

SYNCHROTRON SOURCES. I. EXTENSION OF THEORY FOR SMALL PITCH ANGLES

RICHARD I. EPSTEIN*

Institute for Plasma Research, Stanford University

Received 1972 November 27

ABSTRACT

We examine some of the features of synchrotron radiation which occur when relativistic particles move with small pitch angles in a magnetic field. The emission from individual particles is computed; the general properties of the emission from distributions of particles with small pitch angles are discussed; and, finally, the emissivities and degrees of polarization for several specific source models are presented. The application of these results to astrophysical phenomena are considered in the accompanying paper.

Subject heading: synchrotron radiation

I. INTRODUCTION

The conventional analyses of synchrotron theory, in which it is assumed that the pitch angle Ψ between the particle velocity and the magnetic field is of the order of unity, are adequate for the discussion of many astrophysical phenomena; however, some of the observable radiation from quasars, galactic nuclei, and pulsars may be emitted by particles which have small pitch angles. Several authors (O'Dell and Sartori 1970*a*; Cavaliere, Morrison, and Pacini 1970) have suggested that in some sources the radiating particles may even satisfy the more stringent condition $\Psi < \gamma^{-1} \equiv mc^2/E$ (where E is the energy of the particle). The present investigation deals with the special features of synchrotron radiation which occur when the radiating particles have small pitch angles. The contribution of other workers who have dealt with some aspects of this problem (O'Dell and Sartori 1970*a, b*; Melrose 1971*a*; Tadamaru 1972) are also discussed.

In § II the emission from individual particles is examined. First, we obtain expressions for the emission from a particle which has a very small pitch angle $\Psi \ll \gamma^{-1}$, and then we consider the more general case in which the restriction on the pitch angle of the particle is only $\Psi \ll 1$; exact and approximate expressions are presented. In § III we derive expressions for the emissivity, degree of circular polarization, and maximum degree of linear polarization for optically thin synchrotron sources in which the particles mainly have small pitch angles and stream along somewhat irregular magnetic fields. The emission for several source distributions is computed in § IV, and the main results are summarized in § V. The application of these results to astronomical objects is considered in the accompanying paper (Epstein and Petrosian 1972).

II. SYNCHROTRON RADIATION FROM INDIVIDUAL PARTICLES

a) Radiation from Particles with Very Small Pitch Angles

While the general analysis of the synchrotron radiation from a particle which spirals in a magnetic field is fairly involved, we can gain some understanding of synchrotron radiation from particles with small pitch angles and derive some useful formulae by first limiting the discussion to particles with very small pitch angles.

* Present address: Department of Astronomy, University of Texas at Austin.

For the special case of a single, isolated (in a vacuum), highly relativistic particle ($\gamma \gg 1$) which has a very small pitch angle ($\Psi \ll \gamma^{-1}$), the problem of computing the synchrotron emission is greatly simplified. Designate the inertial frame of the observer by K and call the inertial frame in which the particle orbit is circular the K_0 frame. K_0 moves relative to K with a velocity $c\beta_{\parallel}$, where $\beta_{\parallel}$ is the magnitude of the particle velocity along the magnetic field, $\beta_{\parallel} = |\beta_{\parallel}| = \beta \cos \Psi$, and $\beta = (1 - \gamma^{-2})^{1/2}$. As seen in K_0 , the velocity of the particle is

$$\beta_0 = \frac{(\gamma^2 - 1)^{1/2} \sin \Psi}{(\cos^2 \Psi + \gamma^2 \sin^2 \Psi)^{1/2}} \approx \gamma \Psi \ll 1 \quad (1)$$

(quantities which are measured in K_0 are subscripted with 0). When $\gamma \Psi \ll 1$, the particle energy in K_0 is nonrelativistic so that to observers in this frame the emission appears to be mainly monochromatic cyclotron radiation. The particle emissivity in K_0 can be written as (Bekefi 1966)

$$\eta_0 = \frac{\pi e^2 \beta_0^2 \nu_B^2}{2c} (1 + \cos^2 \theta_0) \delta(\nu_0 - \nu_B) \text{ ergs s}^{-1} \text{ sterad}^{-1} \text{ Hz}^{-1}, \quad (2)$$

where θ_0 is the angle between the direction of emission $\mathbf{k}_0$, and $\mathbf{B}$; $\nu_B = |eB/2\pi mc|$; e is the charge of the particle; and $B = |\mathbf{B}|$.

The radiation is completely polarized, and the major axis of the polarization ellipse is in the $\mathbf{k}_0 \times \mathbf{B}$ direction. The degrees of linear and circular polarization are¹

$$\rho_{l0} = \frac{1 - \cos^2 \theta_0}{1 + \cos^2 \theta_0}, \quad \rho_{c0} = \mu \frac{2 \cos \theta_0}{1 + \cos^2 \theta_0}, \quad (3)$$

where $\mu = \text{sign}(-e\beta_{\parallel} \cdot \mathbf{B})$ and $\rho_{c0} > 0$ for right-hand elliptically polarized radiation (the electric vector moves counterclockwise as seen by the observer). The quantities $F_0 \equiv \int \eta_0 d\nu_0$, $F_{l0} \equiv \int \rho_{l0} \eta_0 d\nu_0$, and $F_{c0} \equiv \int \rho_{c0} \eta_0 d\nu_0$ are shown in figure 1. In terms of these functions the degrees of linear and circular polarization are F_{l0}/F_0 and F_{c0}/F_0 , respectively. While the emission is in general elliptically polarized, it is circularly polarized at $\theta_0 = 0$ and $\theta_0 = \pi$ (with the opposite sense in each hemisphere) and linearly polarized at $\theta_0 = \pi/2$.

By using the transformations

$$\tan \theta = \frac{\sin \theta_0 (1 - \beta_{\parallel}^2)^{1/2}}{\cos \theta_0 + \beta_{\parallel}}, \quad (4)$$

$$\nu = \nu_0 \frac{(1 - \beta_{\parallel}^2)^{1/2}}{1 - \beta_{\parallel} \cos \theta}, \quad (5)$$

$$\eta = \eta_0 \frac{(1 - \beta_{\parallel}^2)^{3/2}}{(1 - \beta_{\parallel} \cos \theta)^2}, \quad (6)$$

we can obtain the expression for the radiation as seen in the K frame. It has been assumed here that $\beta_{\parallel}$ is parallel to $\mathbf{B}$ (when $\beta_{\parallel}$ and $\mathbf{B}$ are antiparallel, θ and θ_0 in eqs. [4]–[9] should be replaced by $\pi - \theta$ and $\pi - \theta_0$, respectively). The effects of these transformations are illustrated in figure 1 where $F = \int \eta d\nu$, $F_l \equiv \int \rho_l \eta d\nu$, and

¹ By “the degree of linear polarization” we mean the apparent degree of linear polarization which would be detected if the emission were analyzed with antennas that were sensitive only to linearly polarized radiation. Similarly, the “degree of circular polarization” refers to the relative difference which would be recorded by antennas that were sensitive to only right- or left-hand circularly polarized radiation. In terms of the Stokes parameters the degree of linear polarization and the degree of circular polarization are given by $(Q^2 + U^2)^{1/2}/I$ and V/I , respectively.

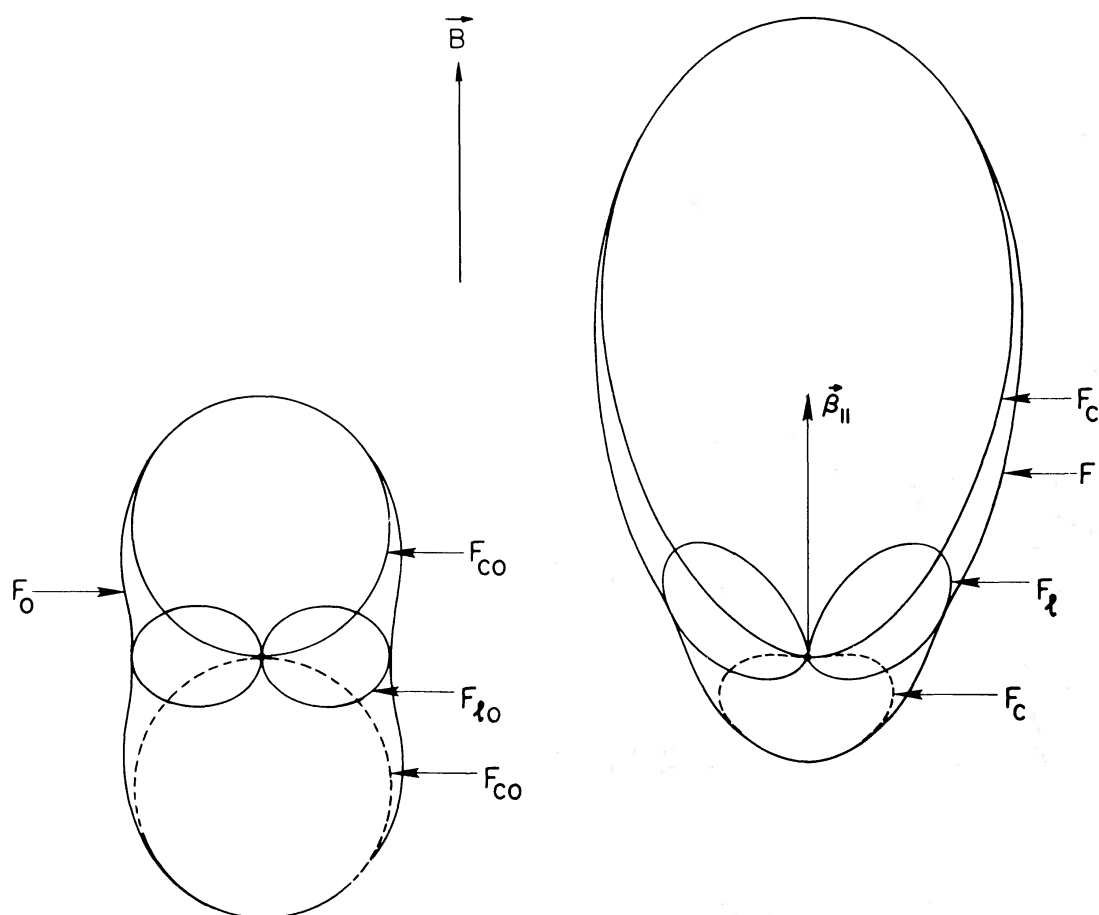


FIG. 1.—Angular distribution of the intensity and polarization for the emission from a particle with a *very* small pitch angle. The figure on the left is the pattern for the nonrelativistic cyclotron radiation which would be observed in K_0 . The figure on the right is the radiation pattern as seen in the laboratory frame, K , when $\beta_{\parallel} = 0.3$. The sense of the elliptical polarization is opposite for the solid and dashed F_c curves.

$F_c \equiv \int \rho_c \eta dv$ are shown for $\beta_{\parallel} = 0.3$. The radiation is beamed and amplified in the direction of $\beta_{\parallel}$, and the frequency is Doppler shifted according to equation (5): at $\theta = 0$ it is “blueshifted” to $\nu_B[(1 + \beta_{\parallel})/(1 - \beta_{\parallel})]^{1/2}$ whereas at $\theta = \pi$ it is redshifted to $\nu_B[(1 - \beta_{\parallel})/(1 + \beta_{\parallel})]^{1/2}$.

The expressions for the emissivity and polarization of the radiation as observed in K are²

$$\eta = \frac{\pi e^2 \gamma^2 \nu^3}{\nu_B c} \left[1 - \frac{\nu}{\gamma \nu_B} + \frac{\nu^2}{2\gamma^2 \nu_B^2} \right] \delta \left(\nu - \frac{2\gamma \nu_B}{1 + \theta^2 \gamma^2} \right), \quad (7)$$

² η is the energy emitted by the particle per unit solid angle, time, and frequency; however, this is not the observed emissivity. The particle is moving toward the observer in K with a mean velocity $c\beta \cos \theta \cos \Psi$ so that the energy which is emitted in a time dt_e is received in a time $dt_r = dt_e (1 - \beta \cos \theta \cos \Psi)$. If the “apparent emissivity” η_a is defined as $\eta_a = \eta / (1 - \beta \cos \theta \cos \Psi)$, then $\eta_a d\Omega$ is the energy received per unit time and frequency by an observer who subtends a solid angle $d\Omega$. To calculate the emission from time independent collections of particles, as is done in § IV, only the emissivity η has to be used. A good discussion of the distinction between the emissivity and the apparent emissivity is given by Ginzburg and Syrovatskii (1969). In their notation $\eta = r^2 p$ and $\eta_a = r^2 \tilde{p}$.

$$\rho_c = \mu \frac{1 - \theta^4 \gamma^4}{1 + \theta^4 \gamma^4} = \frac{2\mu\gamma\nu_B(\nu - \gamma\nu_B)}{\nu^2 - 2\gamma\nu\nu_B + 2\gamma^2\nu_B^2}, \quad (8)$$

$$\rho_l = \frac{2\theta^2\gamma^2}{1 + \theta^4\gamma^4} = \frac{\nu(2\gamma\nu_B - \nu)}{\nu^2 - 2\gamma\nu\nu_B + 2\gamma^2\nu_B^2}. \quad (9)$$

Only the terms of lowest order in $\Psi\gamma$ and θ have been retained.

In any given direction the emission is monochromatic with a frequency $2\gamma\nu_B/(1 + \theta^2\gamma^2)$ and elliptically polarized with the major axis of the polarization ellipse aligned with $\mathbf{k} \times \mathbf{B}$. By integrating η over all directions of emission the power spectrum of the total emission from a single particle is found:

$$\begin{aligned} H(\nu, \gamma, \Psi, B) &\equiv 2\pi \int \eta \sin \theta d\theta \text{ ergs s}^{-1} \text{ Hz}^{-1} \\ &= \frac{2\pi^2 e^2 \Psi^2}{c} \nu \left[1 - \frac{\nu}{\gamma\nu_B} + \frac{1}{2} \left(\frac{\nu}{\gamma\nu_B} \right)^2 \right] \quad \left(\frac{\nu_B}{\gamma} \ll \nu \leq 2\gamma\nu_B \right) \\ &= 0 \quad (\nu > 2\gamma\nu_B). \end{aligned} \quad (10)$$

The rate of energy loss for the particle is obtained by integrating H over all frequencies:

$$-\frac{dE}{dt} = \int_0^\infty H d\nu = \frac{8\pi^2 e^2 \nu_B^2 \gamma^2 \Psi^2}{3c}. \quad (11)$$

Equation (11) is in agreement with the general expression for the power radiated by a particle in a magnetic field (Blumenthal and Gould 1970).

$H(\nu)$, $\rho_l(\nu, \theta)$, and $\rho_c(\nu, \theta)$ are shown in figure 2. The relation of these expressions to observable quantities is discussed in § III.

b) Radiation from Particles with Small Pitch Angles

We will now relax the restriction that $\Psi \ll \gamma^{-1}$ and obtain expressions for the particle emissivity with the weaker condition $\Psi \ll 1$. The radiation field of a highly relativistic charged particle which spirals with arbitrary pitch angle is well known (Ginzburg and Syrovatskii 1969); for emission in the $\mathbf{k}$ direction and at large distance r from the particle the electric field can be expressed as

$$\begin{aligned} \mathcal{E} &= \text{Re} \sum_{n=1}^{\infty} \mathcal{E}_n \exp [2\pi i \nu_n (r/c - t)], \\ \mathcal{E}_n &= \frac{4\pi n e \nu_B \sin \Psi}{c \gamma (1 - \beta \cos \theta \cos \Psi)^2} \left\{ \mathbf{e}_1 J_n'(Z_n) - i \mathbf{e}_2 \frac{(\cos \theta - \beta \cos \Psi)}{\beta \sin \theta \sin \Psi} J_n(Z_n) \right\}, \\ Z_n &= \frac{n \beta \sin \theta \sin \Psi}{1 - \beta \cos \theta \cos \Psi}, \\ \nu_n &= \frac{n \nu_B}{\gamma (1 - \beta \cos \theta \cos \Psi)}. \end{aligned} \quad (12)$$

The unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are in the $\mathbf{B} \times \mathbf{k}$ and $\mathbf{k} \times (\mathbf{B} \times \mathbf{k})$ directions, respectively; $J_n(x)$ is the Bessel function of order n ; and $J_n'(x) = dJ_n(x)/dx$.

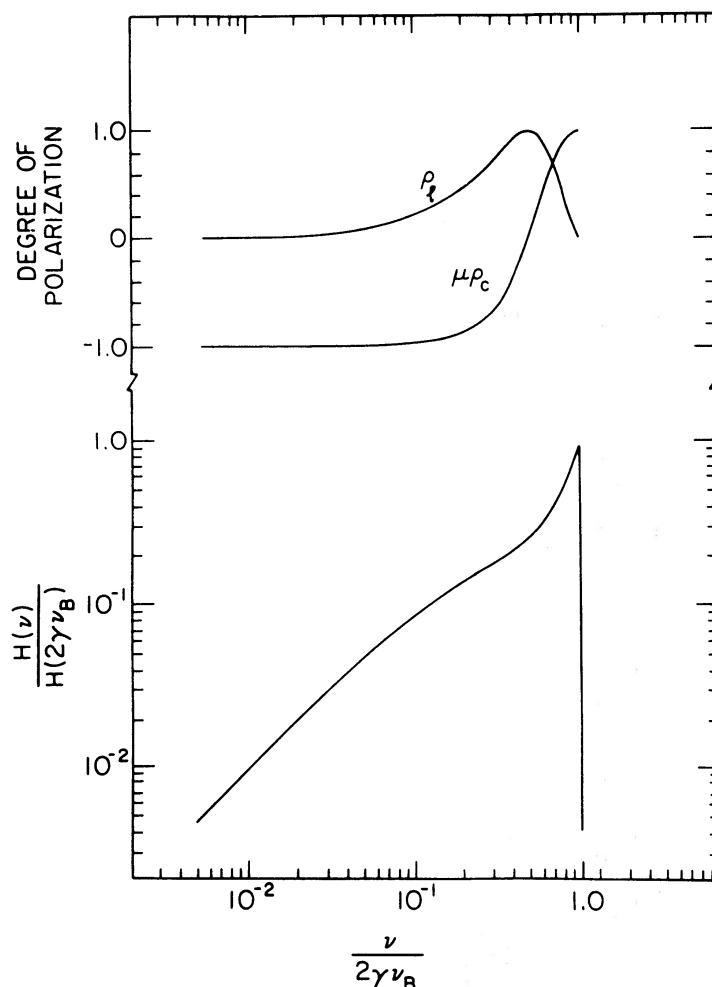


FIG. 2.—The emissivity per unit frequency and the polarization of the emission from relativistic particles which spiral with *very* small pitch angles.

The emissivity of the particle can be expressed as

$$\eta = \frac{r^2 c}{8\pi} (1 - \beta \cos \theta \cos \Psi) \sum_{n=1}^{\infty} |\mathfrak{E}_n|^2 \delta(\nu - \nu_n). \quad (13)$$

The factor $(1 - \beta \cos \theta \cos \Psi)$ adjusts for the average motion of the particle along the magnetic field (see n. 2).

Most of the radiation is beamed within an angle of the order of γ^{-1} of the direction of the particle velocity. Since the pitch angle is also small, the emission is mainly confined to limited range of θ : $\theta \lesssim (\Psi + \gamma^{-1}) \ll 1$. By retaining only the terms of the lowest order in θ , Ψ , and γ^{-1} the expression for the emissivity reduces to

$$\eta = \frac{16\pi e^2 \nu_B^2 \gamma^4 \Psi^2}{c(1 + \theta^2 \gamma^2 + \Psi^2 \gamma^2)} \sum_{n=1}^{\infty} n^2 \left[\{J_n'(Z_n)\}^2 + \left\{ \frac{1 - \theta^2 \gamma^2 + \Psi^2 \gamma^2}{2\theta \Psi \gamma^2} J_n(Z_n) \right\}^2 \right] \delta(\nu - \nu_n),$$

$$\nu_n = \frac{2n\gamma\nu_B}{(1 + \theta^2 \gamma^2 + \Psi^2 \gamma^2)}, \quad Z_n = \frac{2n\gamma^2 \theta \Psi}{(1 + \theta^2 \gamma^2 + \Psi^2 \gamma^2)}, \quad (14)$$

and the total emissivity integrated over all directions of emission is

$$H = \frac{4\pi^2 e^2 \Psi^2 \nu}{c} \sum_{n=1}^{\infty} \{ [J_n'(v_n)]^2 + w_n [J_n(v_n)]^2 \},$$

$$v_n = \left[\frac{\nu \gamma \Psi^2}{\nu_B} \left\{ 2n - \frac{\nu \gamma \Psi^2}{\nu_B} [1 + (\Psi \gamma)^{-2}] \right\} \right]^{1/2},$$

$$w_n = \left\{ \frac{n - (\nu \gamma \Psi^2 / \nu_B) [1 + (\Psi \gamma)^{-2}]}{v_n} \right\}^2. \quad (15)$$

H is shown in figure 3 for several values of $\Psi \gamma$.

For very small pitch angles ($\Psi \gamma \ll 1$) equation (15) reduces to equation (10). When $\nu \gg \nu_B / \gamma \Psi^2$, the approximations for Bessel functions of large order can be used in equation (15) yielding (Ginzburg and Syrovatskii 1969)

$$H = \frac{2\pi 3^{1/2} e^2 \Psi \nu_B}{c} \frac{\nu}{\nu_c} \int_{\nu/\nu_c}^{\infty} K_{5/3}(x) dx, \quad (16)$$

where $\nu_c = \frac{3}{2} \gamma^2 \Psi \nu_B$ and $K_{5/3}(x)$ is the modified Bessel function of order $\frac{5}{3}$.

In many cases the following simple approximations for equation (15) will be adequate: for "very-small-pitch-angle synchrotron radiation," $\gamma \Psi < 0.6$:

$$H = H_{(\text{equation 10})}; \quad (17)$$

for "small-pitch-angle synchrotron radiation," $\gamma \Psi > 0.6$ and $\Psi \ll 1$:

$$H = \frac{2\pi^2 e^2 \Psi^2 \nu}{c}, \quad \nu_B / \gamma \ll \nu < \nu_B / \gamma \Psi^2$$

$$= H_{(\text{equation 16})}, \quad \nu_B / \gamma \Psi^2 < \nu. \quad (18)$$

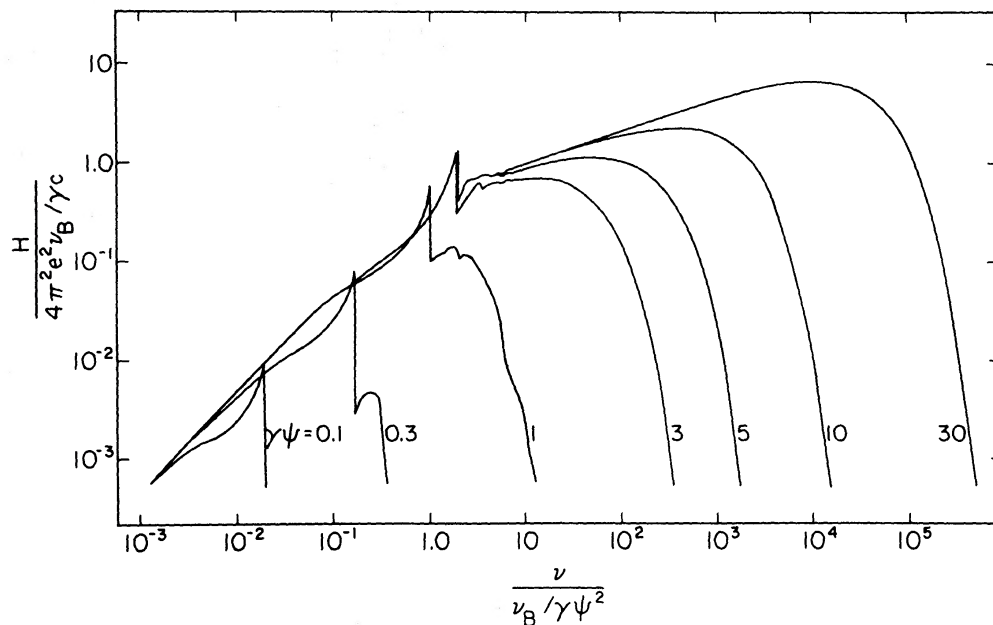


FIG. 3.—The emissivity per unit frequency for relativistic particles which spiral with small pitch angles.

These approximations are accurate to about 1 percent when ν is less than $0.1\nu_B/\gamma\Psi^2$ or greater than $2\nu_B/\gamma\Psi^2$. Large deviations (20–50%) occur only in the frequency range $0.3 \lesssim \nu\gamma\Psi^2/\nu_B < 0.8$ and only when $0.5 \lesssim \gamma\Psi \lesssim 0.8$.

In their study of low-frequency cutoffs O'Dell and Sartori (1970*b*) assumed that H equals zero when $\nu < \nu_B/\gamma\Psi^2$, $\Psi \ll 1$, and $\gamma\Psi \gg 1$; however, since $H \propto \nu$ is a more accurate approximation for this regime (eq. [18]), their conclusions (concerning the occurrence of a steep “cyclotron turnover”) are incorrect. The actual emission spectra for source distributions of the type they considered are discussed in example 2 of § IV.

III. SYNCHROTRON RADIATION FROM COSMIC SOURCES: GENERAL CONSIDERATIONS

We will now consider the emission from optically thin synchrotron sources in which most of the radiating particles have small pitch angles. The formal expressions for the radiation from this type of source are easily found, but to use these expressions to compute observable quantities requires a complete description of the structure of the magnetic field and the particle distributions, information which in fact is never available. This difficulty can be circumvented if the source structure is amenable to certain approximations. Two approaches that can be taken are: (1) one can first compute the radiation from a homogeneous source which is permeated by a uniform field and then make some allowances for the irregularities in the actual sources; or (2) one can at the outset assume that the magnetic field is tangled or irregular, and then assess how the results should be altered to account for a small degree of uniformity. The first approach is usually appropriate for conventional synchrotron sources; but when the emission is mainly from particles with small pitch angles, the second approach is likely to be more useful. In what follows we will discuss the conditions for which each of these approximation schemes is applicable, and we will then derive expressions for the emission from sources in which the radiating particles have small pitch angles and the magnetic field structure is somewhat irregular.

First consider the emission from a small volume element in a synchrotron source. If the radiating particles mainly have small pitch angles, most of the emission will be concentrated within a small angle, θ_R , of the direction of the local magnetic field (θ_R is defined below in the text following eq. [30]). The magnitude of θ_R can be estimated as follows: For sources in which the characteristic pitch angle $\bar{\Psi}$ is large compared to the typical width of the radiation beam for the individual particles $\bar{\gamma}^{-1}$, θ_R is of the order of $\bar{\Psi}$; but when $\bar{\Psi} \ll \bar{\gamma}^{-1}$, θ_R is fixed by the size of the individual radiation pattern. These estimates can be summarized as

$$\theta_R \approx \bar{\Psi} + \bar{\gamma}^{-1}. \quad (19)$$

Now consider the source as a whole. The direction of the magnetic field will vary throughout the source, and the magnitude of these variations can be characterized by an angle θ_B . By this we mean that the magnetic field throughout the source is aligned for the most part within θ_B of some given direction: if the field is absolutely uniform, θ_B is zero; and if it is very tangled, θ_B is of the order of unity (the precise definition for θ_B is given by eq. [31] below). Figure 4 shows a schematic of a portion of a synchrotron source in which θ_B and θ_R are indicated. As before, θ is the angle between the direction of observation and the direction of the local magnetic field.

When θ_R is much greater than θ_B , the variations in the direction of the magnetic field broaden the radiation pattern by only a small amount and the net emission from the source is only slightly altered by the irregularities in the magnetic field. In this case one can compute the main features of the emission by assuming that the field is completely uniform. When $\theta_B \gg \theta_R$, however, the radiation from different regions of the source is beamed in widely different directions (relative to the size of the beam) so that the net emission in any one direction, say $\mathbf{k}$, is proportional to the volume

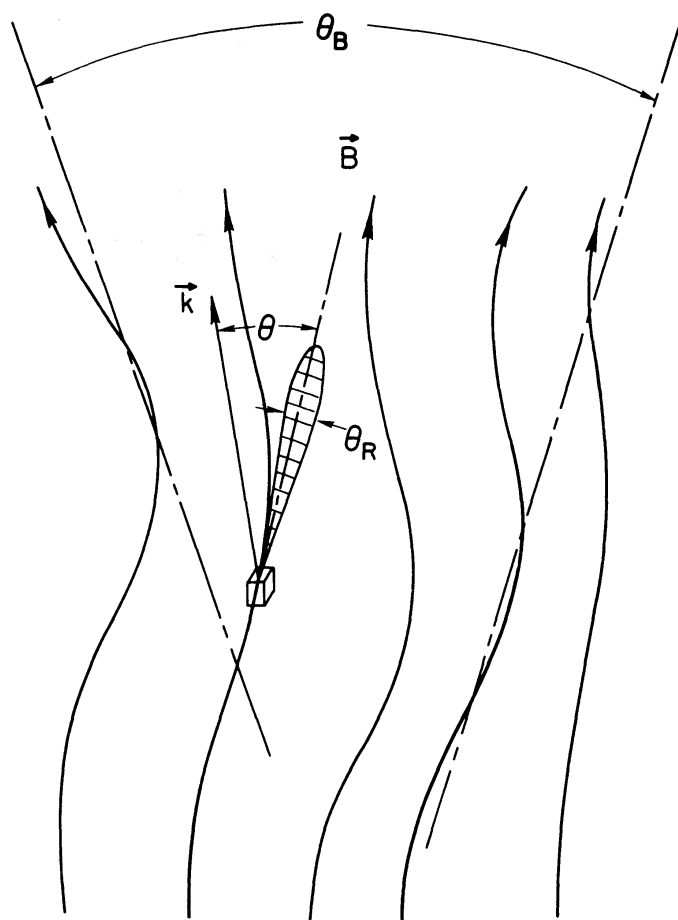


FIG. 4.—The structure of a portion of a synchrotron source. The characteristic variation in the direction of the magnetic field, θ_B , is taken to be much greater than the width of the radiation pattern for the emission from a small volume element, θ_R .

emissivity averaged over a number of the possible orientations of $\mathbf{B}$ relative to $\mathbf{k}$. Since the irregularities in the source structure are predominant in this case, the second approximation procedure can be used for computing the total source emission.

If a synchrotron source mainly contains particles with small pitch angles, then θ_R will be so small that even a moderate deviation from a strictly uniform magnetic structure will insure that $\theta_B \gg \theta_R$. Assuming that this condition is fulfilled, we will derive expressions for the emissivity and polarization of an optically thin source. The main results of these calculations can be anticipated as follows: (1) As we mentioned above, the net emission in the direction $\mathbf{k}$ is the sum of the contributions from a number of regions in which $\mathbf{B}$ has various orientations relative to $\mathbf{k}$. This will enable us to express the total source emissivity, E , and the total emissivity of the circularly polarized radiation, E_c , in terms of the single particle emissivity integrated over all directions of emission (see eqs. [38] and [39] below). (2) Since the major axis of the polarization ellipse is in the $\mathbf{k} \times \mathbf{B}$ direction, its orientation varies for the emission from different regions of the source so that the total amount of linearly polarized radiation is relatively small; equation (42) gives an upper limit for the net degree of linear polarization. The remainder of this section is concerned with deriving equations

(38), (39), and (42); in the next section these results will be used to compute the emission for specific source models.

Let $n(\gamma, \Psi, \mathbf{r})d\gamma d\Psi$ be the number of particles per unit volume at point $\mathbf{r}$ with energies between γ and $\gamma + d\gamma$ and pitch angles between Ψ and $\Psi + d\Psi$ ($0 \leq \Psi \leq \pi/2$). The emissivity of a single particle is expressed as $\eta(\nu, \theta, \Psi, \gamma, B)$ so that for incoherent radiation the emissivity per unit volume in the $\mathbf{k}$ direction and at frequency ν is

$$\epsilon(\nu, \mathbf{k}, \mathbf{r}) = \int_1^\infty \int_0^{\pi/2} n\eta d\Psi d\gamma \text{ ergs(Hz s sterad cm}^3)^{-1}. \quad (20)$$

Similarly, the volume emissivity of the circularly polarized radiation can be written as

$$\epsilon_c(\nu, \mathbf{k}, \mathbf{r}) = \int_1^\infty \int_0^{\pi/2} n\eta_c d\Psi d\gamma, \quad (21)$$

where $\eta_c(\nu, \theta, \Psi, \gamma, B) = \eta\rho_c$ and $\rho_c(\nu, \theta, \Psi, \gamma, B)$ is the degree of circular polarization for the emission from an individual particle. The variable θ in the arguments of η and η_c is a function $\mathbf{r}$ since it is the angle between $\mathbf{k}$ and $\mathbf{B}(\mathbf{r})$.

Assuming that the source is optically thin and that the ambient medium within the source does not significantly affect the transfer of the radiation, we can express the total emissivity E and the circularly polarized emissivity E_c as

$$E_{(c)}(\nu, \mathbf{k}) = \int_V \epsilon_{(c)} d^3r \text{ ergs(Hz s sterad)}^{-1}, \quad (22)$$

where V is the total volume of the source or, if the source is sufficiently well resolved, the portion of the source which is being observed.

The degree of circular polarization is now given by

$$P_c(\nu, \mathbf{k}) = E_c/E. \quad (23)$$

If the quantities n and $\mathbf{B}$ were known throughout the source, equation (22) could be used in its present form; however, since this is not normally the case, it would be more useful to have expressions which involve only the gross or average properties of the source. For the type of sources which we are considering here (ones for which θ_B is much greater than θ_R), these expressions can be obtained by changing the variables of integration in equation (22) and integrating over the orientations of the magnetic field.

The magnetic field at each point in the source can be expressed as

$$\mathbf{B}(\mathbf{r}) = B \sin \theta \cos \phi \mathbf{i} + B \sin \theta \sin \phi \mathbf{j} + B \cos \theta \mathbf{k}, \quad (24)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a set of three mutually perpendicular unit vectors, $\mathbf{k}$ is in the direction of the observer, and B , θ , and ϕ are functions of $\mathbf{r}$. The volume element in equation (22) can be written in terms of the B , θ , ϕ variables by using

$$d^3r = \left| \frac{d^3r}{d^3B} \right| B^2 dB \sin \theta d\theta d\phi. \quad (25)$$

This expression for d^3r represents that portion of the source volume in which the magnetic field strength is between B and $B + dB$ and the field direction is between θ and $d\theta$ in one of the angular coordinates and between ϕ and $d\phi$ in the other.

By defining

$$N(\gamma, \Psi, B, \theta, \phi) \equiv n \left| \frac{d^3 r}{d^3 B} \right| B^2 \text{ particles (gauss sterad erg rad)}^{-1} \quad (26)$$

and using equation (25), we can transform equation (22) to

$$E_{(c)}(\nu, \mathbf{k}) = \int N \eta_{(c)} \sin \theta d\theta d\phi dB d\Psi d\gamma. \quad (27)$$

The position $\mathbf{r}$ in the argument of n in equation (26) is now a function of $\mathbf{B}$. It may be that $\mathbf{r}(\mathbf{B})$ is not a single-valued function of $\mathbf{B}$, but this can be allowed for by modifying the definition of N as follows: Consider the source divided into, say, j regions, in each of which $\mathbf{r}(\mathbf{B})$ is a single valued function. The right-hand side of equation (26) is evaluated separately in each of these regions, and N is taken to be the sum of these j terms.

The quantity N describes the net properties of a source. It gives the total number of particles that are in a given interval of energy and pitch angle and which are located in regions of the source in which the magnetic field has a certain range of directions and field strengths. Since we are concerned with sources in which the average pitch angle of the particles is small and the magnetic field is fairly irregular, N is peaked near $\Psi = 0$ and it is a slowly varying function of θ and ϕ . To simplify the analysis we will limit our discussion to sources for which N is separable:

$$N(\gamma, \Psi, B, \theta, \phi) = \lambda(\gamma, B, \theta, \phi) \xi(\gamma, B, \Psi). \quad (28)$$

With the definitions

$$I_{(c)}(\nu, \theta, \gamma, B) \equiv \int_0^{\pi/2} \eta_{(c)} \xi d\Psi, \quad (29)$$

equation (27) now becomes

$$E_{(c)}(\nu, \mathbf{k}) = \int \lambda_{(c)} \sin \theta d\theta d\phi dB d\gamma. \quad (30)$$

Since ξ is peaked in the vicinity of $\Psi = 0$, and η and η_c are small everywhere except for θ near Ψ and $\pi - \Psi$, the integrals I and I_c are peaked about $\theta = 0$ and $\theta = \pi$. Furthermore, I and I_c are symmetric and antisymmetric functions of $(\theta - \pi/2)$, respectively (since η_c depends on the sign of $\mathbf{k} \cdot \mathbf{B}$ while η does not). A measure of the angular width of I is given by $\theta_R(\nu, \gamma, B)$, which we now define as the value of $\theta < \pi/2$ at which I is one half of its maximum value.

The overall structure of the source is described by λ . The quantity θ_B , which characterizes the scale of the irregularities in the source, is defined in terms of λ by

$$\theta_B(\gamma, B, \theta, \phi) \equiv \frac{\lambda}{|\partial \lambda / \partial \theta|}. \quad (31)$$

As we noted earlier, if the mean pitch angles of the relativistic particles in a synchrotron source is sufficiently small and if the magnetic field is somewhat irregular, then θ_B is much greater than θ_R . In the present context this means that λ is nearly constant over the ranges of the values of θ and ϕ for which I and I_c make significant contributions to the integrands in equation (30) so that by excluding λ from the

angular integrations we only introduce relative errors of the order of (θ_R/θ_B) . To this accuracy the expressions for E and E_c can be reduced to

$$\begin{aligned} E(\nu, \mathbf{k}) &= \int_0^\infty \int_1^\infty \{\lambda_{\theta=0} + \lambda_{\theta=\pi}\} Q d\gamma dB, \\ E_c(\nu, \mathbf{k}) &= \int_0^\infty \int_1^\infty \{\lambda_{\theta=0} - \lambda_{\theta=\pi}\} Q_c d\gamma dB, \end{aligned} \quad (32)$$

where

$$Q_{(c)}(\nu, \mathbf{k}, \gamma, B) \equiv \int_0^{2\pi} \int_0^{\pi/2} I_{(c)} \sin \theta d\theta d\phi = \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\pi/2} \eta_{(c)} \xi \sin \theta d\theta d\phi d\Psi, \quad (33)$$

and we have used the symmetries of I and I_c mentioned above.

Since $\xi(\gamma, B, \Psi)$ is independent of θ and ϕ , Q and Q_c can be evaluated by first integrating over all directions of emission; this yields

$$Q_{(c)}(\nu, \gamma, B) = \int \xi H_{(c)} d\Psi, \quad (34)$$

where

$$H_{(c)}(\nu, \gamma, \Psi, B) = \int \eta_{(c)} \sin \theta d\theta d\phi. \quad (35)$$

H is given by equation (15), or it can be approximated by equations (17) and (18). For $\gamma\Psi \ll 1$, $H_c = H\rho_c(\nu)$, where $\rho_c(\nu)$ is given by equation (8). When $\nu \ll \nu_B/(\gamma\Psi^2)$, H_c nearly equals $-\mu H$ (since this is mainly emission from the first harmonic and at large θ); and when $\gamma\Psi \gg 1$ and $\nu \gg \nu_B/(\gamma\Psi^2)$, H_c can be expressed as (Legg and Westfold 1968; Melrose 1971b)³

$$\begin{aligned} H_c &= \frac{4\pi\mu e^2\nu_B}{3^{1/2}c} \left[\int_{\nu/\nu_c}^\infty K_{1/3}(x) dx + 2 \frac{\nu}{\nu_c} K_{1/3}\left(\frac{\nu}{\nu_c}\right) \right. \\ &\quad \left. - \frac{d}{d\Psi} \left\{ \tan \Psi \left[\int_{\nu/\nu_c}^\infty K_{1/3}(x) dx + 2 \frac{\nu}{\nu_c} K_{1/3}\left(\frac{\nu}{\nu_c}\right) \right] \right\} \right]. \end{aligned} \quad (36)$$

Combining equations (32) and (34) and the definitions

$$\begin{aligned} N_E(\gamma, \Psi, B, \mathbf{k}) &\equiv \{\lambda_{\theta=0} + \lambda_{\theta=\pi}\} \xi(\gamma, B, \Psi), \\ N_c(\gamma, \Psi, B, \mathbf{k}) &\equiv \{\lambda_{\theta=0} - \lambda_{\theta=\pi}\} \xi(\gamma, B, \Psi), \end{aligned} \quad (37)$$

we obtain our final expressions for the source emissivities:

$$E(\nu, \mathbf{k}) = \int_0^\infty \int_1^\infty \int_0^{\pi/2} N_E H d\Psi d\gamma dB \quad (38)$$

$$E_c(\nu, \mathbf{k}) = \int_0^\infty \int_1^\infty \int_0^{\pi/2} N_c H_c d\Psi d\gamma dB. \quad (39)$$

³ Integration of equation (34) by parts yields the integrand

$$\frac{4\pi\mu e^2\nu_B\xi}{3^{1/2}c} \left[\left(1 + \frac{\tan \Psi}{\xi} \frac{d\xi}{d\Psi} \right) \int_{\nu/\nu_c}^\infty K_{1/3}(x) dx + 2 \frac{\nu}{\nu_c} K_{1/3}\left(\frac{\nu}{\nu_c}\right) \right].$$

When ξ is replaced by $\phi(\Psi) \sin \Psi/2$, this expression can be written as $8\pi^2 i \mathfrak{E}^{21}(E, \omega, \Psi) \sin \Psi$, where $\mathfrak{E}^{21}$ is given by equation (37) of Melrose (1971b).

Equations (38) and (39) give the emission and degree of circular polarization in the $\mathbf{k}$ direction in terms of the functions N_E and N_c , which describe the net properties of those portions of the source in which the magnetic field is parallel to $\mathbf{k}$, and in terms of the functions H and H_c , which describe the average contribution of each particle to the total emission. N_E represents the total flux of particles in the direction of the observer, while N_c is equal to the particle flux in the regions in which $\mathbf{B} \cdot \mathbf{k}$ is positive minus the particle flux in the regions in which $\mathbf{B} \cdot \mathbf{k}$ is negative.

While it was possible to express the circularly polarized emissivity simply as an integral of $N_c H_c$, a similar expression cannot be obtained for the linearly polarized emissivity, because, for the type of source with which we are concerned here ($\theta_B > \theta_R$), the degree and direction of linear polarization depend very sensitively on the magnetic field structure; nevertheless, we can still set an upper limit to the degree of linear polarization. First consider the emission from a source for which $N(\gamma, \Psi, B, \theta, \phi)$ is isotropic. Since there is no preferred direction along which the polarization ellipse could be aligned, the degree of linear polarization for this emission must be zero. For the more general cases where the source distribution can be described by the sum of an isotropic term, N_0 , and a nonisotropic term, N_1 , the emission which is related to N_0 will not exhibit any linear polarization, while the emission from the portion of the source which is represented by N_1 will, in general, be somewhat linearly polarized. In the vicinity of $\theta = 0$ the total source distribution function can be expressed as

$$\begin{aligned} N(\gamma, \Psi, B, \theta, \phi) &= N_0(\gamma, \Psi, B) + N_1(\gamma, \Psi, B, \theta, \phi), \\ N_0 &= N(\gamma, \Psi, B, \theta = 0), \\ N_1 &= \theta \times \lim_{\theta \rightarrow 0} \frac{\partial N}{\partial \theta}. \end{aligned} \quad (40)$$

The source distribution function near $\theta = \pi/2$ can be expanded in the same way, but to simplify the present discussion the contributions from these terms will not be included here.

The total emissivity for the source, E , and the emissivity for the anisotropic portion of the source distribution, E_1 , are given by equation (27), and the angular derivative of N is related to θ_B by equation (31) so that an upper limit to the degree of linear polarization can be written as

$$P_l(\nu, \mathbf{k}) \lesssim \frac{E_1(\nu, \mathbf{k})}{E(\nu, \mathbf{k})} = \frac{\int (\theta/\theta_B) N \eta \sin \theta d\theta d\phi dB d\Psi d\gamma}{\int N \eta \sin \theta d\theta d\phi dB d\Psi d\gamma}. \quad (41)$$

Since the integrands in equation (41) are small everywhere except for $\theta \lesssim \theta_R$, the upper limit for P_l becomes

$$P_l(\nu, \mathbf{k}) \lesssim \theta_R/\theta_B \ll 1. \quad (42)$$

For sources in which the relativistic particles have small pitch angles and the magnetic field is somewhat nonuniform, the radiation will exhibit only a small degree of linear polarization; the actual amount will depend on the details of the source structures and particle distributions. For these reasons we cannot derive any useful quantitative expressions for the linear polarization and have not attempted to compute the degrees of linear polarization for the simple source models considered in the next section.

IV. SYNCHROTRON SOURCES: EXAMPLES

We will consider the emission from source distributions of the form

$$\begin{aligned}
 N_E(\gamma, \Psi, B, \mathbf{k}) &= G\gamma^{-s}\Psi \exp[-\Psi^2/\tilde{\Psi}^2]\delta(B - B_0), & \gamma_1 < \gamma < \gamma_2, \\
 &= 0, & \text{otherwise;} \\
 N_c(\gamma, \Psi, B, \mathbf{k}) &= \mu G\gamma^{-s}\Psi \exp[-\Psi^2/\tilde{\Psi}^2]\delta(B - B_0), & \gamma_1 < \gamma < \gamma_2, \\
 &= 0, & \text{otherwise,} \quad (43)
 \end{aligned}$$

where $\gamma_1 \gg 1$, $\tilde{\Psi} \ll 1$, and $s > 1$. The dependence of N_E and N_c on $\mathbf{k}$ is not explicitly shown, but the parameters G , s , $\tilde{\Psi}$, B_0 , γ_1 , and γ_2 may be slowly varying functions of $\mathbf{k}$. The sources described by these distributions have (1) energy spectra which follow power laws within certain intervals and are zero elsewhere and (2) pitch-angle distributions which are peaked at small angles. We have assumed that the magnetic field can be characterized by single values of the field strength and μ ; $\mu \equiv \text{sign}(-e\mathbf{B} \cdot \mathbf{k})$.

The source emissivity and the degree of circular polarization for a given distribution function are obtained by using equations (23), (38), and (39). We have computed these quantities for three sets of parameters: (1) $\gamma_2\tilde{\Psi} \ll 1$, (2) $\gamma_1\tilde{\Psi} \gg 1$, and (3) $\gamma_1\tilde{\Psi} \ll 1$ and $\gamma_2\tilde{\Psi} \gg 1$. These examples illustrate sources which respectively emit predominantly very-small-pitch-angle synchrotron radiation (eq. [17]), small-pitch-angle synchrotron radiation (eq. [18]), and a combination of the two.

EXAMPLE 1. $\gamma_2\tilde{\Psi} \ll 1$: *Very-small-pitch-angle synchrotron radiation.*

These distribution functions describe sources in which essentially all of the particles have $\gamma\tilde{\Psi} \ll 1$. The appropriate single-particle emissivities are H as given by equation (17) and $H_c = \rho_c(\nu)H$, where $\rho_c(\nu)$ is given by the last expression in equation (8). The source emissivity and the degree of circular polarization are

$$\begin{aligned}
 E(\nu, \mathbf{k}) &= A(\gamma_2\tilde{\Psi})^{2-s} \left[\left[\frac{\nu}{\nu_2} \left\{ \left(\frac{\nu_2}{\nu_1} \right)^{s-1} \left[\frac{1}{s-1} - \frac{2}{s} \frac{\nu}{\nu_1} + \frac{2}{s+1} \left(\frac{\nu}{\nu_1} \right)^2 \right] \right\} \right] \right. \\
 &\quad \left. - \left[\frac{1}{s-1} - \frac{2}{s} \frac{\nu}{\nu_2} + \frac{2}{s+1} \left(\frac{\nu}{\nu_2} \right)^2 \right] \right] \right], \quad \nu_B/\gamma_1 \ll \nu < \nu_1, \\
 &= A(\gamma_2\tilde{\Psi})^{2-s} \left[\left(\frac{\nu}{\nu_2} \right)^{2-s} \left\{ \left[\frac{1}{s-1} - \frac{2}{s} + \frac{2}{s+1} \right] \right. \right. \\
 &\quad \left. \left. - \left(\frac{\nu}{\nu_2} \right)^{s-1} \left[\frac{1}{s-1} - \frac{2}{s} \frac{\nu}{\nu_2} + \frac{2}{s+1} \left(\frac{\nu}{\nu_2} \right)^2 \right] \right\} \right] \right], \quad \nu_1 < \nu < \nu_2, \quad (44)
 \end{aligned}$$

$$\begin{aligned}
 P_c(\nu, \mathbf{k}) &= \frac{\mu A(\gamma_2\tilde{\Psi})^{2-s}}{E(\nu, \mathbf{k})} \left[\left[\frac{\nu}{\nu_2} \left\{ \left(\frac{\nu_2}{\nu_1} \right)^{s-1} \left[\frac{2}{s} \frac{\nu}{\nu_1} - \frac{1}{s-1} \right] - \left[\frac{2}{s} \frac{\nu}{\nu_2} - \frac{1}{s-1} \right] \right\} \right] \right] \\
 &\quad \nu_B/\gamma_1 \ll \nu < \nu_1, \\
 &= \frac{\mu A(\gamma_2\tilde{\Psi})^{2-s}}{E(\nu, \mathbf{k})} \left[\left(\frac{\nu}{\nu_2} \right)^{2-s} \left\{ \left[\frac{2}{s} - \frac{1}{s-1} \right] - \left(\frac{\nu}{\nu_2} \right)^{s-1} \left[\frac{2}{s} \frac{\nu}{\nu_2} - \frac{1}{s-1} \right] \right\} \right] \right], \\
 &\quad \nu_1 < \nu < \nu_2, \quad (45)
 \end{aligned}$$

where $A \equiv (2\pi^2 e^2 \nu_B G \tilde{\Psi}^{s+2})/c$, ν_B is evaluated at $B = B_0$, $\nu_1 = 2\gamma_1\nu_B$, $\nu_2 = 2\gamma_2\nu_B$, and s is not exactly equal to 2. The functions E and P_c are shown in figure 5 for $\gamma_1\tilde{\Psi} = 10^{-4}$

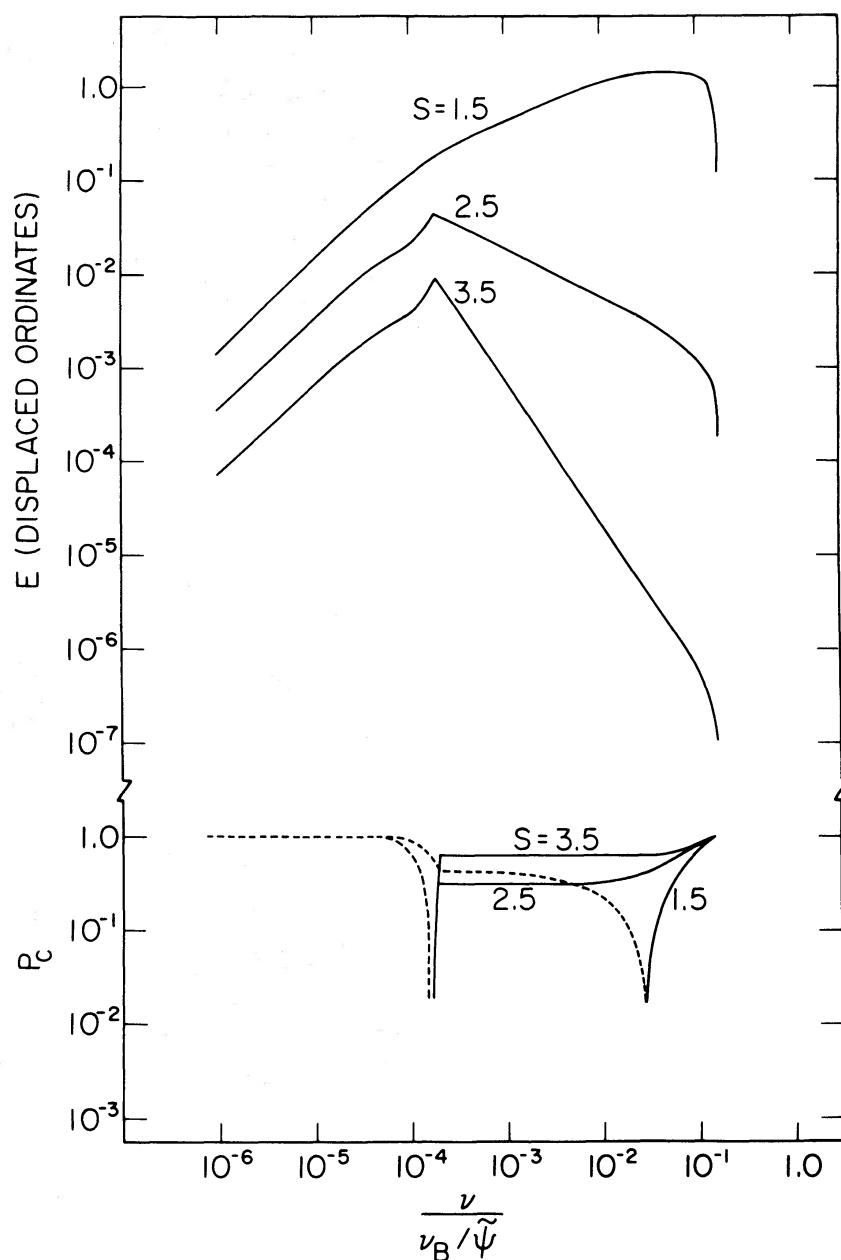


FIG. 5.—The source emissivity and the degree of circular polarization. Example 1: $\gamma_1 \tilde{\Psi} = 10^{-4}$; $\gamma_2 \tilde{\Psi} = 0.1$. The solid and dotted P_c curves indicate opposite sense of circular polarization.

and $\gamma_2 \tilde{\Psi} = 0.1$. The spectrum is mainly composed of two regions: a high-frequency region where

$$\nu_1 \ll \nu \ll \nu_2, \quad E \propto \nu^{2-s}, \quad P_c \approx \mu \frac{(s-2)(s+1)}{s^2 - s + 2}, \quad (46)$$

and a low-frequency region where

$$\nu_B/\gamma_1 \ll \nu \ll \nu_1, \quad E \propto \nu, \quad P_c \approx -\mu. \quad (47)$$

At the higher frequencies the spectrum follows a power law, but the power-law index differs from that which is obtained by conventional synchrotron theory; in the present case the logarithmic slope of the emissivity is $2 - s$ instead of the usual result of $(1 - s)/2$.

From figure 5 it can be seen that the sense of the circular polarization reverses near the frequency at which the emission is a maximum. This is a striking feature by which it may be possible to identify this type of emission; however, there are other processes whereby similar effects can be produced. Several authors (Ramaty 1969; Ramaty and Holt 1970; Pacholczyk and Swihart 1970; Melrose 1971*b*) have noted that synchrotron sources which are self-absorbed at low frequencies may also exhibit a reversal in the sense of the circular polarization. If a reversal of this sort were ever observed, it would be necessary to rely on other features of the emission, such as the shape of the spectrum and the variations of the polarization with frequency, to distinguish between self-absorption and the optically thin emission considered here.

Recently Tadamaru (1972) and Melrose (1971*a*) have studied the emission from particles with very small pitch angles ($\gamma\tilde{\Psi} \ll 1$). Tadamaru's work was concerned with the emission from distributions which are significantly different from those considered in the present investigation. Melrose computed the quantities r_l and r_c which, in the present notation, are equivalent to

$$r_{l,c} = \int_0^\infty \rho_{l,c}(\nu) H d\nu / \int_0^\infty H d\nu, \quad (48)$$

where ρ_l , ρ_c , and H are given by equations (8), (9), and (10), respectively. It is not clear how r_l and r_c can be related to observable quantities. Melrose claimed that $|r_c|$, which equals 0.5, represents an upper limit to the degree of circular polarization which can be exhibited by a source; however, as one can see from figure 5, this clearly is not true.

EXAMPLE 2. $\gamma_1\tilde{\Psi} \gg 1$: *Small-pitch-angle-synchrotron radiation.*

For sources described by these distribution functions most of the particles have $\gamma\tilde{\Psi} \gg 1$ and $\tilde{\Psi} \ll 1$ so that the single-particle emissivity H can be approximated by equation (18). At high frequencies the function H_c is given by equation (36), and at very low frequencies it is equal to $-H$. In those frequency intervals for which equations (38) and (39) can be easily evaluated the source emissivity and the degree of circular polarization are given by

$$\begin{aligned} E(\nu, \mathbf{k}) &= Ac_1(\gamma_1\tilde{\Psi})^{1-s}\nu\tilde{\Psi}/\nu_B, & \nu_B/\gamma_1 \ll \nu \ll \nu_B/\gamma_1\tilde{\Psi}^2, \\ &= Ac_2(\gamma_1\tilde{\Psi})^{1/3-s}(\nu\tilde{\Psi}/\nu_B)^{1/3}, & \nu_B/\gamma_1\tilde{\Psi}^2 \ll \nu \ll \nu_B\gamma_1^2\tilde{\Psi}, \\ &= Ac_3(\nu\tilde{\Psi}/\nu_B)^{(1-s)/2}, & \nu_B\gamma_1^2\tilde{\Psi} \ll \nu \ll \nu_B\gamma_2^2\tilde{\Psi}, \end{aligned} \quad (49)$$

$$\begin{aligned} P_c(\nu, \mathbf{k}) &= -\mu, & \nu_B/\gamma_1 \ll \nu \ll \nu_B/\gamma_1\tilde{\Psi}^2, \\ &= \mu c_4(\gamma_1\tilde{\Psi})^{-5/3}(\nu\tilde{\Psi}/\nu_B)^{1/3}, & \nu_B/\gamma_1\tilde{\Psi}^2 \ll \nu \ll \nu_B\gamma_1^2\tilde{\Psi}, \\ &= \mu c_5(\nu\tilde{\Psi}/\nu_B)^{-1/2}, & \nu_B\gamma_1^2\tilde{\Psi} \ll \nu \ll \nu_B\gamma_2^2\tilde{\Psi}. \end{aligned} \quad (50)$$

The functions $c_i(s)$ are defined and tabulated in the Appendix. The quantities E and P_c are shown in figure 6 for $\gamma_1\tilde{\Psi} = 10$ and $\gamma_2\tilde{\Psi} > 10^3$. For frequencies much higher than $\nu_B/\gamma_1\tilde{\Psi}^2$ our results for the source emissivity and the polarization are the same as those of conventional synchrotron theory (Ginzburg and Syrovatskii 1965; Legg

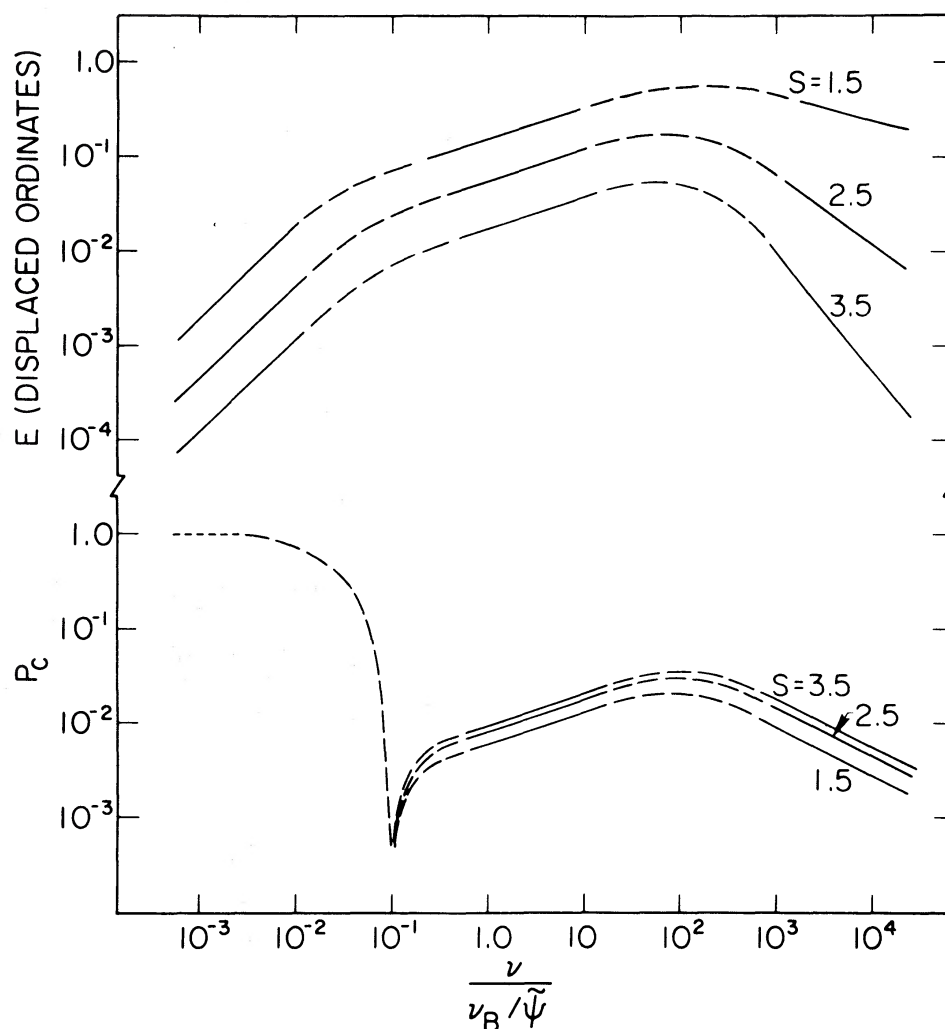


FIG. 6.—Same as fig. 5. Example 2: $\gamma_1\tilde{\Psi}' = 10$; $\gamma_2\tilde{\Psi}' > 10^3$. The dashed curve segments are rough interpolations between the computed portions of the curves.

and Westfold 1968). At frequencies lower than $\nu_B/\gamma_1\tilde{\Psi}'^2$, however, the emission decreases linearly with frequency and becomes strongly circularly polarized. O'Dell and Sartori (1971b) were the first to note that synchrotron spectra should have unusual features in this frequency range; however, their actual analysis was incorrect because, as we mentioned in § II, they used a poor approximation for the single-particle emissivity.

EXAMPLE 3. $\gamma_1\tilde{\Psi}' \ll 1$ and $\gamma_2\tilde{\Psi}' \gg 1$: *Small-pitch-angle and very-small-pitch-angle synchrotron radiation.*

These distributions are essentially superpositions of the distributions of examples 1 and 2. At low frequencies the emission is similar to that of example 1, and at high frequencies it is similar to that of example 2. For intermediate frequencies, $\nu \approx \nu_B/\tilde{\Psi}'$, the spectra are either concave or convex (on a log E versus log ν display) for distribution indices s greater or less than 3, respectively. The emissivity and degree of circular polarization for this example are shown in figure 7 where $\gamma_1\tilde{\Psi}' = 10^{-3}$ and $\gamma_2\tilde{\Psi}' > 10^3$.

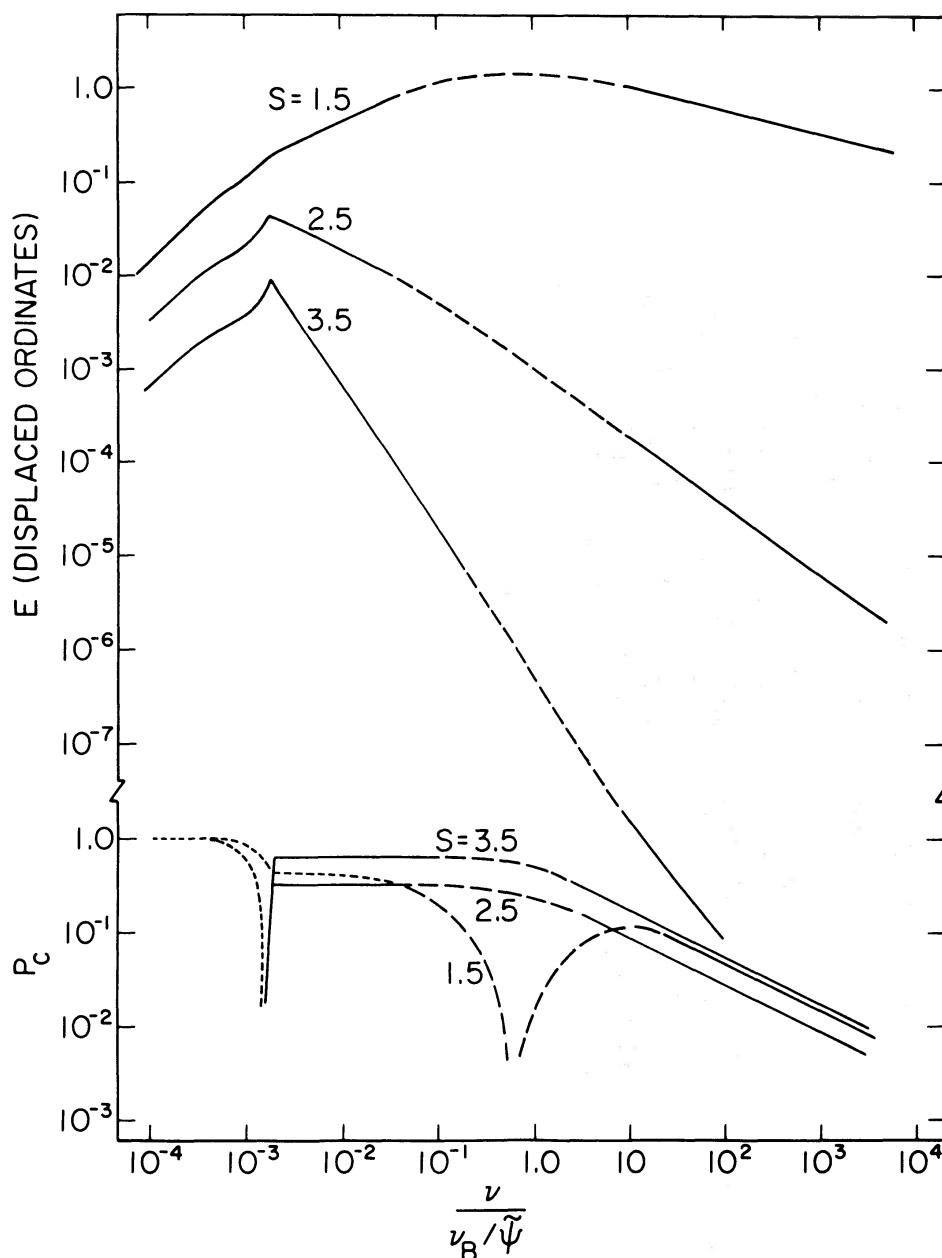


FIG. 7.—Same as fig. 5. Example 3: $\gamma_1 \tilde{\Psi} = 10^{-3}$; $\gamma_2 \tilde{\Psi} > 10^3$.

V. SUMMARY

The emission from a relativistic particle which spirals with a small pitch angle [$\Psi \lesssim O(\gamma^{-1})$] differs qualitatively from the usual synchrotron emission ($\Psi \gg \gamma^{-1}$): the particle spectrum peaks around $\nu \approx \gamma \nu_B$ (instead of at $\nu \approx \gamma^2 \nu_B \sin \Psi$) and falls off linearly with frequency as $\nu \rightarrow 0$ (rather than as $\nu^{1/3}$). For sources which mainly contain particles with small pitch angles, the degree of circular polarization will normally be high, whereas, if the magnetic field is somewhat irregular, the degree of linear polarization will be small. If the particle distribution also has a low-energy

cutoff, the emissivity will vary linearly with frequency at low frequencies. Applications of these results to specific astrophysical problems are considered in Paper II.

I wish to thank Peter Sturrock for suggesting portions of this study and Bill Adams, Spiro Antiochos, Evelyn Carr, B. Fahr, Chris McKee, Chuck Newman, and Vahé Petrosian for valuable discussions and for help in the preparation of the manuscript. This work has been supported by the National Science Foundation under grant GP-23840, the National Aeronautics and Space Administration under grant NGL 05-020-272, and the Office of Naval Research under contract N00014-67-A-0112-0062.

APPENDIX

$$c_1 = \frac{1}{2(s-1)},$$

$$c_2 = \frac{2}{3^{1/3}(3s-1)},$$

$$c_3 = (8\pi)^{-1} \left[3^{s/2} \Gamma\left(\frac{3s-1}{12}\right) \Gamma\left(\frac{3s+19}{12}\right) \Gamma\left(\frac{s+1}{4}\right) \right],$$

$$c_4 = \frac{3s-1}{3^{1/3}(3s+4)},$$

$$c_5 = s \Gamma\left(\frac{3s+8}{12}\right) \Gamma\left(\frac{3s+4}{12}\right) \Gamma\left(\frac{s}{4}\right) / \left[3^{1/2} 2 \Gamma\left(\frac{3s-1}{12}\right) \Gamma\left(\frac{3s+19}{12}\right) \Gamma\left(\frac{s+1}{4}\right) \right].$$

TABLE A1
COEFFICIENTS FOR EQUATIONS (49) AND (50)

s	c_1	c_2	c_3	c_4	c_5
1.0.....	...	0.69	0.64	0.20	0.18
1.5.....	1.00	0.40	0.39	0.29	0.30
2.0.....	0.50	0.28	0.32	0.35	0.40
2.5.....	0.33	0.21	0.31	0.39	0.48
3.0.....	0.25	0.17	0.33	0.43	0.56
3.5.....	0.20	0.15	0.39	0.45	0.63
4.0.....	0.16	0.13	0.48	0.48	0.69
4.5.....	0.14	0.11	0.63	0.50	0.74
5.0.....	0.13	0.10	0.88	0.51	0.80

REFERENCES

- Bekefi, G. 1966, *Radiation Processes in Plasmas* (New York: Wiley).
 Blumenthal, G. R., and Gould, R. J. 1970, *Rev. Mod. Phys.*, **42**, 237.
 Cavaliere, A., Morrison, P., and Pacini, F. 1970, *Ap. J. (Letters)*, **162**, L133.
 Epstein, R. I., and Petrosian, V. 1972, *Ap. J.*, **183**, 611 (Paper II).
 Ginzburg, V. L., and Syrovatskii, S. I. 1969, *Ann. Rev. Astr. and Ap.*, **7**, 375.
 Legg, M. P. C., and Westfold, K. C. 1968, *Ap. J.*, **154**, 499.
 Melrose, D. B. 1971a, *Ap. Letters*, **8**, 35.
 ———. 1971b, *Ap. and Space Sci.*, **12**, 172.
 O'Dell, S. L., and Sartori, L. 1970a, *Ap. J. (Letters)*, **161**, L63.
 ———. 1970b, *ibid.*, **162**, L37.
 Pacholczyk, A. G., and Swihart, T. L. 1970, *Ap. J.*, **170**, 405.
 Ramaty, R. 1969, *Ap. J.*, **158**, 753.
 Ramaty, R., and Holt, S. S. 1970, *Nature*, **226**, 68.
 Tadamaru, E. 1972, *Ap. J.*, **172**, 327.