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# THE RATE OF SEPARATION OF MAGNETIC LINES OF FORCE IN A RANDOM MAGNETIC FIELD

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## ABSTRACT

The mixing of magnetic lines of force, as represented by their rate of separation, as a function of distance along the magnetic field, is considered with emphasis on neighboring lines of force. This effect is particularly important in understanding the transport of charged particles perpendicular to the average magnetic field. The calculation is carried out in the approximation that the separation changes by an amount small compared with the correlation scale *normal* to the field, in a distance *along* the field of a few correlation scales. It is found that the rate of separation is very sensitive to the precise form of the power spectrum. For a power-law spectrum $\propto k^{-\alpha}$ with $\alpha < 2$, the rate of separation can be quite rapid, and is insensitive to the initial value of the separation. Application to the interplanetary and interstellar magnetic fields is discussed, and it is shown that in some cases field lines, much closer together than the correlation scale, separate at a rate which is effectively as rapid as if they were many correlation lengths apart. In such cases the scale of coherent flux tubes is much less than the correlation scale of the field.

*Subject headings:* hydromagnetics — interplanetary medium — interstellar medium

## I. INTRODUCTION

One of the important concepts in the study of random or turbulent magnetic fields is the concept of field-line random walk. By this is meant the stochastic meandering of a magnetic line of force brought about by turbulent motions of the medium. The concept is particularly important in understanding the transport of charged particles in the irregular magnetic fields of space (see e.g., Jokipii 1966; Jokipii and Parker 1968, 1969; Jones 1971). One finds that in many cases of interest the random walk of magnetic lines of force dominates the transport of charged particles normal to the magnetic lines of force. Indeed, in the solar wind one finds that $\kappa_\perp/\kappa_\parallel$ may *increase* with decreasing energy down to $\sim 20$ MeV for protons, where $\kappa_\perp/\kappa_\parallel$ is perhaps as large as 0.2–0.4 (Jokipii 1972). But at these low energies the particle cyclotron radius is less than $\sim 2 \times 10^{10}$ cm, which is an order of magnitude smaller than the $\sim 2 \times 10^{11}$ cm coherence length of the magnetic fluctuations. Thus, it is of importance to consider the behavior of magnetic lines of force on scales small compared with the coherence length.

The meandering of the lines of force contributes to $\kappa_\perp$ in that a particle moving along a line of force in general also moves normal to the average field. This gives rise to a transport normal to the average field which contributes to $\kappa_\perp$. If, when a particle is scattered in pitch angle, it finds itself on a new, independent line of force, then the field-line random walk contributes fully to $\kappa_\perp$ and one has the standard result quoted above. This will be true, for example, if the particle cyclotron radius in the average field is of the same order as the coherence-scale normal to the average field, $L_{c\perp}$, or larger. But one readily sees that if the average cyclotron radius is much smaller than $L_{c\perp}$, the contribution of the random walk to $\kappa_\perp$ depends critically on the rate at which *nearby* field lines separate, and this is sensitive to the form of the magnetic-field power spectrum. If, for example, field lines initially $\sim r_c$ apart separate to $\sim L_c$ apart



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1973ApJ...183.1029J



in one pitch-angle scattering mean free path or less, clearly the random walk will contribute fully to $\kappa_\perp$.

In the following it is shown that reasonable forms for the power spectrum give rates of field-line separation which are quite large, so that $\kappa_\perp$ may be large for even low-rigidity particles. The results provide valuable insight into the concept of flux tubes and the general structure of random magnetic fields.

The organization of the rest of this paper is as follows. In the next section the mean-square rate of separation of field lines is calculated in terms of a magnetic-field power spectrum. In § III an appropriate Fokker-Planck equation is derived, and in §§ IV and V applications are discussed.

## II. FIELD-LINE SEPARATION

It is the purpose of this paper to study the problem of the rate of separation of two magnetic lines of force which are initially a distance $x_{20} - x_{10}$ apart, as a function of distance along the line of force.

Consider a spatially random magnetic field $\boldsymbol{B}(\boldsymbol{r}) = B_0\hat{e}_z + \boldsymbol{B}_1(\boldsymbol{r})$, with $\langle \boldsymbol{B}_1 \rangle = 0$ and $\langle \boldsymbol{B}_1{}^2 \rangle^{1/2} \ll B_0$. Define $L_{c_\perp}$ as the correlation scale of $\boldsymbol{B}$ normal to $z$, and $L_{c\parallel}$ as the scale parallel to $z$. The lines of force are the family of solutions to the differential equations

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dx}{B_z} . \tag{1}$$

Now, as shown by Jokipii and Parker (1969), to first order in $B_1/B_0$, one has

$$\Delta x = x(z) = x_0(z_0) = \frac{1}{B_0}\int_{z_0}^{z} B_{1x}(x_0, y_0, z')dz' . \tag{2}$$

It is readily deduced that $\langle \Delta x \rangle = 0$ and, if $R_{xx}(\xi)$ is the correlation function of $B_{1x}$,

$$\langle \Delta x^2 \rangle = \frac{2}{B_0{}^2}\int_0^{\Delta z} (\Delta z - z')R_{xx}(0, 0, z')dz' , \tag{3}$$

which in the limit that $\Delta z \gg L_{c_\perp}$ assumes the simple form

$$\langle \Delta x^2 \rangle = \frac{\Delta z}{B_0{}^2}\int_{-\infty}^{\infty} R_{xx}(0, 0, z')dz' . \tag{4}$$

It is clear that equation (4) is correct in the limit that $\langle \Delta x^2 \rangle^{1/2} \ll L_{c_\perp}$ for $\Delta z \sim L_{c\parallel}$. Consider next the quantity $x_2 - x_1$, the separation of two lines of force, where for the moment $y_{20} - y_{10} = 0$. From equation (2) we have to the lowest order

$$x_2(z) - x_1(z) = x_{20} - x_{10} + \frac{1}{B_0}\int_{z_0}^{z} [B_{1x}(x_{20}, y_0, z') - B_{1x}(x_{10}, y_0, z')]dz' . \tag{5}$$

Now, defining $x_2(z) - x_1(z) - (x_{20} - x_{10}) = \Delta(x_2 - x_1)$, we have the ensemble average

$$\langle \Delta(x_2 - x_1)^2 \rangle = \frac{4}{B_0{}^2}\int_0^{\Delta z} (\Delta z - z')[R_{xx}(0, 0, z') - R_{xx}(x_{20} - x_{10}, 0, z')]dz' . \tag{6}$$

If the increment $[\Delta(x_2 - x_1)^2]^{1/2}$ is small compared with $L_{c_\perp}$ for $\Delta z \sim L_{c\parallel}$, we have

$$\langle \Delta(x_2 - x_1)^2 \rangle = \frac{2\Delta z}{B_0{}^2}\int_{-\infty}^{\infty} [R_{xx}(0, 0, z') - R_{xx}(x_2 - x_1, 0, z')]dz' , \tag{7}$$

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1973ApJ...183.1029J



where $x_{20} - x_{10}$ has been replaced by $x_2 - x_1$, since the change in $(x_2 - x_1)$ is small. Equation (7) is a valid approximation provided that

$$\frac{2L_{c\parallel}}{B_0^2}\int_{-\infty}^{\infty}[R_{xx}(0, 0, z') - R_{xx}(x_2 - x_1, 0, z')]dz' \ll L_{c\perp}^2 .$$

In that which follows, it will be assumed that the magnetic fluctuations are such that this inequality is satisfied. One may conclude immediately from equations (3) and (6) that if $x_2 - x_1$ is large compared with the coherence scale of $B_x$ in the $x$ direction, $L_{c\perp}$, then

$$\langle\Delta(x_2 - x_1)^2\rangle/\Delta z = 2\langle\Delta x^2\rangle/\Delta z . \tag{8}$$

This is precisely what is expected on intuitive grounds since if $x_2 - x_1$ is large, the two field lines random-walk independently and the rate of separation grows, on the average, twice as fast as a single line random-walks.

Now consider the problem in the $(x, y)$-plane. Suppose that the field is statistically symmetric about the $z$-axis in which case, for a given value of $z$, the correlation tensor of $B_x$ and $B_y$ as a function of $x$ and $y$ must be of the form

$$R_{ij}^{\perp}(x, y) = \begin{pmatrix} ax^2 + b & axy \\ axy & ay^2 + b \end{pmatrix}, \tag{9}$$

where $a(r, z)$ and $b(r, z)$ are functions of $r^2 = (x^2 + y^2)$ and $z$. Then, defining the magnitude of the separation of the lines of force

$$\rho = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2},$$

and noting that $\langle\Delta(y_1 - y_2)^2\rangle$ and $\langle\Delta(x_1 - x_2)\Delta(y_1 - y_2)\rangle$ are given by equations similar to equation (7) with $R_{yy}$ and $R_{xy}$, respectively, replacing $R_{xx}$, one has immediately

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = \frac{2}{B_0^2}\int_{-\infty}^{\infty}[b(0, z') - b(\rho, z') - a(\rho, z')\rho^2]dz' , \tag{10}$$

where the symmetry about the $z$-axis is explicitly displayed. Equation (10) is readily seen from examination of equation (9) to be equivalent to the form

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = \frac{2}{B_0^2}\int_{-\infty}^{\infty}[R_{xx}(x = 0, y = 0, z') - R_{xx}(x = \rho, y = 0, z')]dz' , \tag{10a}$$

which we will use below. Explicit evaluation of these coefficients for specific power spectra or correlation functions will be deferred to a later section. First we consider the probability distribution of $\rho$.

## III. A FOKKER-PLANCK EQUATION AND THE FOKKER-PLANCK COEFFICIENTS

In order to apply the above to the statistics of field-line separation, first consider the appropriate equation for $P(\rho, z)$, the probability distribution for the separation of two lines of force. Again following the discussion of Jokipii and Parker (1969), we assume $P$ satisfies a Fokker-Planck equation:[1]

$$\frac{\partial P}{\partial z} = \frac{1}{2}\frac{\partial^2}{\partial\rho^2}\left(\frac{\langle\Delta\rho^2\rangle}{\Delta z}P\right) - \frac{\partial}{\partial\rho}\left(\frac{\langle\Delta\rho\rangle}{\Delta z}P\right) . \tag{11}$$

[1] One may use a Fokker-Planck equation because, under the assumptions leading to equation (10), $\rho(z) - \rho(z_0)$ may be regarded as the sum of a large number of independent increments $\Delta\rho$.

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1973ApJ...183.1029J



In general, it is difficult to evaluate $\langle \Delta\rho \rangle / \Delta z$ by integrating equation (1). Instead it is more convenient to obtain $\langle \Delta\rho \rangle / \Delta z$ in terms of $\langle (\Delta\rho)^2 \rangle / \Delta z$ as follows: Clearly, at large $z$ the distribution becomes uniform over the $(x, y)$-plane and $\partial P / \partial z$ goes to zero. Hence $P \propto \rho$ must be the invariant distribution so that

$$\frac{1}{2} \frac{\partial^2}{\partial \rho^2} \left( \frac{\langle \Delta\rho^2 \rangle}{\Delta z} P \right) - \frac{\partial}{\partial \rho} \left( \frac{\langle \Delta\rho \rangle}{\Delta z} P \right) = 0 \tag{12}$$

if $P \propto \rho$. This yields immediately

$$\frac{\langle \Delta\rho \rangle}{\Delta z} = \frac{1}{2\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\langle \Delta\rho^2 \rangle}{\Delta z} \right) + \frac{C}{\rho}, \tag{13}$$

where $C$ is an integration constant which is evaluated as follows. First note that $\rho \langle (\Delta\rho)^2 \rangle / \Delta z \to 0$ as $\rho \to 0$, so that $C = -\frac{1}{2} [\langle (\Delta\rho)^2 \rangle / \Delta z]_{\rho \to 0}$. Then make use of the fact that for any continuous magnetic field two field lines with zero separation must not separate, so that $[\langle (\Delta\rho)^2 \rangle / \Delta z]_{\rho \to 0} = 0$. Another way of seeing this is to note from equation (7) that $\langle \Delta(x_2 - x_1)^2 \rangle$ and $\langle \Delta(y_2 - y_1)^2 \rangle$ go to zero as $x_2 - x_1$ and $y_2 - y_1$ go to zero. More explicitly, if $\boldsymbol{B}$ is continuous, the reader may expand $\boldsymbol{B}$ near $\rho = 0$ in a Taylor series and note that for $\rho$ small enough it is sufficient to keep only the first term. The question of just how small $\rho$ must be for this to be valid is subtle and will be discussed further below, in connection with specific correlation functions. It is sufficient here to note that for small enough $\rho$ this can be done if the field is continuous. Then one finds that $\Delta\rho$ is linear in $\rho$ and $\langle \Delta\rho^2 \rangle$ is proportional to $(\rho^2)$. Thus $C = 0$ and our Fokker-Planck equation becomes

$$\frac{\partial P}{\partial z} = \frac{1}{2} \frac{\partial}{\partial \rho} \left[ \frac{\langle (\Delta\rho)^2 \rangle}{\Delta z} \left( \frac{\partial P}{\partial \rho} - \frac{P}{\rho} \right) \right], \tag{14}$$

where $\langle (\Delta\rho)^2 \rangle / \Delta z$ is given in equation (10).

Equation (14) may be solved to find the distribution of probabilities, subject to given initial conditions. However, analytical solutions are difficult to obtain and recourse must usually be made to numerical techniques.

In the present paper it is of primary interest to determine the distance $z$ required for field lines, initially separated by a small fraction of $L_c$, to separate to $\sim L_c$ where they can be regarded as independent. One approach to answer this question is to perform a Monte Carlo calculation. One starts $\rho$ at an initial value $\rho_0$ and finds successive values of $\rho$ at points separated by steps $\Delta z_*$ from the relation

$$\rho_{n+1} = \rho_n \pm \left( \frac{\langle \Delta\rho^2 \rangle}{\Delta z} \Delta z_* \right)^{1/2} + \frac{\langle \Delta\rho \rangle}{\Delta z} \Delta z_*, \tag{15}$$

where the plus or minus sign is random. For example, the mean distance for $\rho$ to approach $L_c$, $\langle \Delta z \rangle_1$, can be determined for a given initial value $\rho_0 \ll L_c$, by performing this calculation a number of times on a computer, using a random-number generator to pick the plus or minus sign. Below, this approach is used to discuss the rate of separation of lines of force for specific power spectra.

## IV. EVALUATION OF THE COEFFICIENT $\langle (\Delta\rho)^2 \rangle / \Delta z$.

Now consider the evaluation of the Fokker-Planck coefficient for certain specific forms of the fluctuating magnetic field. Observations of the interplanetary magnetic field indicate that the *observed* power spectrum as a function of wavenumber $k$ is of the form $A/(k^2 + 1/L_c^2)^\alpha$ with $\alpha \sim 0.75$ and $L_c \sim 10^{11}$ cm (Jokipii 1971). This rather flat power-law spectrum extends down to very small scales (large $k$) less than $10^8$–$10^9$ cm.

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Probably it steepens or cuts off at some smaller scale. Now, this observed spectrum corresponds to a one-dimensional spectrum for wavenumbers parallel to the direction of the solar wind. The full three-dimensional form of the spectrum is not yet determined observationally.

Suppose, for purposes of illustration, that the power spectrum of the fluctuating field has a form corresponding to isotropic fluctuations (Batchelor 1960), so that one may write

$$P_{xx}(\mathbf{k}) = \int d^3r \exp(i\mathbf{k}\cdot\mathbf{r}) R_{xx}(\mathbf{r}) = G(|\mathbf{k}|)(k_y^2 + k_z^2), \tag{16}$$

with particular functional form $G(|\mathbf{k}|) = A[k^2 + (1/L_c)^2]^{-[(\alpha/2)+2]}$. Possible high wavenumber cutoffs to this spectrum will be discussed below. Thus the one-dimensional spectrum of $B_x$ which is observed on spacecraft is proportional to $k^{-\alpha}$ for $k \gg 1/L_c$. Observations require that $\alpha \sim 1.5$–2. The correlation scale of the fluctuations is $L_c$ both normal and parallel to the average field.[2]

Substituting equation (16) into equation (10) yields

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = \frac{1}{B_0^2}\frac{1}{2\pi^2}\int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_y P_{xx}(k_x, k_y, 0)[1 - \exp(ik_x\rho)]$$

$$= \frac{A}{2\pi B_0^2}\frac{1}{\Gamma(\alpha/2+2)}\left[\frac{\Gamma(\alpha/2)}{2}L_c^{\alpha} - \left(\frac{L_c\rho}{2}\right)^{\alpha/2} K_{\alpha/2}(\rho/L_c)\right], \tag{17}$$

where $K_\nu(z)$ is the Bessel function of an imaginary argument. It is more informative to normalize this to the random-walk coefficient $\langle\Delta x^2\rangle/\Delta z$ defined in equation (4). For the above spectrum, one finds

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = 2\frac{\langle\Delta x^2\rangle}{\Delta z}\left[1 - \frac{2}{\Gamma(\alpha/2)}\left(\frac{\rho}{2L_c}\right)^{\alpha/2} K_{\alpha/2}\left(\frac{\rho}{L_c}\right)\right], \tag{18}$$

which has the asymptotic limits (for $0 < \alpha < 2$)

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = 2\frac{\langle\Delta x^2\rangle}{\Delta z} \qquad \text{for } \frac{\rho}{L_c} \gg 1, \tag{19}$$

and

$$\frac{\langle(\Delta\rho)^2\rangle}{\Delta z} = 2\frac{\langle\Delta x^2\rangle}{\Delta z}\left(\frac{\rho}{2L_c}\right)^{\alpha}\frac{\Gamma(1-\alpha/2)}{\Gamma(1+\alpha/2)} \qquad \text{for } \frac{\rho}{L_c} \ll 1. \tag{20}$$

(Note that for $\alpha \geq 2$, the leading term in the $\rho/L_c \ll 1$ expansion for $\langle(\Delta\rho)^2\rangle/\Delta t$ is proportional to $\rho^2$.) Equations (18), (19), and (20) contain the desired result for the assumed (reasonable) spectrum. In figure 1 $\langle(\Delta\rho)^2\rangle/\Delta z$, normalized to the large-distance random walk $2\langle(\Delta x)^2\rangle/\Delta z$, is plotted as a function of $\rho$ for various values of $\alpha$.

As an illustration of the use of the coefficients defined in equations (18), (19), and (20), consider the problem outlined in the discussion of equation (15). We consider an initial value $\rho_0 = 0.1L_c$ and compute by a Monte Carlo technique the distance

[2] Note that for $\alpha < 2$ as is observed, the expectation value of $\partial B_x/\partial x$ is not finite. One has $\langle(\partial B_x/\partial x)^2\rangle = \int d^3k k_x^2 P_{xx}(k)$, and this expression diverges. For this reason a pure power-law spectrum such as given in equation (16) may not be physically acceptable. One may expect, therefore, that such a power-law spectrum has a cutoff at some high wavenumber $k_c$. The present discussion will be applicable for $\rho \gg 1/k_c$.

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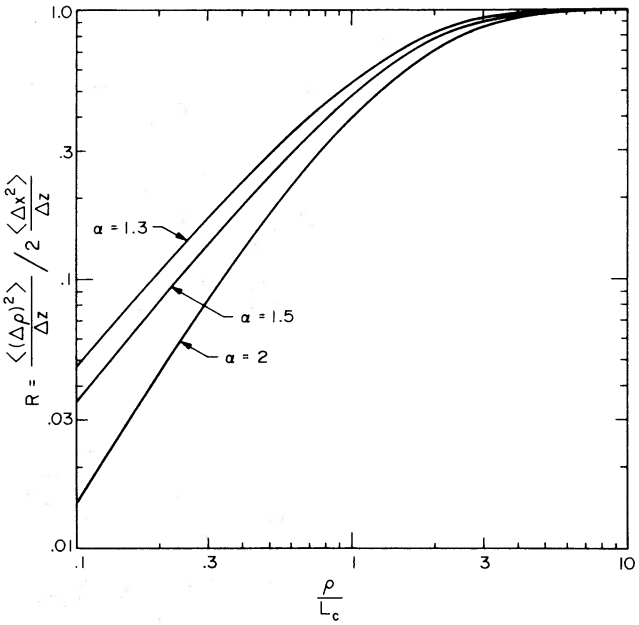



FIG. 1.—Plot of $\langle(\Delta\rho)^2\rangle/\Delta z$ normalized to the large-distance random walk $2\langle(\Delta x)^2\rangle/\Delta z$ as a function of $\rho$ for various values of $\alpha$.

along the average field required for $\rho$ to reach $\sim L_c$. Because $\rho < L_c$ is of prime interest in the problem, equation (20) was used for $\langle(\Delta\rho)^2\rangle/\Delta z$. For each value of $\alpha$, and for each fluctuating field amplitude, $\langle\Delta z\rangle_1$ was found from averaging the results of 100 runs. The values of $\langle\Delta z\rangle_1/L_c$ so found were plotted versus $\langle(\Delta x)^2\rangle/(\Delta z L_c)$, and a smooth curve was drawn through the data points. The results are displayed in figure 2.

Note the surprising result, that the effective rate of change of $\rho$ for field lines closer together than $L_c$ is essentially the same as for field lines separated by more than $L_c$. This may be seen by the following considerations. For $\alpha = 1.5$ and $\langle\Delta x^2\rangle/\Delta z \approx 0.05$

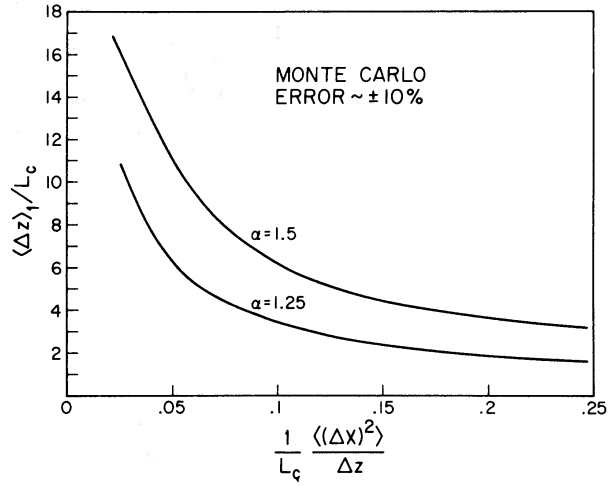



FIG. 2.—Mean distance $\langle\Delta z\rangle_1$ required for two lines of force initially $0.1L_c$ apart to random-walk to a separation of $\sim L_c$, as a function of the power spectral amplitude represented by $\langle(\Delta x)^2\rangle/\Delta z$, for various values of $\alpha$. The approximate expression (eq. [20]) was used in the calculations. Also, the theory breaks down if $\langle\Delta x^2\rangle/\Delta z$ is not small compared with unity.

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1973ApJ...183.1029J



field lines starting at $\rho_0 \sim 0.1L_c$ gained an amount $\sim L_c$ in separation a distance of only $\sim 12L_c$ along the field, whereas field lines which are many times $L_c$ apart change separation by an rms amount $\sim L_c$ in a distance of $10L_c$. There clearly is not much difference between $\rho \lesssim L_c$ and $\rho \gg L_c$.

Two further points should be made concerning the computer results. First, the values of $\langle \Delta z \rangle_1$ were found to be remarkably insensitive to the initial separation $\rho_0$. For example, if $\rho_0$ were 0.01 instead of 0.1, the curves would be shifted by less than 30 percent. Second, the values of $\langle \Delta z \rangle_1$ *are* critically sensitive to the slope of the power spectrum and increase rapidly as $\alpha$ approaches 2. This indicates that the rapid separation discussed here occurs only when the power spectrum is relatively flat, with a value of $\alpha$ somewhat less than 2.

Now consider briefly the question of a cutoff to the power-law spectrum discussed above. First, as noted in footnote 2, if $\alpha < 2$, the rms gradient of $\boldsymbol{B}_1$ is not finite in the absence of a cutoff. If one requires the rms gradient to be finite, either the spectrum must have a $\alpha > 2$ or there must be a cutoff. We have noted above that for $\alpha \gtrsim 2$, the rate of separation is slow. If the spectrum had $\alpha < 2$, but there were a cutoff at some wavenumber $k_c \gg 1/L_c$, then the above conclusions should go through for separations $\rho \gg 1/k_c$. For $\rho \lesssim 1/k_c$ there would be much less rapid separation.

We may explore this latter effect more fully by noting that for a value of $\alpha < 2$ with a cutoff at $k_c$, the expectation value of the gradient of $\boldsymbol{B}_1$ would be of order $\langle \boldsymbol{B}_1^2 \rangle^{1/2} k_c$, so that a Taylor series expansion would be valid only for separations $\rho \lesssim 1/k_c$. It is readily shown (see discussion following eq. [13] above) that retaining only the first term in a Taylor series yields $\langle (\Delta\rho)^2 \rangle \propto \rho^2$, so this is the form to be expected at small separations $\rho \lesssim 1/k_c$. Note also the interesting fact that for $\alpha > 2$ (so that the gradients are finite) the leading term in the expansion of equation (18), corresponding to equation (20), is proportional to $\rho^2$ independent of the value of $\alpha$. Apparently a Taylor series is adequate for steep spectra.

The above conclusions suggest the following general picture of magnetic field-line mixing. If the spectrum of fluctuations as a function of wavenumber $k_\perp$ normal to the average field is a power law for $k_\perp \gtrsim 1/L_{c_\perp}$ with index $\alpha$ less than 2, then field lines mix rapidly even for separations much less than $L_{c_\perp}$. One does not have coherent flux tubes. However, if the spectrum has $\alpha$ greater than 2 for wavenumbers larger than $1/L_{c_\perp}$, then one has coherent flux tubes with transverse scale $\sim L_{c_\perp}$, and field lines closer together than $L_{c_\perp}$ do not mix rapidly. For the case of interest ($\alpha < 2$) there must be a high-wavenumber cutoff to the spectrum if the field-gradient expectation values are to be finite. If this cutoff occurs at wavenumber $k_c$, then the scale of coherent flux tubes is of the order of $1/k_c$. That is, field lines much closer together than $1/k_c$ separate slowly. In the case of flat spectra, the scale of coherent lines of force can be much less than the coherence scale of the fluctuating field. The mixing of lines of force is determined by the coherence scale of the *gradients* of $\boldsymbol{B}$ (which is $\sim 1/k_c$) and not by the coherence scale of $\boldsymbol{B}$ itself.

## V. APPLICATIONS

We now consider the application of the preceding results to astrophysical plasmas. In interplanetary space the observed power spectrum of the magnetic field is essentially flat (zero slope) at small wavenumbers (less than $\sim 10^{-11}$ cm$^{-1}$ and is a power law $\propto k^{-\alpha}$ with index $\alpha$ at large wavenumbers, with $\alpha \sim$ 1.3–1.8 (Jokipii 1971). Any cutoff to the spectrum occurs at scales less than about $10^8$–$10^9$ cm. However, these measurements refer to what is effectively a one-dimensional spectrum along the direction of the solar-wind velocity, so that the full three-dimensional spectrum is not known.

First suppose that the spectrum normal to the field is approximately as given in

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equation (16), with $\alpha \simeq 1.5$ as observed. Jokipii and Parker (1969) analyzed the data and concluded that

$$\langle(\Delta x)^2\rangle/\Delta z \simeq 5 \times 10^{10}\ \text{cm} \simeq \tfrac{1}{4}L_c\,. \tag{21}$$

This corresponds to a rather large-amplitude fluctuating field, and the lowest-order results of the preceding section are probably not strictly applicable. However, if one assumes that they are not too seriously in error, one finds from figure 2 that $\langle\Delta z\rangle_1 \sim 4$ in the solar wind. This should be correct to perhaps a factor of 3 or so. One may state, following the discussion at the end of the preceding section, that field lines mix extremely rapidly.

It is possible, of course, that the spectrum *normal* to the field is much steeper than the observed spectrum without contradicting the observations, in which case one would not have the rapid field-line mixing.

These results indicate that field-line random-walk, under some conditions, may be surprisingly efficient in transporting even low-rigidity particles normal to the interplanetary magnetic field. The conclusions of Jokipii and Parker (1969) concerning $\kappa_\perp$ in the solar wind may be appropriate even in cases where the cyclotron radius is much less than $L_c$. The relatively large values of $\kappa_\perp/\kappa_\parallel$ implied by these arguments (Jokipii 1972) are also supported by recent studies of solar-flare events which appear to require a rapid longitudinal spread of $\sim$10-MeV protons about the Sun (McKibben 1972). Further, because of the sensitivity to the steepness of the spectrum, if the spectrum changes with time, it may happen that at certain times $\kappa_\perp$ is large and that at other times it is small. This possibility is attractive in view of the fact that some observers report slow longitudinal spread of solar cosmic rays.

At this point it also is of interest to note that Jokipii and Parker (1969) found that the fluctuations in the *interstellar* magnetic field were such that $\langle(\Delta x)^2\rangle/\Delta z \simeq \frac{1}{3}L_c$, where $L_c \sim 100$ pc. Thus, if the interstellar fluctuations had a power-law spectrum, one would also expect rapid field-line mixing in the interstellar medium. This could help explain the apparently uniform distribution of cosmic rays in the galactic disk. Our knowledge of the structure of the interstellar magnetic field is not yet sufficient to either confirm nor deny this possibility.

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