

In computing the angular power spectrum, one is invited to evaluate quantities like

$$Y_l^m(\Omega_1) Y_{l'}^{-m'}(\Omega_2)\,, \tag{B1}$$

where the indices in the two factors in some cases may be different. According to the isotropy assumption, expected values should be averaged over a random choice of the orientation of coordinates. The result of averaging the expression (B1) can only depend on the angle $\theta_{12}$ between the two directions $\Omega_1$ and $\Omega_2$,

$$\langle Y_l^m(\Omega_1) Y_{l'}^{-m'}(\Omega_2)\rangle = \delta_{ll'}\delta_{mm'}P_l(\cos\theta_{12})/4\pi\,. \tag{B2}$$

The following is a derivation of this equation.

Starting with a given direction specified by polar angles $\theta$, $\phi$ with respect to a chosen coordinate system, we can generate a new direction by rotating the coordinate axes about the $z$-axis through angle $\hat{\phi}$ and then about the new $y$-axis by $\hat{\theta}$. The new direction specified by the old quantities $\theta$, $\phi$ in the new coordinate system has coordinates $\theta'$, $\phi'$ relative to the original coordinates. The spherical harmonics (in the original coordinates) evaluated at the two directions are related by the equation (Edmonds 1957, p. 54)

$$Y_l^m(\theta', \phi') = \sum_{m''} C_l^{mm''}(\hat{\theta}, \hat{\phi}) Y_l^{m''}(\theta, \phi)\,, \tag{B3}$$

where the rotation matrix satisfies

$$Y_l^m(\theta, \phi) = C_l^{m0}(\theta, \phi) Y_l^0(0, 0)\,. \tag{B4}$$

In evaluating the average of the term (B1), we can start with coordinates chosen so that $\Omega_2$ is along the $z$-axis, and $\Omega_1$ is in the $(x, z)$-plane. Then we can average the expression over a random orientation by the following steps: (1) rotate through a randomly chosen angle $\phi$ about the $z$-axis; (2) rotate the polar axis to a randomly chosen part of the sky, $\hat{\theta}$, $\hat{\phi}$, as in equation (B3). The result is

$$\langle Y_l^m(\Omega_1) Y_{l'}^{-m'}(\Omega_2)\rangle = \int \frac{d\phi}{2\pi}\frac{\sin\hat{\theta}d\hat{\theta}d\hat{\phi}}{4\pi} \times \sum_{m''} C_l^{mm''}(\hat{\theta}, \hat{\phi}) Y_l^{m''}(\theta_{12}, \phi) Y_{l'}^{-m'}(\hat{\theta}, \hat{\phi})\,. \tag{B5}$$

Equation (B2) follows on evaluating the integral over $\phi$ and using equations (A2) and (B4).

## APPENDIX C

## INTEGRAL IDENTITIES

The purpose of this Appendix is to derive two integral identities for spherical Bessel functions. If in the formula

$$(2\pi)^3\delta(\boldsymbol{r} - \boldsymbol{r}') = \int d^3k \exp[i\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')] \tag{C1}$$

one substitutes the spherical wave expansion for the plane waves $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$ and $\exp(i\boldsymbol{k}\cdot\boldsymbol{r}')$ (eq. [A5]) and makes use of equation (A2), one finds

$$\sum_{l,m}\int k^2dkj_l(kr)j_l(kr')Y_l^m(\Omega_r)Y_l^{-m}(\Omega_{r'}) = \tfrac{1}{2}\pi\delta(\boldsymbol{r} - \boldsymbol{r}')\,. \tag{C2}$$

The result of multiplying this equation by $Y_l^m(\Omega_{r'})$ and integrating over $\Omega_{r'}$ is

$$\int_0^\infty k^2dkj_l(kr)j_l(kr') = \tfrac{1}{2}\pi\frac{\delta(r - r')}{r^2}\,, \tag{C3}$$

where we have used

$$\delta^3(\boldsymbol{r} - \boldsymbol{r}') = \delta(\Omega, \Omega')\delta(r - r')/r^2 . \tag{C4}$$

Next consider the integral over the product of three Bessel functions (eq. [30]). Since (eq. [17])

$$j_0(ky) = \int \frac{d\Omega}{4\pi} \exp(i\boldsymbol{k}\cdot\boldsymbol{y}) , \tag{C5}$$

equations (C1) and (C4) give

$$\int d^3k j_0(ky) \exp[i\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')]$$

$$= 2\pi^2 \int d\Omega_y \delta^3(\boldsymbol{y} + \boldsymbol{r} - \boldsymbol{r}') = \frac{2\pi^2}{y^2} \delta(y - |\boldsymbol{r} - \boldsymbol{r}'|) . \tag{C6}$$

When the plane-wave expansion (A5) is substituted into the left-hand side of this equation, one finds

$$\sum_{l,m} I_l(y, r, r') Y_l^m(\Omega_r) Y_l^{-m}(\Omega_{r'}) = \delta(y - |\boldsymbol{r} - \boldsymbol{r}'|)/8y^2 , \tag{C7}$$

where $I_l$ is defined by equation (30). The addition theorem gives

$$\sum \frac{2l + 1}{4\pi} I_l(y, r, r')P_l(\mu) = \frac{\delta[y - (r^2 + r'^2 - 2rr'\mu)^{1/2}]}{8y^2} , \tag{C8}$$

where $\mu$ is the cosine of the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. On multiplying this equation by a Legendre polynomial and integrating over $\mu$, one finds

$$I_l(y, r, r') = \frac{\pi}{4yrr'} P_l[(r^2 + r'^2 - y^2)/2rr'] , \qquad |r - r'| \leq y \leq r + r' , \tag{C9}$$

and $I_l = 0$ outside these bounds. It will be noted that, by equation (C3),

$$I_l(0, r, r') = \pi\delta(r - r')/2rr' . \tag{C10}$$

## APPENDIX D

## ESTIMATING $w(\theta)$

The purpose of this appendix is to describe in more detail the estimate of $w(\theta)$ when the survey does not cover the whole sky, and to indicate the relation of this estimate to the apparent spectrum.

In equation (18) let $\sigma$ be a sum of delta functions representing the observed positions of $N_a$ objects in the survey area $\Delta\Omega$. Then

$$\delta N_p = 4\pi^2 \int_{\theta_a}^{\theta_b} K(\theta) d\cos\theta \tag{D1}$$

is the number of different pairs with angular separation in the range

$$\theta_a \leq \theta_{ij} \leq \theta_b . \tag{D2}$$

The expected number of pairs for a random distribution of objects is estimated by setting

$$\sigma(\Omega) = \mathcal{N}_a W(\Omega)\,, \qquad \mathcal{N}_a = N_a/\Delta\Omega\,, \tag{D3}$$

in equations (18) and (D1). Here $W = 1$ inside the survey region and $W = 0$ elsewhere. An estimate of $1 + w$ is the ratio of the observed to expected number of pairs in the range of equation (D2). It will be convenient to write

$$1 + w'(\theta) = K(\theta)/K_n(\theta)\,, \tag{D4}$$

where (eqs. [18], [19], [20], [D3])

$$\frac{8\pi^2 K_n(\theta)}{\mathcal{N}_a^{\,2}} = 2\pi \sum_{l,m} |I_l^m|^2 P_l(\cos\theta) = L(\theta)\,,$$

$$L(\theta) = \int_{\Delta\Omega} d\Omega_1 d\Omega_2 \delta(\cos\theta_{12} - \cos\theta)\,. \tag{D5}$$

$K_n$ is proportional to the expected number of pairs per increment of $\cos\theta$ for a random distribution. Since $K_n$ is a smooth function of $\theta$, an actual estimate of $w$ would very nearly correspond to smoothing equation (D4) over a small range of $\cos\theta$. Equation (D4) is convenient here because it is easy to compute its expected value.

Equations (19), (20), (46), (D4), and (D5) yield

$$\mathcal{N}_a^{\,2} L(\theta) w'(\theta) = 2\pi \sum_{l,m} [|c_l^m|^2 + 2\mathcal{N}_a \operatorname{Re} I_l^{-m} c_l^m] P_l(\cos\theta)\,. \tag{D6}$$

The expected value of the first term on the right-hand side of this expression is given by equation (48), and the expected value of the second term may be computed in a similar way. After some manipulation, one finds

$$\langle \mathcal{N}_a^{\,2} w'(\theta)\rangle = \mathcal{N}^2\left[w(\theta) - \int_{\Delta\Omega} \frac{d\Omega_1 d\Omega_2}{\Delta\Omega^2} w(\theta_{12})\right]\cdot \tag{D7}$$

The second term on the right-hand side is a zero error. This effect aside, equation (D7) shows that $w'$ is a reasonable estimate.

Finally, let us consider the relation between the apparent spectrum and the covariance function. In estimating the spectrum a convenient approximation is to neglect the variation of the true spectrum across the spectral window function. However, if the distribution has appreciable structure on angular scales comparable to the size of the survey region, then the estimate of the spectrum on this assumption is in error at small $l$. This causes a systematic error in the value of $w$ estimated from the transform of the spectrum (eq. [5]).

The estimate of the spectrum is

$$Z_l'' = 4\pi \sum_m |c_l^m|^2/(2l + 1)\Delta\Omega\,, \qquad l > 0\,, \tag{D8}$$

and this quantity with equation (5) gives an estimate $w''(\theta)$ for the covariance function. The expected value for this estimate may be computed in the same way as for equation (D7), with the result

$$\begin{aligned}\langle w''(\theta)\mathcal{N}_a^{\,2} - \mathcal{N}_a/4\pi\rangle &= \frac{\mathcal{N}^2}{2\pi\Delta\Omega}\left[w(\theta)L(\theta) + L(\theta)\int_{\Delta\Omega} \frac{d\Omega_1 d\Omega_2 w(\theta_{12})}{\Delta\Omega^2}\right.\\ &\quad \left. - 2\int_{\Delta\Omega} \frac{d\Omega_1 d\Omega_2 d\Omega_3}{\Delta\Omega} w(\theta_{13})\delta(\cos\theta_{12} - \cos\theta)\right]\cdot\end{aligned} \tag{D9}$$

If $\theta$ is small compared to the survey area, then $L(\theta) \simeq 2\pi\Delta\Omega$, and the right-hand side of this expression reduces to a sensible estimate of the covariance function, but with a nearly constant zero error. For larger $\theta$, $L(\theta)$ is less than $2\pi\Delta\Omega$, decreasing approximately linearly with increasing $\theta$. If the survey covers the area above latitude $b = +40°$, then the error in $w''$ arising from the $L(\theta)$ factor is 15 percent at $\theta = 10°$.

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