THE ASTROPHYSICAL JOURNAL, **181**: 865–874, 1973 May 1

INTEGRAL KERNELS AND EXACT SOLUTIONS TO SOME RADIATIVE PROBLEMS IN SPHERICAL GEOMETRIES

JOHANNES SCHMID-BURGK*
Lehrstuhl für Theoretische Astrophysik, University of Heidelberg
Received 1971 March 13; revised 1972 November 2

ABSTRACT

Problems of radiative transfer in spherical geometries are investigated in the gray approximation by using a two-parameter family of absorption coefficients as functions of radial depth. The parameters describe the extent to which the atmosphere is actually dominated by geometric effects. With these absorption coefficients, kernels for the Fredholm equation for the mean intensity are calculated in analytic form.

Exact solutions to the resulting Wiener-Hopf equation are presented for the case that the "degree of sphericity" remains constant throughout the atmosphere. Milne's problem is used for demonstration.

The extent to which the radiation field remains anisotropic down to arbitrarily large optical depths (thus invalidating the unmodified Eddington approximation) is shown as well as the difference in surface intensities between plane-parallel and extended atmospheres. An approximate expression for the mean intensity is derived from the exact solution.

Subject headings: atmospheres, stellar — radiative transfer

I. INTRODUCTION

Two major aspects distinguish problems of radiation transfer in spherically symmetric geometries from their plane-parallel counterparts: the strong forward peaking of the intensity at small optical depths in extended atmospheres, and the impossibility of replacing the radial-coordinate variable in the spherical transfer equation by an optical-depth variable in a general manner. Both aspects prohibit the unmodified application of mathematical methods efficient in plane geometries to spherical problems.

Chapman (1966) has discussed the first aspect and tentatively proposed an approximate expression for the mean intensity as solution to spherical Milne problems. Recently, several algorithms have been developed (Cassinelli 1971; Hummer and Rybicki 1971; Schmid-Burgk 1969) that allow computation of spherically symmetric radiation fields, each one abandoning the neglect of forward peaking that is inherent in the discrete-ordinate method of Chandrasekhar (1950). It would seem desirable to supplement these algorithms by some pilot models whose solutions can be given with as little recourse to numerical approximations as possible, exact solutions in analytical form being the optimum.

To construct such pilot models, spherical transfer problems are here formulated in terms of Fredholm equations the kernels of which do not require numerical integration for their construction.

This approach has been used in the treatment of homogeneous spheres and shells where the "sphericity" of the problem enters only through the shape of the boundaries while the absorption coefficient itself is not spherically stratified. Heaslet and Warming (1965) and Gruschinske and Ueno (1971) among others have determined

* Work completed while on leave of absence at the Department of Astronomy, University of Washington.

solutions for this case by iterative methods. The absence of known integral kernels for inhomogeneous media has prevented the extension of such procedures.

We present here an investigation into some cases of absorption coefficients that vary with radius in a manner simple enough to permit a rigorous treatment by means of Fredholm equation techniques. Although this requires a high degree of idealization over realistic atmospheres, the study has several purposes to fulfill. First, as yet no exact solutions to transfer problems of spherical symmetry, with surface conditions appropriate to the description of extended stellar atmospheres, have been published; the exact solutions to be developed here will therefore be of use in determining the efficiency of algorithms like the ones mentioned above. Second, the dependence of the solutions on a few parameters well defined in terms of the geometrical characteristics of the atmosphere will permit a clear insight into the extent to which these characteristics influence the redistribution of photons as compared to the plane-parallel situation.

This paper treats absorbing spheres of finite surface radius R. The absorption coefficient varies with the radial coordinate r according to $\kappa\rho = c_1(r^2 \pm c_2^2)^{-1/2}$, where c_1 and c_2 are positive constants. Fredholm equation kernels are given for this general ansatz for $\kappa\rho$, while exact solutions can be presented only for the subset $c_2 = 0$. This special case corresponds to a constant rate of change of the optical depth (measured radially inward) with relative change in radius; therefore, those aspects of the transfer problem which derive from the sphericity of the geometry should not (even asymptotically for large optical depths) approach the plane-parallel situation in the deep interior of the atmosphere. For $c_2 \neq 0$, solutions must be constructed with the help of approximate methods (in analogy to plane-parallel slabs of absorptive material); for the simple expressions for the Fredholm kernels pertaining to this general form of $\kappa\rho$, such methods are readily available. Such solutions, although useful in the discussion of spherical shells surrounding transparent or opaque cores, will not be presented here.

To discuss the effect of geometry on radiative transfer, we here define the "degree of sphericity" s at each point in the atmosphere by $s = \ln (R/r)/\tau(r)$, where $\tau(r)$ is the optical depth at r, measured radially inward from the surface. Large values of s indicate a large influence of spherical geometry. The subset $c_2 = 0$, treated in the sequel, describes opacity laws of a special nature, namely, $s = \text{const}$. It therefore presents aspects not common to all spherical transfer problems; for example, the power laws $\kappa\rho \propto r^{-n}$ with $n > 1$, discussed previously, lead to some results quite different from ours. From this it should not be concluded, however, that the following results are limited to this subset. In particular, the fact that isotropy of the radiation field is not achieved even in the regions closest to the center holds true for a much wider class of opacity laws, among them those giving the sphere a finite optical diameter. Laws leading to infinite optical diameters cannot *a priori* be assumed to entail radiation isotropy. Thus, especially in cases in which the radiation field itself influences the opacity law by determining the temperature of the transmitting gas, algorithms for computing the photon field should allow for some flexibility with respect to the choice of interior boundary conditions. Whenever s can be estimated to decrease to small values near the center, however, the geometrical aspects of transfer will approach those of the plane-parallel case rather than those presented here.

II. INTEGRAL REPRESENTATION: FREDHOLM EQUATIONS

The equation of transfer for a medium with spherical geometry is

$$\left(\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu}\right) I(r, \mu) = \kappa\rho[S(r) - I(r, \mu)] , \tag{1}$$

where the standard notations signify $I(r, \mu)$, intensity; r, radial coordinate; $\cos^{-1}\mu$, angle between direction of propagation and outward radius vector; $\kappa\rho$, volume absorption coefficient (possibly containing angle-independent scattering contributions); and $S(r)$, source function.

Since the procedure to include contributions to the radiative flux that are generated outside the medium is straightforward, such contributions will be neglected. The outer boundary condition for a spherical shell of outer radius R, inner radius a (possibly zero) is then

$$I(R, \mu) = 0 \qquad \text{for } -1 \leq \mu \leq 0\,. \tag{2}$$

The inner boundary condition will depend on the transmission properties of the central "hole," to be discussed later.

The formal solution of equation (1) is given by

$$I(r, \mu) = \int_0^{\tau_{r,\mu}(R)} S(\xi) \exp\,[-\tau_{r,\mu}(\xi)] d\tau_{r,\mu}(\xi)\,, \tag{3}$$

where $\tau_{r,\mu}(\xi)$ is the optical path length of a ray connecting points on radii r and ξ, respectively, and passing at angle $\cos^{-1}\mu$ through r:

$$d\tau_{r,\mu}(\xi) = \left|\frac{\kappa\rho d\xi}{(1 - r^2(1 - \mu^2)/\xi^2)^{1/2}}\right|\cdot \tag{4}$$

In the case that the source function contains the mean intensity J in a linear relationship, integration of equation (3) over μ leads to an integral equation of Fredholm type for J, the solution of which permits the calculation of $I(r, \mu)$ from equation (3) as a second step.

For $\kappa\rho$ given, to obtain such a Fredholm equation involving a kernel of analytical form, the integration of equation (4) and the subsequent one over μ have to be performed analytically. This is possible for functions of the form

$$\begin{aligned} \kappa\rho &= \frac{m}{(r^2 + \sigma a^2)^{1/2}} && \text{for } a^* \leq r \leq R \\ &= 0 \text{ or } \infty && \text{for } 0 \leq r < a^* \\ &= 0 && \text{for } r > R\,, \end{aligned} \tag{5}$$

to be studied in the sequel. Here, m and a are arbitrary positive numbers; σ can take the values 0, ± 1; and $a^* = 0$ for $\sigma = 0, +1$; $a^* = a$ for $\sigma = -1$. In equation (5), the case $a = 0$ corresponds to a constant degree of sphericity, $s = 1/m$, discussed above; for $a \neq 0$, the sign of σ determines whether the sphericity increases or decreases inward. The two problems investigated in earlier work are contained in equation (5) as limiting cases: $m \to \infty$ leads to the plane-parallel situation, $\sigma a^2 \to +\infty$ with m/a fixed to the homogeneous sphere. For $\sigma = -1$ we have a shell of finite optical thickness that surrounds a core either transparent or opaque.

From equations (4) and (5) follows the optical path length

$$\tau_{r,\mu}(\xi) = m|\log\langle\{(\xi^2 + \sigma a^2)^{1/2} + [\xi^2 - r^2(1 - \mu^2)]^{1/2}\}/[(r^2 + \sigma a^2)^{1/2} + r|\mu|]\rangle| \tag{6}$$

if ξ is the point closest to the center along the μ-ray from ξ to r; a similar expression holds if it is not.

If R is chosen as the unit of length, then the intensities are, in the region $r \leq 1$,

$$I^-(r, \mu) = m \int_r^1 [A(\xi) - B(\xi)]^m C_m(\xi, r, \mu) S(\xi) d\xi \qquad \text{for } -1 \leq \mu \leq 0\,,$$

$$I^+(r, \mu) = m \int_\zeta^r [A(\xi) + B(\xi)]^m C_m(\xi, r, \mu) S(\xi) d\xi$$

$$+ mF(\mu, \mu_0) \int_\zeta^1 [A(\xi) - B(\xi)]^m C_m(\xi, r, \mu) S(\xi) d\xi \qquad \text{for } 0 \leq \mu \leq 1\,, \quad (7a)$$

where $A(\xi) = (\xi^2 + \sigma a^2)^{1/2}$, $B(\xi) = [\xi^2 - r^2(1 - \mu^2)]^{1/2}$, $C_m(\xi, r, \mu) = \xi/\{[A(r) + r\mu]^m A(\xi) B(\xi)\}$, $\zeta = \text{Max}\,[r(1 - \mu^2)^{1/2}, a^*]$, and $\mu_0 = (1 - a^2/r^2)^{1/2}$. The factor $F(\mu, \mu_0)$ distinguishes the case of an opaque core from the others: $F(\mu, \mu_0) = 1$ except for values $\mu > \mu_0$ in the opaque-core case where $F(\mu, \mu_0) = 0$.

For the region $r > 1$ the upper limit of the first integral contributing to $I^+(r, \mu)$ has to be replaced by 1, and

$$I^-(r, \mu) = 0\,. \quad (7b)$$

The splitting into two terms of the intensity for positive μ reflects the fact that in a spherical situation part of the "far side" of the sphere is seen.

Quite generally, integration of equation (7a) over μ yields

$$W(\tau) = \int_0^{\tau_c} G(t, \tau) S[\xi(t)] \xi(t) dt\,, \quad (8)$$

where we define $W(\tau) = r(\tau) J[r(\tau)]$; τ_c is the radial optical depth of a point on the inner surface, and τ and t are the radial optical depths of r and ξ, respectively.

For the case of integer values of m the integration of equation (7a) is straightforward and leads to kernels $G(t, \tau)$ of the form

$$G(t, \tau) = K(|t - \tau|) - \sigma K(2\tau_c - t - \tau) \quad (9)$$

in all cases under consideration here except the opaque core where

$$G(t, \tau) = K(|t - \tau|)\,. \quad (10)$$

The function K is defined by

$$K(z) = -\tfrac{1}{2} \log (1 - e^{-2z/m}) \quad (11a)$$

for $m = 1$;

$$K(z) = -\tfrac{1}{2} \log (1 - e^{-2z/m}) - \sum_{l=1}^{(m-1)/2} \frac{e^{-2lz/m}}{2l} \quad (11b)$$

for $m > 1$, odd; and

$$K(z) = -\tfrac{1}{2} \log [(1 - e^{-z/m})/(1 + e^{-z/m})] - \sum_{l=1}^{m/2} \frac{e^{-(2l-1)z/m}}{2l - 1} \quad (12)$$

for m even. In passing we note that similar kernels are obtained when $S(r)$ is generalized to contain simple phase functions as factors (Rayleigh scattering).

Inserting

$$S(r) = \lambda_1 J(r) + \lambda_2 f(r) \quad (13)$$

(with λ_i constants, $f(r)$ a given function), and the appropriate form of $G(t, \tau)$ into equation (8), we obtain the Fredholm equation for $W = rJ$, homogeneous or inhomogeneous depending on whether λ_2 is zero or not.

As long as τ_c has a finite value, kernel (9) can be reduced to (10), depending on the variable $|t - \tau|$ only, by noting the significance of the quantity $2\tau_c - t - \tau$. The upper limit of integration in equation (8) then has to be changed from τ_c to $2\tau_c$, and the source function in the region $\tau_c \leq t \leq 2\tau_c$ to be defined (for $0 \leq v \leq \tau_c$) by

$$\xi(\tau_c + v)S[\xi(\tau_c + v)] = -\sigma\xi(\tau_c - v)S[\xi(\tau_c - v)]\,. \tag{14}$$

III. EXACT SOLUTIONS. DISCUSSION

Modern procedures of constructing approximate solutions to equations of the type (8) are readily available (e.g., Bellman 1968). Exact solutions, to be presented here, can be obtained when equation (8) is reducible to an integral equation of Wiener-Hopf type (Hopf 1934). This is possible for $\tau_c \to \infty$, or, equivalently, $a \to 0$, $\sigma = 0$, when the second term in kernel (9) vanishes.

The inclusion of other sources in $S(r)$ being straightforward, we restrict this investigation to source functions of the form

$$S(r) = J(r) + T(0)\,, \tag{15}$$

where $T(0)$ is a singular source of luminosity L, placed at the center.[1] To solve the resulting Fredholm equation, we follow Morse and Feshbach (1953) in method and nomenclature, searching first for the Fourier transform $F\{\Psi_+\}$ of $\Psi_+(x)$. Here, we use the definitions $x = \tau/m$, and $\Psi_+(x) = W(mx)$ for $x \geq 0$, $\Psi_+(x) = 0$ for $x < 0$. In the sequel, we sketch the steps in the calculation for odd values of m; for m even, they are entirely analogous.

The Fourier transform of kernel (11) is found to be

$$V(k) \equiv F\{K(|mx|)\} = (2\pi)^{-1/2}\left[\frac{\pi}{2k}\coth\frac{\pi k}{2} - \frac{1}{k^2} - 2E(m)\sum_{l=1}^{(m-1)/2}\frac{1}{4l^2 + k^2}\right], \tag{16}$$

with $E(1) = 0$, $E(m > 1) = 1$, k being the complex Fourier variable. Hence we have to factorize the quantity

$$Y(k) = Y_+(k)/Y_-(k) = 1 - (2\pi)^{1/2}mV(k)\,, \tag{17}$$

analytic in the strip $|\text{Im}\,k| < m + 1$ of the k-plane, to obtain

$$F\{\Psi_+(x)\} = P(k)/Y_+(k)\,, \tag{18}$$

where $P(k)$ is a polynomial of degree less than $Y_+(k)$.

Since the only zeroes of $Y(k)$ occur at $k = \pm i$, of single order, we have

$$Y_+(k) = \frac{k^2 + 1}{k + (m + 1)i}\exp(-q_+) \tag{19}$$

with

$$q_+(k) = \frac{1}{2\pi i}\int_{-\infty - i\beta}^{\infty - i\beta}\log\left\{\frac{\eta^2 + (m + 1)^2}{\eta^2 + 1}[1 - (2\pi)^{1/2}mV(\eta)]\right\}\frac{d\eta}{\eta - k} \quad (\beta < m + 1)\,. \tag{20}$$

[1] As pointed out by a referee, the following mathematics is contained in Aamodt (1962).

This shows that $Y_+ \to k$ for $k \to \infty$, which requires $P(k)$ to be a constant, Λ, determined by the luminosity L of the central source. Establishing the relation between L and Λ and Fourier-retransforming $F\{\Psi_+\}$ completes construction of the exact solution.

The mean intensity at large optical depths will be governed by small values of Re k in the Fourier transform. Expanding equations (19) and (20) near $k_r = \text{Re}\, k = 0$, we get (with Im $k = 1$)

$$F\{\Psi_+(x \to \infty)\} \to \Lambda \frac{m+2}{2k_r} \exp(q_0) \tag{21}$$

with

$$q_0 = \frac{1}{\pi} \int_0^\infty \log \left\{ \frac{\eta^2 + (m+1)^2}{\eta^2 + 1} [1 - (2\pi)^{1/2} m V(\eta)] \right\} \frac{d\eta}{\eta^2 + 1}, \tag{22}$$

and thus

$$\Psi_+(x \to \infty) \to -i\Lambda(m+2)(\pi/2)^{1/2} \exp(q_0 + x); \tag{23}$$

that is,

$$J(r \to 0) \to \text{const.}/r^2, \tag{24}$$

a result that conforms with the first-order moment of the transfer equation,

$$\frac{dK}{dr} + \frac{1}{r}(3K - J) = -\kappa\rho \frac{L}{(4\pi r)^2}, \tag{25}$$

in the limit of a constant ratio K/J, where K is the second-order moment of the intensity.

Inserting expression (24) into equation (7a) and computing the moments for $r \to 0$ leads to the exact expressions for the asymptotic values of their ratio:

$$K/J = 1 + m^3 \left[2 - \tfrac{1}{4}\pi^2 + E(m) \sum_{l=1}^{(m-1)/2} \frac{4}{(4l^2 - 1)^2} \right] \quad \text{for } \tau \to \infty. \tag{26}$$ [2]

The quantity J/K is a good measure for the anisotropy of the radiation field, the isotropic value being 3. We see from equation (26) that even for arbitrary large optical depths, J/K does not attain this limiting value in the cases under study here. This is all the more true for $a \neq 0$, ($\sigma = +1$), when the absorption coefficient at the center remains finite. With increasing m, J/K does of course approach 3 for $\tau \to \infty$, as can be seen in fig. 1 where the τ-dependence of this quotient for different degrees of sphericity $1/m$ is compared to the plane-parallel case. Asymptotic values, equation (26), are attached to the curves. The curves resemble each other closely after application of scale factors equal to these asymptotic values. The figure makes plausible the success of Eddington's approximation in plane geometry; to a similar extent an Eddington approximation will be useful for the present cases of spherical geometries after modification by these scale factors. It should be noted that the optical depths around which the radiation field attains its asymptotic degree of anisotropy are rather independent of the degree of sphericity.

[2] We use

$$\sum_{l=1}^{\infty} 4/(4l^2 - 1)^2 = \pi^2/4 - 2, \quad \text{and} \quad \sum_{l=(m+1)/2}^{\infty} 4/(4l^2 - 1)^2 \to 2/(3m^3)$$

for $m \to \infty$ to obtain $K/J \to \frac{1}{3}$ for $m \to \infty$.

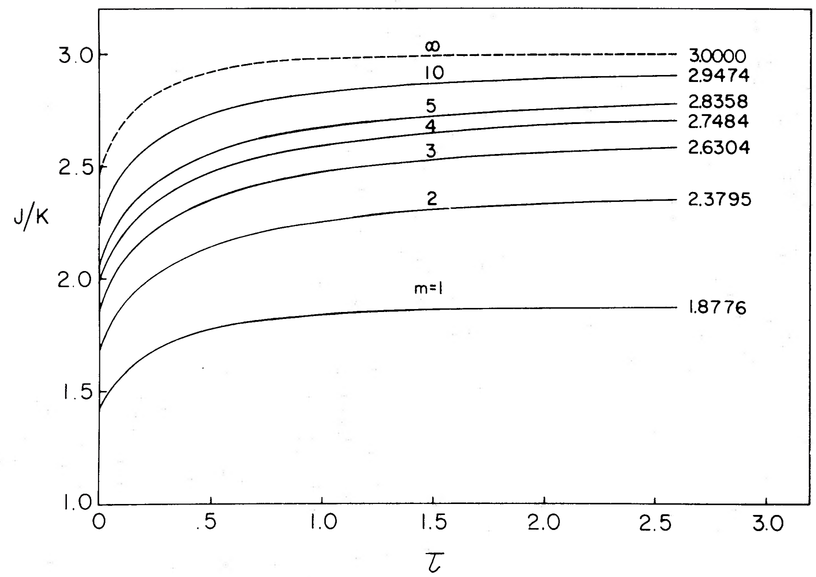

FIG. 1.—J/K, the measure of the anisotropy of the radiation field, for $\kappa\rho = m/r$, as a function of optical depth for different degrees of sphericity, $1/m$. The plane-parallel limit is given by the broken line. Exclude the point $r = 0$ as the position of the singular source.

Equations (25) and (26) determine the relation between Λ and L, so that the solutions are completely determined. We get

$$L/\Lambda = i(m + 2)m^2(2\pi)^{5/2}\left[4 + E(m)\sum_{l=1}^{(m-1)/2}\frac{8}{(4l^2 - 1)^2} - \frac{\pi^2}{2}\right]\exp(q_0)\,, \quad (27)$$

which is equivalent to the Hopf-Bronstein relation.

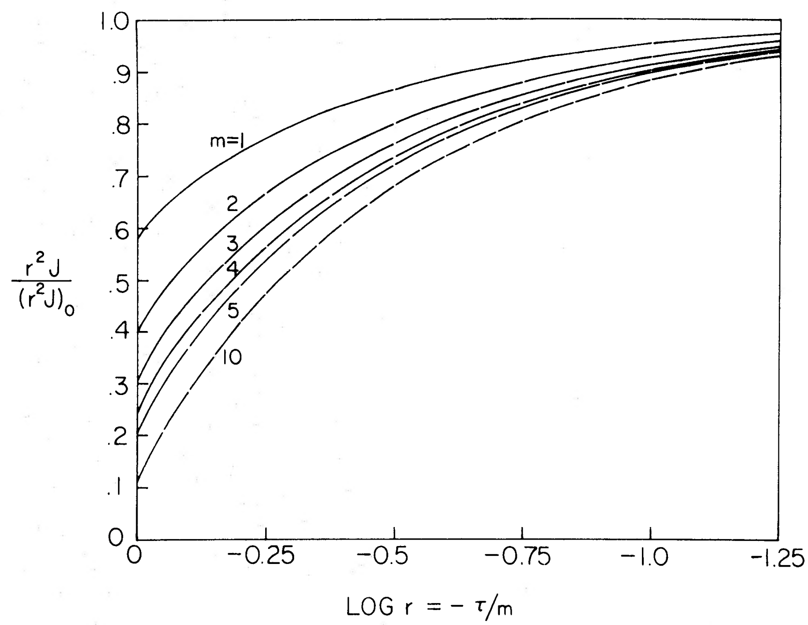

FIG. 2.—The quantity r^2J, normalized to its value at small radii, versus the natural logarithm of radius in units of surface radius. Difference in optical depth between neighboring markers on curves is 1/2.

In figures 2–5 the results are presented by comparing models of different degrees of sphericity but of the same luminosity (and outer radius). Figure 2 shows the quantity $p(r)$, with

$$J(r) = \text{const.}\, p(r)/r^2\,, \tag{28}$$

where the constant is chosen such that $p(r \to 0) \to 1$. The plane-parallel equivalent to $p(r)$ is the expression $\tau + q(\tau)$ in Hopf's solution. Obviously, a representation linear in τ does not approximate $p(r)$ satisfactorily over any larger portion of optical depth; for small values of m, i.e., high degrees of sphericity, this representation is seen to be particularly weak. To obtain a more realistic approximation, we note that $q_+(k)$ in equation (20) is a slowly varying function of k which we can tentatively set equal to a constant. The ensuing expression for Y_+ then leads to an expression of the form

$$J(\tau) \sim \text{const.} \left(1 - \frac{m}{m+2}\, e^{-2\tau/m}\right) \Big/ r^2(\tau)\,, \tag{29}$$

an approximation that improves with increasing degree of sphericity (and is correct for $m = 0$!). Mutliplying the second term in the bracket by a factor that reproduces

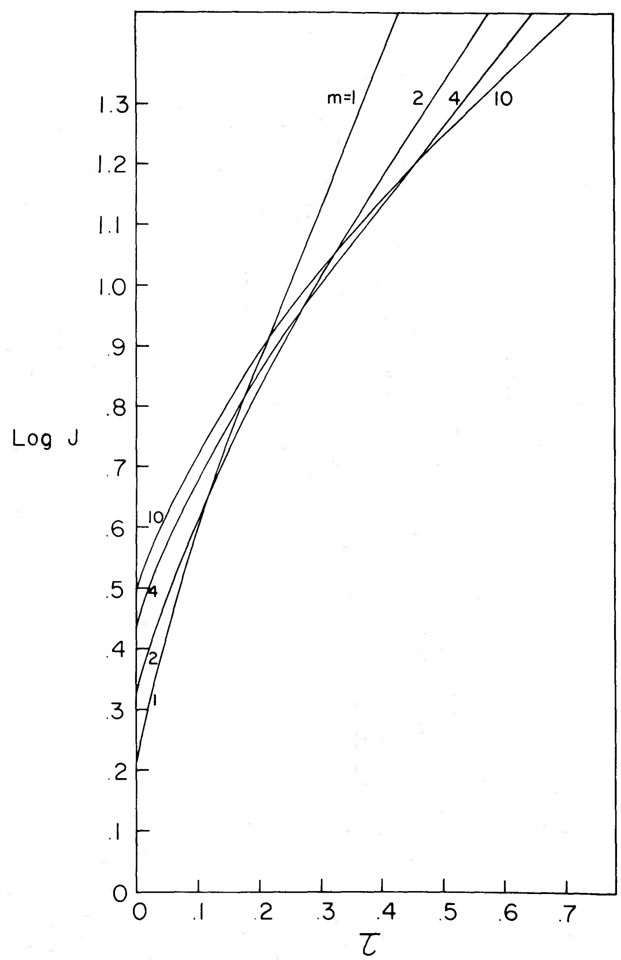

FIG. 3.—Natural logarithm of the mean intensity versus optical depth for different values of m. All curves correspond to the same values of luminosity and of surface radius.

J on the surface exactly, we arrive at a formula that nowhere deviates from the correct solution by more than about 4 percent ($m \leq 10$).

As the models become more and more "spherical," the surface value of the mean intensity decreases as does the spatial gradient of J; the opposite is true for the τ-gradient of J (fig. 3). This can be interpreted in terms of the cone from which any point in the atmosphere receives a considerable contribution to its flux: the solid angle of this cone decreases with increasing degree of sphericity, and therefore the strength of the sources within this cone has to go up concurrently in order to provide the required luminosity.

Figure 4 details the angular dependence of the intensity at different optical depths for the case $m = 3$. Since the ratio J/K always remains below the isotropy value, the intensity shows a marked peak at $\mu = +1$ even at arbitrarily large optical depths. The asymptotic angular distribution of photons is essentially attained around $\tau = 2$. The choice of our models as being of constant degrees of sphericity throughout leads to the existence of one value of μ, for each value of m, at which the relative intensity remains equal to unity, independent of optical depth, as can be seen in figure 4. This angle is well suited to define the cone mentioned above.

The laws of limb darkening for models with different values of m are demonstrated in fig. 5. The strong forward peaking for extended atmospheres, as well as the transition to the plane-parallel case, is obvious. With regard to the transfer equation (1), the only difference between spherical and plane-parallel geometries appears in the term $(1 - \mu^2)(\partial I/\partial \mu)/r$ which should be most important around $\mu = 0$. The increase, for small μ, of this term as m approaches the plane-parallel limit can be seen in figure 5. In the vicinity of $\mu = 0$, the surface intensity is shown to vary like $m\mu J$.

The computations were done at the computer facilities of the Universities of Heidelberg and Washington and supported in part by NSF grant GP-19573. The hospitality of the University of Washington, Department of Astronomy, is gratefully acknowledged.

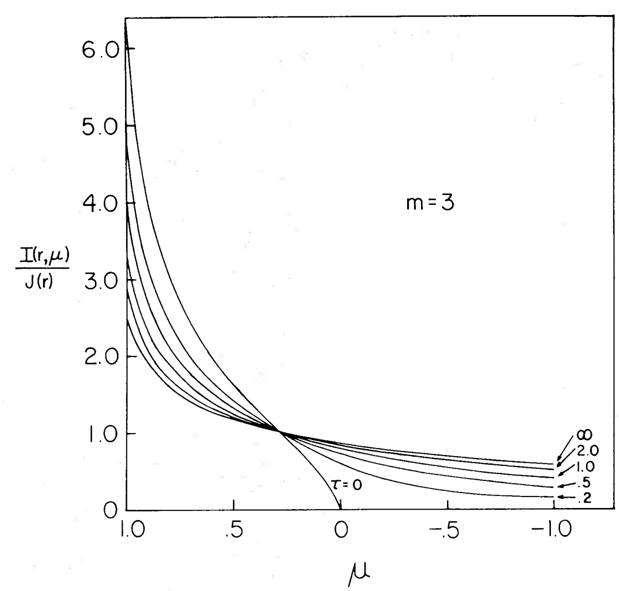

FIG. 4.—Angular dependence of intensity, in units of mean intensity, at different values of optical depth, for the case $m = 3$. Note the "cone angle" $\mu = 0.286$.

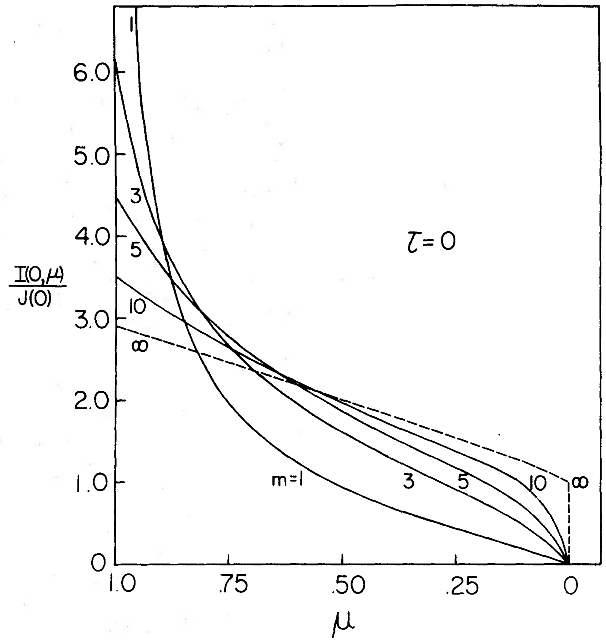

FIG. 5.—Center-to-limb darkening for different degrees of sphericity: surface intensity, in units of mean intensity, versus μ.

REFERENCES

Aamodt, R. E. 1962, thesis, University of Michigan.
Bellman, R. E. 1968, *Computing*, **3**, 131.
Cassinelli, J. P. 1971, *Ap. J.*, **165**, 265.
Chandrasekhar, S. 1950, *Radiative Transfer* (London and New York: Oxford University Press).
Chapman, R. D. 1966, *Ap. J.*, **143**, 61.
Gruschinske, J., and Ueno, S. 1971, to be published.
Heaslet, M. A., and Warming, R. F. 1965, *J. Quant. Spectrosc. and Rad. Transf.*, **5**, 669.
Hopf, E. 1934, *Mathematical Problems of Radiative Equilibrium* (Cambridge: Cambridge Mathematical Tract No. 31).
Hummer, D. G., and Rybicki, G. B. 1971, *M.N.R.A.S.*, **152**, 1.
Morse, P. M., and Feshbach, H. 1953, *Methods of Theoretical Physics* (New York: McGraw-Hill Book Co.).
Schmid-Burgk, J. 1969, unpublished thesis, University of Heidelberg.