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ADVANCED EVOLUTION OF MASSIVE STARS. I. HELIUM BURNING

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ABSTRACT

This is the first in a series of papers which will describe a survey of the later stages of evolution of massive stars, beginning at helium burning and proceeding through the final hydrodynamic stages. In this paper, (1) the motivation and astrophysical background for the survey are developed and discussed, (2) the methods and approximations to be used are described, (3) the initial models for the survey are developed, (4) the techniques to be used are compared with other investigations of helium stars, (5) the "single-star" approximation is examined by explicit comparison with published evolutionary sequences of core helium burning, (6) the relation between helium core mass M_α and total mass M is calibrated, and (7) helium-burning nucleosynthesis is reexplored using a revised rate for $3\alpha \rightarrow {}^{12}C$ and various values for the rate of ${}^{12}C(\alpha, \gamma){}^{16}O$. The latter results are fitted by a simple function of M_α and of θ_α^2 for the 7.12-MeV level in ${}^{16}O$.

I. INTRODUCTION

This is the first paper in a series dealing with the thermonuclear evolution of massive stars, beginning at helium burning and proceeding through the final hydrodynamic stages. Section II contains a discussion of the theoretical framework within which the investigation will be conducted. Sections III and IV describe in some detail the input physics and certain mathematical constraints used in constructing the evolutionary models, with emphasis on the first (helium burning) stage. Section V will present the results of evolutionary calculations of helium burning, with emphasis on comparison with previous work, and on aspects of importance for nucleosynthesis theory and for further evolution.

II. THEORETICAL FRAMEWORK

Historically the observational testing of evolutionary models of stars has primarily involved a comparison of the theoretical and observational H-R diagrams. Massive stars are rare and evolve swiftly; observational H-R diagrams emphasize stars which are numerous and evolve slowly. Nevertheless, massive stars may be of more importance to astrophysics than their small numbers would suggest. Their large luminosities imply large rates of nucleosynthesis; they may be the primary site of nucleosynthesis in a galaxy. If this is true, then the observed nuclear abundances (which seem similar to those of the solar system for a surprising variety of locations) may provide important clues as to the evolution of massive stars.

This is the first in a series of papers which will be concerned with the evolution of massive stars, with emphasis on the thermonuclear aspects of that evolution. Of particular interest are the final state of these stars and the composition of any matter ejected. It is hoped that comparison of these theoretical models with observation will provide new insight into nucleosynthesis and the evolution of massive stars.

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Such a program is limited by the theory available, by the quality of experimentally determined parameters which must be used, and, in this case, by the computer time available. As befits an initial survey of a complex problem, this investigation will use simple, standard techniques whenever they do not appear to distort the essential physics of the situation. Besides requiring less time and money, such an investigation is easier to interpret and can serve as a first "iterative" step, to be followed by a more sophisticated analysis. Of course, a considerable effort has been made to keep the models as realistic as possible within the framework of these constraints.

The thermonuclear evolution through each burning stage must be considered carefully. At a given instant the energy-generation rate is of primary importance, but the resulting *abundances* are vital for a correct treatment of subsequent evolution. In the general case errors can accumulate and can even be amplified. As a massive star evolves, more and more nuclear reactions become possible during each burning stage. The limit of this increasing complexity is a thermal-equilibrium state in which the use of statistical mechanics makes the theory simple again. Unfortunately we will be concerned primarily with the difficult intermediate regime, and must consider carefully the complexity of the reaction kinetics. By using the evolutionary properties of realistic nuclear reaction networks (computed for simplified astrophysical situations), it is possible to define useful "abbreviated" (or "equivalent") networks. These very small networks are designed to approximate the evolution of major abundances reasonably well, but be swiftly soluble by computer. Using such approximate networks with a stellar evolutionary code gives a first guess at the density-temperature (ρ-T) history during that burning stage (the adequacy of the thermonuclear approximations can then be checked by evolving a realistic reaction network over the same ρ-T history).

In order to complete the program outlined above it will be necessary to evolve stellar models from a stage in which the hydrostatic approximation is valid (i.e., mechanical inertia neglected) to a stage in which the hydrodynamic motion is violent (for example, explosion or gravitational collapse). This poses no small difficulty. By its very nature the hydrostatic approximation suppresses hydrodynamic behavior. Further, any implicit scheme of numerical hydrodynamics which can take time steps large compared to the local sound-travel time must also suppress hydrodynamic modes to some extent. There is a fundamental problem here: models evolved to the onset of hydrodynamic instability with the hydrostatic approximation may be superficially plausible, yet completely misleading. Considerable care must be taken in joining hydrostatic and hydrodynamic computational schemes.

Initially we are concerned with the hydrostatic case. Following the simple and most common approach, we assume spherical symmetry. The numerical technique was chosen so that in subsequent hydrodynamic stages the *same* physics (e.g., equation of state, opacity, nuclear reactions, convection theory, etc.) and the *same* difference equations could be used to the maximum extent possible. The explicit scheme of numerical hydrodynamics which was chosen is similar to that of Christy (1964). For consistency in zoning, the hydrostatic scheme of Henyey *et al.* (1959), but without the special variables, was adopted. These references provide a general basis for constructing and solving difference equations; they describe some of the basic mathematical methods used, but *not* the input physics.

A considerable reduction in the effort expended can be obtained if we consider only that part of the star which has been processed by hydrogen burning. Hydrogen burning is the most exhaustively investigated aspect of stellar evolution and as such hardly needs to be repeated here. After core hydrogen burning we have an evolved core and a hydrogen-rich envelope. This envelope can be neglected if we can represent the hydrogen-burning shell by imposing appropriate boundary conditions on the evolved core. The pressure and temperature at the hydrogen-burning shell are so small compared to the values in the core that they will be considered to be zero. There are two possible problems with such a "boundary condition" procedure, however.

First, the hydrogen-burning shell processes matter so that the mass of the core is not constant. After core hydrogen burning a massive star spends most of its remaining time in core helium burning. Let us estimate the change in mass of the helium core during this stage. If the average luminosity due to hydrogen burning in the core-hydrogen-burning stage is denoted by $L_H(H)$, then the lifetime of core hydrogen burning is

$$t_H = X_H Q_H M_\alpha / L_H(H) , \tag{1}$$

where Q_H is the energy release per unit mass of hydrogen consumed, X_H is the hydrogen abundance, and M_α is the mass of the helium core at hydrogen depletion. Similarly, during core helium burning, we have

$$t_\alpha L_H(\alpha) = X_H Q_H \delta M_\alpha , \tag{2}$$

where $L_H(\alpha)$ is the average luminosity due to hydrogen burning in the core-helium-burning stage. We have

$$\delta M_\alpha / M_\alpha = [L_H(\alpha)/L_H(H)][t_\alpha/t_H] , \tag{3}$$

but Stothers and Chin (1968) find for masses in the range $M = 15$–$100\ M_\odot$ that $L_H(\alpha) \simeq L_H(H)$ and $t_H/t_\alpha \simeq 9$, so $\delta M_\alpha/M_\alpha \simeq 0.1$. In a preliminary investigation such as this, such a small change may be ignored, provided M_α is not near some critical mass. Accordingly we must restrict our investigation to $M_\alpha \gg 1.45\ M_\odot$ (the Chandrasekhar limit) if the constant-core-mass approximation is to be valid. This implies a total stellar mass of $M \gtrsim 9\ M_\odot$ or so. The evolution of less massive stars has already been considered in some detail (see Arnett 1969, 1971*a;* Paczyński 1970*a*, *b;* and references therein).

A second possible problem with this "boundary condition" procedure involves the intrusion of the surface convection zone inside the hydrogen-burning shell. Stothers and Chin (1968) found that such a process would reduce the core mass for massive stars ($M > 15\ M_\odot$) for late evolutionary stages. In obtaining this result they assumed that the radiative flux is constant in a chemically homogeneous, radiative zone. Sugimoto (1970*a*) has pointed out that this approximation breaks down when neutrino loss is taken into account because then the radiative time scale for the base of the envelopes is no longer negligible with regard to the evolutionary time scale for the core. Sugimoto (1970*b*) finds that consideration of this effect suggests that the core-envelope mixing is small during carbon burning and probably negligible in later stages, at least for stars of $M/M_\odot = 12$ and 30 ($M_\alpha/M_\odot = 3$ and 10). Also, although Paczyński (1970*a*, 1971*a*) found such mixing for $M/M_\odot = 7$, he did *not* find it for $M/M_\odot = 10$ and 15. As a first guess, subject to revision, we assume that core-envelope mixing may be neglected for massive stars.

The foregoing discussion suggests that the advanced evolution of massive stars may be investigated by evolving initially pure helium "stars" with zero boundary conditions at the "surface." There are two particular advantages in such an approach. First, since the evolution of pure helium stars (and of helium burning in massive Population I stars) has been explored in detail by several investigators, it can be ascertained how well the computational techniques employed here will reproduce the standard results for a standard case. The importance of such checking of computer codes to be used to investigate new and exotic situations cannot be overemphasized. Second, the nucleosynthesis occurring in core helium burning can be reexamined in the light of new nuclear data for the triple-alpha reaction and of continued uncertainty in the rate of $^{12}C(\alpha, \gamma)^{16}O$. A discussion of some of the implications of the evolutionary models to be presented here has already appeared (Arnett 1971*b*).

III. MICROSCOPIC INPUT PHYSICS

a) Equation of State

Except where stated otherwise, the equation of state used in this investigation has three components: (1) a blackbody radiation gas, (2) a Maxwell-Boltzmann gas of "noninteracting" ions, and (3) a positron-electron gas of noninteracting fermions. The positron-electron component was included in tabular form; variable order interpolation (second and third) was used. As the complexities of the equation of state are irrelevant for helium burning, a more detailed discussion will be deferred until more advanced stages of evolution are considered. See also Arnett (1969).

b) Opacity

The conditions to be explored involve high temperatures, and low to moderate densities. Even under conditions more favorable for the process than these, electron conduction is not important (Arnett 1969). The dominant source of opacity is Compton scattering. Since we begin at helium burning, and neutrino energy loss dominates above $T_9 > 0.5$ ($T_m \equiv T/10^m$ ° K), we are interested in the range $0.05 \lesssim T_9 \lesssim 0.5$. Following Sampson (1959), the opacity for electron scattering in this range is

$$\kappa_s = \kappa_{\rm Th}/(1 + 2.2T_9) , \qquad (4)$$

where $\kappa_{\rm Th}$ is the Thomson opacity. In order to approximately correct for the small contribution from bound-free and free-free interactions, the expression

$$\kappa_k = 75.3\left(\sum_i Z_i^2 Y_i\right)\frac{\rho}{\mu_e}(T_6)^{-3.5} \qquad (5)$$

was used (cgs units). Here Z_i is the proton number of species i, is the number of nucleons per electron, and Y_i is related to the number density N_i by

$$N_i = \rho \mathfrak{A} Y_i , \qquad (6)$$

where $\mathfrak{A}$ is Avogadro's number and ρ is the mass density. The "mass fraction" X_i is just $Y_i A_i$, where A_i is the nucleon number of species i. The numerical factor in expression (5) comes from examining the tables of Cox, Stewart, and Eilers (1965) in the appropriate range of ρ and T. This choice of κ_k appears to have no significant effect on this investigation.

c) Neutrino Radiation

The analytic fitting formulae for energy loss by neutrino radiation of Beaudet, Petrosian, and Salpeter (1967) were used throughout. Neutrino radiation from nuclear processes will be discussed explicitly as part of the thermonuclear evolution as the need arises. Neutrino radiation was unimportant during helium burning.

d) Nucleosynthesis

The equations governing the change in abundance of the dominant nuclei during helium burning may be written as:

$$\dot{Y}_\alpha = -\tfrac{1}{2}[\alpha\alpha\alpha]Y_\alpha^3 - Y_\alpha(Y_{\rm C}[^{12}{\rm C}\alpha]_\gamma + Y_0[^{16}{\rm O}\alpha]_\gamma) , \qquad (7)$$

$$\dot{Y}_{\rm C} = \tfrac{1}{6}[\alpha\alpha\alpha]Y_\alpha^3 - Y_\alpha Y_{\rm C}[^{12}{\rm C}\alpha]_\gamma , \qquad (8)$$

$$\dot{Y}_{\rm O} = Y_\alpha(Y_{\rm C}[^{12}{\rm C}\alpha]_\gamma - Y_{\rm O}[^{16}{\rm O}\alpha]_\gamma) , \qquad (9)$$

and

$$\dot{Y}_{\rm Ne} = Y_\alpha Y_{\rm O}[^{16}{\rm O}\alpha]_\gamma , \qquad (10)$$

where Y_i is defined in equation (6), the reaction rates are specified in the notation of Fowler, Caughlan, and Zimmerman (1967), and $^{20}\mathrm{Ne}(\alpha, \gamma)^{24}\mathrm{Mg}$ is assumed to be negligible (an excellent approximation, see below). Recent experimental results of Professor C. A. Barnes suggest a revision of the rate of the triple-alpha reaction (W. A. Fowler, private communication):

$$[\alpha\alpha\alpha] = 2.13 \times 10^{-5} \rho^2 (T_8)^{-3} \exp(-44.10/T_8)\ \mathrm{s}^{-1} \quad (11)$$

(see also Austin, Trentelman, and Kashy 1971). This revision reduces the rate at any given temperature. The rate $[^{12}\mathrm{C}\alpha]_\gamma$ for the reaction $^{12}\mathrm{C}(\alpha, \gamma)^{16}\mathrm{O}$ is taken from Fowler *et al.* (1967); in evaluating this expression a value for the reduced alpha width, in the 7.12-MeV level in $^{16}\mathrm{O}$, of $\theta_\alpha^2 = 0.085$ was used. This width is quite uncertain at present, so the effect of using different values of θ_α^2 was explored (see Arnett 1971*b* for discussion and references). The reaction rate $[^{16}\mathrm{O}\alpha]_\gamma$ for $^{16}\mathrm{O}(\alpha, \gamma)^{20}\mathrm{Ne}$ has recently been reinvestigated by Toevs (1971; Toevs *et al.* 1971 [TFBL]). This differs from the older values used by Deinzer and Salpeter (1964, due to Reeves) by

$$\log_{10}\left(\frac{\text{old value}}{\text{TFBL value}}\right) \simeq 0.9 - 0.22(3/T_8)\ . \quad (12)$$

Now, from equations (9) and (10) it can be shown that little $^{20}\mathrm{Ne}$ will be produced unless

$$\frac{[^{16}\mathrm{O}\alpha]_\gamma}{[^{12}\mathrm{C}\alpha]_\gamma} > 0.1 \quad (13)$$

or so. Using the Toevs *et al.* (1971) value of $[^{16}\mathrm{O}\alpha]_\gamma$ and the Fowler *et al.* (1967), value of $[^{12}\mathrm{C}\alpha]_\gamma$, equation (13) is an equality at $T_8 \simeq 3.1$. By the time the central regions of a star can reach this temperature, however, $^4\mathrm{He}$ is almost entirely consumed, so little $^{20}\mathrm{Ne}$ should be formed. In the range $2 \lesssim T_8 \lesssim 3$, equation (12) gives

$$(\text{old value/TFBL value}) \simeq 4\ . \quad (14)$$

Using this and noting that equation (10) is linear in $[^{16}\mathrm{O}\alpha]_\gamma$, we can estimate the $^{20}\mathrm{Ne}$ production on the basis of the models of Deinzer and Salpeter (1964). At $M_\alpha = 100\ M_\odot$, the most massive star we will consider, the mass fraction of $^{20}\mathrm{Ne}$ would have been $X(^{20}\mathrm{Ne}) \simeq 0.07$. At $M_\alpha = 30\ M_\odot$, $X(^{20}\mathrm{Ne}) \simeq 0.03$ would have been obtained. These estimates agree with recent calculations of Vidal, Shaviv, and Koslovsky (1971). As will be seen later, such small abundances of $^{20}\mathrm{Ne}$ have no significant effect on subsequent evolution (this might not be true if $\theta_\alpha^2 \gg 0.085$, however). Therefore, we take $[^{16}\mathrm{O}\alpha]_\gamma$ to be small during helium burning.

In solving equations (7), (8), and (9), it must be ensured (1) that nucleons are conserved and (2) that abundances of nuclei go smoothly to zero as they are depleted. The latter requirement suggests that a differencing scheme which is partially implicit be used, at least for $^4\mathrm{He}$.

Define:

$$A = \tfrac{1}{2}[\alpha\alpha\alpha] Y_\alpha^2\ , \quad B = [^{12}\mathrm{C}\alpha]_\gamma Y_\mathrm{C}\ , \quad \text{and } C = 1 + \delta t(3A + B)\ , \quad (15)$$

where δt is the integration time step and the Y_i refers to abundances at the beginning of that time step. The reaction rates are evaluated with the best current guesses for ρ and T at the end of the time step, although centered values might be as good or better. Then

$$\frac{\delta Y_\alpha}{\delta t} = -Y_\alpha (A + B)/C\ , \quad (16)$$

$$\frac{\delta Y_{\mathrm{C}}}{\delta t} = Y_\alpha (A/3 - B)/C\,, \tag{17}$$

and

$$\frac{\delta Y_{\mathrm{O}}}{\delta t} = Y_\alpha B/C\,, \tag{18}$$

where the δY's are the changes in abundance over δt. This scheme of differencing is fast and stable; even though the ^{12}C equation (eq. [8]) was not differenced in a fully backward manner, no difficulties were encountered as Y_{C} went toward zero. Unless δt is chosen to be suspiciously large, the scheme is more than accurate enough. From the point of view of the mathematics of nucleosynthesis, helium burning is easy; many other schemes could be devised which would work.

This formalism conserves nucleons identically. Since $\Sigma_i X_i = \Sigma_i Y_i A_i = 1$, it must also be true that

$$\Sigma_i A_i \delta Y_i = 0\,, \tag{19}$$

which may be verified from equations (16), (17), and (18). In converting ^{4}He to ^{12}C, the energy released per gram of fuel consumed is $q_1 = 5.85 \times 10^{17}$ ergs g^{-1}, and for ^{12}C$(\alpha, \gamma)^{16}$O is $q_2 = 4.31 \times 10^{17}$ ergs g^{-1}. Therefore, the rate of energy generation may be written as

$$\epsilon = 4 Y_\alpha \frac{A}{C} q_1 + 16 Y_\alpha \frac{B}{C} q_2\,. \tag{20}$$

IV. MACROSCOPIC PHYSICS AND MATHEMATICAL CONSIDERATIONS

a) Stellar Evolutionary Equations

Sugimoto (1970*c*) has pointed out that some generalizations of the Henyey *et al.* (1959) method give rise to difficulties when advanced evolutionary stages (e.g., carbon burning) are investigated; he suggests a special formulation which avoids the difficulty. It appears that solution for the radiation flux L_r becomes unstable in the limit of rapid evolution (i.e., small time steps). Since such difficulties were also avoided in the formulation used here, a brief discussion of some aspects of this formulation may be of interest. Such discussion is also suggested by the unconventional treatment of convection. Let us denote a spatial difference by Δ and a time difference by δ. Then the evolutionary equations are

$$\Delta m - \frac{4\pi}{3}\frac{\Delta(r^3)}{V} = 0 \tag{21}$$

$$\delta E + P\delta V + \delta t\left(\frac{\Delta L}{\Delta m} - \epsilon\right) = 0\,, \tag{22}$$

$$\ddot{r} + \frac{Gm}{r^2} + A\frac{\Delta P}{\Delta m} = 0\,, \tag{23}$$

$$L - A(F + B) = 0\,, \tag{24}$$

where $A = 4\pi r^2$, $V = 1/\rho$. Further, we have

$$F = \frac{ac}{3}\frac{A}{\kappa}\frac{\Delta(T^4)}{\Delta m} \tag{25}$$

and

$$B = -u(\Delta E + P\Delta V)/V\,. \tag{26}$$

Here u is the "convective speed" (see below). Other symbols not defined above have their conventional meaning. The independent variables are T, ρ, r, and L—that is, temperature, density, radius, and luminosity. The luminosity includes a convective flux B, as well as a conductive and radiative flux F. With the acceleration $\ddot{r}$ set to zero and appropriate designation of B, these equations are essentially the standard equations of stellar structure. Composition change is computed in a parallel (coupled) calculation by the technique described previously. The equations are linearized (using "zone boundary–zone center" differencing like Christy 1964) and solved as described by Henyey *et al.* (1959).

It should be noted that setting $\ddot{r}$ to zero suppresses pulsational instability, which could occur for homogeneous helium stars of mass $M \gtrsim 13\ M_\odot$ (Stothers and Simon 1970). This is justified by the assumption that outside the helium region is a hydrogen-rich envelope, which would damp the instability.

b) *Treatment of Convection*

It is desirable to use a technique of calculating convective motions which (1) reduces to a conventional formulation during slow evolution and (2) avoids pathological fluxes during rapid evolution. This may be accomplished by properly defining the "convective speed" u (a scalar) in equation (26). By using the generalization of mixing-length convection theory discussed before (Arnett 1969, eq. [11], with the factor 2 in the right place), we have

$$\dot{u} = (\langle u_{\rm ML}\rangle^2 - u^2)/(2l)\,, \tag{27}$$

where l is the mixing length (taken to be one pressure scale-height) and $\langle u_{\rm ML}\rangle$ is the usual time-independent expression for the convective speed in mixing-length theory. Using a backward difference in time, an estimate for the new convective speed is

$$u^* = [y^2 + \langle u_{\rm ML}\rangle^2 + 2u(t)y]^{1/2} - y\,, \tag{28}$$

where $y = l/\delta t$ and $u(t)$ is the convective speed at the beginning of the time step. Convergence difficulties arise if the convective speed is reduced too abruptly; by taking the new estimated convective speed to be

$$u(t + \delta t) = 0.9u^* + 0.1u^{**}\,, \tag{29}$$

where u^{**} was $u(t + \delta t)$ for the previous iteration, this problem disappeared.

In helium burning, $\langle u_{\rm ML}\rangle \gg l/\delta t$ so $u^* \simeq \langle u_{\rm ML}\rangle$, and $u(t + \delta t)$ approaches u^* rapidly. Since $\langle u_{\rm ML}\rangle^2$ is proportional to $\Delta E + P\Delta V$, a nearly adiabatic gradient is set up when convection occurs (i.e., $\Delta E + P\Delta V$ approaches zero). The results are virtually identical to those obtained using the usual prescription in this case, as was desired. However, more than 7 significant figures are required in the arithmetic; the scheme worked well with 12 significant figures.

The technique just described is not intended to represent a solution to the difficult problem of time-dependent convection; it was devised primarily on the basis of its limit behavior, not on the basis of valid physics. Even within this conceptual model of convection a more realistic (but more complicated) technique could be devised. Nevertheless this technique is much to be preferred over the standard time-independent one for those last, rapid phases of evolution where the standard scheme does not "turn itself off," but predicts pathological fluxes (e.g., convective speeds greater than light and other nonsense).

Finally, one further aspect of the treatment of convection should be mentioned: it involves the question of whether, in the actual computer algorithm, the mixing of compositions occurs during or at the end of the iterative cycle. In the former case mathematical instabilities can occur (Iben 1965) which are related to the position of convective-radiative boundaries. This is a complex problem (Paczyński 1970*b*); for simplicity the compositions were mixed *after* the iterations were complete.

c) *Convergence Criteria and Time-Step Choice*

Of the four independent variables T, ρ, r, and L, the luminosity L is unique in that it refers to a flow of energy, and is therefore a time derivative. If L is not a directly observable quantity (as it is not for the problem considered here), its physical importance lies in the product $L\delta t$ (see eq. [22]). Therefore, while the last iterative corrections for T, ρ, and r were required to be less than a given fraction (10^{-4} or 10^{-5}) of their final value, the last iterative correction to L was required to be less than a fraction of the maximum of the pair L and $E\Delta m/\delta t$. The latter expression ensures that energy is conserved to good accuracy,[1] but allows δL to have a less stringent convergence criterion in the case of small δt (rapid evolution) when δL has little importance. With this criterion no convergence problems occurred (see also Sugimoto 1970*a*).

The time steps were chosen so that the expected fractional change in T and ρ were less than some small value (usually 0.01 and 0.03, respectively). It is worth noting at this point that in the late stages of stellar evolution time derivatives become more important, so that instead of solving a series of eigenvalue problems in space ("stellar structure"), it is preferable to think of solving an initial-value problem in space and time ("stellar evolution"). Accordingly, although convergence could be obtained with larger time steps, they were kept fairly small for a more accurate integration in time. The time-step constraint on composition change was

$$\delta t < C_n \left| \frac{Y_i + 0.4/A_i}{\dot{Y}_i} \right| \tag{30}$$

for each of the species i, where A_i is the nucleon number of nucleus i and C_n was usually 0.01. This constrains abundant species to change by less than 1 percent per time step, and the mass fraction X_i of less abundant species ($X_i \ll 0.4$) to change by less than 0.004. Further, the time step could not increase by more than a factor of 2, and for no convergence in 10 iterations the time step was reduced by 0.7 and the calculation attempted again. Usually convergence was attained in one to three iterations.

Over the inner 95 percent of the star, an automatic rezoning was performed so that (1) no zone was greater than about 0.03 of the stellar mass and (2) the fractional change in density in adjacent zones was less than a prescribed value (chosen to give about five zones per decade). The mass in the inner zone was 0.001 of the total stellar mass, and the change in mass of adjacent zones was smooth. This gave a minimum of 35 mass boundaries.

While there are many descriptions of stellar evolutionary codes in the literature, nevertheless many discussions of stellar evolution (especially of exotic stages such as those to be discussed here) are of considerably reduced value because some details of the calculation are unclear. It is hoped that the foregoing discussion is sufficiently explicit.

V. RESULTS OF HELIUM BURNING

a) *Comparison of the Evolution of Helium Stars and Helium Cores*

Following the line of reasoning of § II, it was decided to examine the evolution of stars of mass $M_\alpha/M_\odot = 2$, 4, 8, 16, 32, 64, and 100, having a composition initially of pure ^{4}He. Except where stated otherwise, $\theta_\alpha{}^2$ for the 7.12-MeV level in ^{16}O was taken to be 0.085. Although strictly speaking these objects would correspond to the hydrogen-depleted cores of extreme Population II stars, after helium burning the distinction between Populations I and II is not so important. A detailed consideration of the important question of differential effects between the evolution of massive stars of metal-poor and

[1] Strictly speaking, E should be the *thermal* energy density; otherwise the expression is not appropriate for high degeneracy of electrons. For example, the expression $T(\partial E/\partial T)_V$ has proven satisfactory for the situations to be investigated here.

Also an independent "energy check" is continuously performed to monitor accuracy.

metal-rich initial composition will be deferred (although this investigation can provide some insight into that question).

The initial models were $n = 3$ polytropes with central temperatures less than 10^8 ° K. The evolutionary paths in the temperature-density plane of the centers of these stars are shown by the solid lines in figure 1. The curves are labeled by $M_\alpha/M_\odot$. The locus of points corresponding to the ignition of the triple-alpha reaction is indicated by a dashed line. The models of lower mass show an S-shaped crook at triple-alpha ignition. This is due to the development of a convective core. For lower-mass models the convective gradient differs more strongly from the radiative gradient, and the consequent structural rearrangement is more pronounced.

Also plotted in figure 1 are selected evolutionary points for the centers of some models of Population I stars. The most extensive investigation of helium burning in massive stars is that of Stothers (1966) and Stothers and Chin (1968), in which the evolution of stars of mass $M/M_\odot$ = 15, 30, 60, and 100 was considered. Also shown are representative points in the evolution of the $M/M_\odot$ = 7, 10, and 15 models of Paczyński (1970*a*, 1971*b*).

Figure 1 illustrates the point made in § II, that is, helium stars mimic the behavior of helium cores of the same mass if that mass is not near some critical value. The evolution of the stars of $M/M_\odot$ = 30, 60, and 100 rapidly converges to that of the helium stars after helium ignition. When ^{4}He is nearly exhausted ($X_\alpha \simeq$ a few percent), the helium cores of these stars are $M_c/M_\odot$ = 13, 30, and 55 respectively. These values are close to $M_\alpha/M_\odot$ = 16, 32, and 64, as the similarity of evolution suggests. Further, the 15 $M_\odot$ model of Paczyński, which has a core mass $M_c/M_\odot \simeq 3.9$ at helium depletion, follows the evolution of the $M_\alpha/M_\odot$ = 4 helium star after helium ignition. The exception to the rule of similarity of evolution is the $M_\alpha/M_\odot$ = 2 helium star, which begins helium burning like Paczyński's 10 $M_\odot$ star but evolves more like his 7 $M_\odot$ star as helium is depleted. At that point the carbon-oxygen core mass is closer to that for

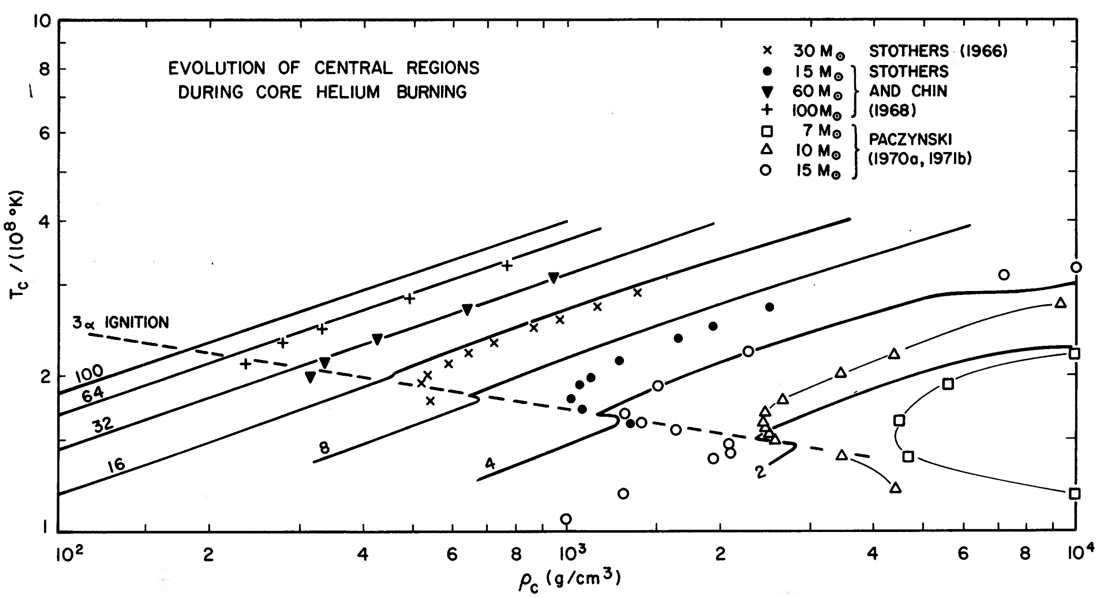

FIG. 1.—Evolution of central temperature and density during core helium burning. The heavy solid lines, labeled by $M_\alpha/M_\odot$, refer to models of this investigation. The locus of points for ignition of helium burning is shown as a dashed line and labeled "3α ignition." The 30 $M_\odot$ model of Stothers (1966) is represented by crosses; the 15, 60, and 100 $M_\odot$ models of Stothers and Chin (1968), by solid circles, solid inverted triangles, and pluses, respectively. Paczyński's (1970*a*, 1971*b*) 7, 10, and 15 $M_\odot$ models are denoted by open squares, triangles, and circles, respectively.

Paczyński's 7 $M_\odot$ star. As the mass of the helium core is about the same as the Chandrasekhar mass limit for electron-degenerate objects, small changes in core mass can cause qualitative differences in the evolution. The assumption of constant core mass begins to break down for stars in this mass range ($M \gtrsim 10\ M_\odot$). A better representation of the hydrogen-burning shell is necessary for lower-mass stars.

In figure 1, the evolution of a 15 $M_\odot$ star is indicated as calculated by Stothers and Chin (1968) and by Paczyński (1970*a*, 1971*b*). The paths do not agree due to the differing sizes of the helium core. Because some of the assumptions of Stothers and Chin (1968) regarding the input physics (equation of state, opacity, etc.) may be suspect for a star as small as 15 $M_\odot$, the Paczyński model has been adopted.

It is interesting to compare also the calculations of Iben (1966) for 15 $M_\odot$, Chiosi and Summa (1970) for 20 $M_\odot$, and Simpson (1971) for 15 and 30 $M_\odot$. To avoid cluttering figure 1, these models were not plotted. These predictions for the size of the convective core during helium burning are all smaller than the Stothers-Chin results, and agree better with Paczyński. The difference might also represent the theoretical uncertainty in the mass of the helium core at this stage of evolution, however.

Despite these inconsistencies in stellar evolutionary results, it is of considerable conceptual assistance to have a table relating helium star masses, M_α, to the masses of stars, M, which will develop equivalent helium cores. Such a listing is given in table 1. Besides the ambiguity for $M_\alpha/M_\odot = 2$, the entry for $M_\alpha/M_\odot = 100$ is especially uncertain because it was obtained by extrapolation. As other calculations of hydrogen and helium burning in massive stars appear, table 1 can be recalibrated. The entries for M are clearly not accurate to the number of significant figures given, but do suggest the range of variation implied by published models.

b) Comparison with Other Calculations of Helium Stars

The evolution of helium stars has been considered by a number of authors; the work of Deinzer and Salpeter (1964), Paczyński (1971*a*) and Divine (1965) is chosen for detailed numerical comparison here. Table 2 lists a number of properties, as calculated in this paper and by the authors above, for the "initial main sequence" for helium burning. There are three main reasons for differences in the results of this paper: (1) the input physics was different (primarily opacity and [$\alpha\alpha\alpha$]), (2) the "initial main sequence" (i.e., when $L = \int_0^M \epsilon dm$ for the first time) was not chosen precisely, and (3) the outer zone was fairly large, containing about 1.2 percent of the mass. The first effect is small, tending to give higher T_c and ρ_c; such behavior is indicated in the table. The second effect causes L to be too large (but not by much, say 10 percent); this behavior is also evident. The large outer zone affects the outer radius R and thereby the effective temperature ($T_e \propto R^{-1/2}$). If instead of blindly using the difference equations, the last zone is integrated analytically (assuming $\rho \propto T^3$ and constant opacity), a better estimate of R (and T_e) is obtained. These corrected values appear in parenthesis in table 2. The extent of the convective core M_c is not too accurately determined because of relatively coarse zoning; an estimate of the error for M_c/M_α is given. It seems

TABLE 1

EQUIVALENT TOTAL MASSES

$M_\alpha/M_\odot$	$M/M_\odot$	$M_\alpha/M_\odot$	$M/M_\odot$
2........	(7 to 10)?	32.......	70 to 80
4........	~15	64.......	110 to 125
8........	20 to 24	100.......	160 to 185
16........	34 to 40		

TABLE 2

COMPARISON OF HELIUM-STAR COMPUTATIONS

(on "helium-burning main sequence")

Quantity	Source: Deinzer and Salpeter (1964)	Divine (1965)	Paczyński (1971)	This Paper
		$M_\alpha=4\ M_\odot$		
$L/L_\odot$..........	1.49 (4)	1.62 (4)	1.61 (4)	1.70 (4)
$R/R_\odot$..........	0.467	0.5023	0.512	0.401 (0.491)
log T_e..........	4.971	4.965	4.959	5.019 (4.957)
log T_c..........	8.218	8.219	8.215	8.227
log ρ_c..........	3.040	3.044	3.046	3.056
M_c/M..........	0.492	0.431	0.453	0.43 ± 0.02
		$M_\alpha=8\ M_\odot$		
$L/L_\odot$..........	...	8.47 (4)	8.00 (4)	8.52 (4)
$R/R_\odot$..........	...	0.7635	0.757	0.61 (0.79)
log T_e..........	...	5.0538	5.049	5.102 (5.047)
log T_c..........	...	8.2562	8.252	8.263
log ρ_c..........	...	2.8044	2.790	2.815
M_c/M..........	...	0.5962	0.602	0.61 ± 0.02

certain that more similar zoning and input physics would give virtually identical models of the helium-burning initial main sequence. Even as they stand, these models are in good agreement.

Using the more accurate integration of the outer zone as described above, a comparison of the evolution in the theoretical H-R diagram of the models of this paper and those of Paczyński and Divine was made. Figure 2 shows the evolution of Paczyński's 4, 8, and 16 $M_\odot$ models by solid lines, Divine's 6 $M_\odot$ model by a dashed line, and the 4, 8, and 16 $M_\odot$ models of this paper by lines of filled circles. Basically the agreement is quite good, but the differences are significant. First, the Divine model is intermediate in behavior between these models and Paczyński's. For the same mass M_α, the size of the initial convective core is nearly identical for all three calculations (using logarithmic interpolation to establish a comparison for Divine's model). However, during helium burning Paczyński's convective cores grow by 30, 21, and 12 percent of their initial size for $M_\alpha/M_\odot$ = 4, 8, and 16 respectively. Divine finds an increase of about 9 percent for $M_\alpha/M_\odot$ = 6, which is less than half as large an increase as Paczyński's results would imply for that mass. Probably because of the relatively coarse zoning employed in this paper at the outer edge of the convective core, no increase in the convective core was found. This is of relatively minor importance for helium burning since the main effect, that of a small change in the size of the carbon-oxygen core for a given mass M, has already been taken into account by the technique used to derive the equivalent masses in table 2. Moreover, it is not expected that the evolution of a convective core in a later, neutrino-dominated regime should exhibit the same sort of core growth as in the photon-dominated regime discussed here. This core growth is related to semiconvection, and reflects the composition dependence of the opacity (Paczyński 1970*b;* Robertson and

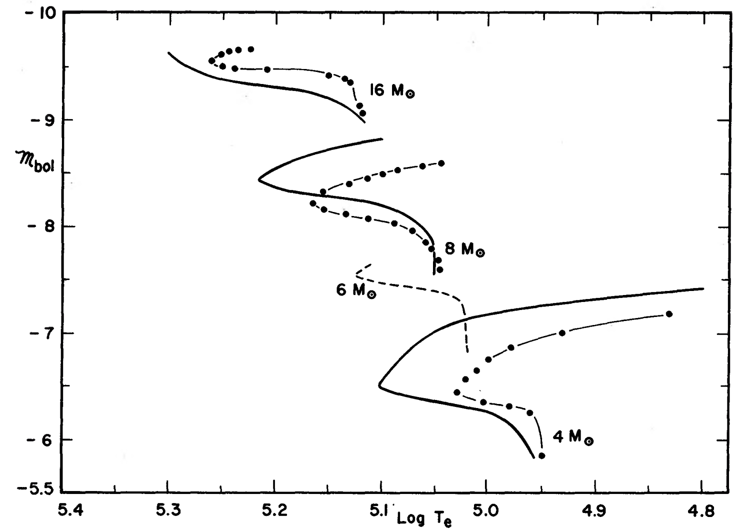

FIG. 2.—Comparison of the evolutionary calculations of helium stars in the H-R diagram. The bolometric magnitude and effective temperature for models of this investigation are represented by filled circles. The evolution of the 6 $M_\odot$ model of Divine (1965) is denoted by a dashed line, while Paczyński's (1971*a*) results for M_α = 4, 8, and 16 $M_\odot$ are represented by solid lines. The growth of the convective core causes Paczyński's models to travel farther to the blue before hooking back.

Faulkner 1972). In a neutrino-dominated situation the thermal structure would be independent of the opacity and (to the extent Urca processes can be neglected) the composition.

While many aspects of the arguments given above deserve further investigation, it appears that the computational techniques and astrophysical approximations used here are adequate for the purposes of this survey.

c) Evolutionary and Structural Properties

Some interesting properties of helium stars at the point of helium ignition (or more precisely, near the "helium-burning main sequence") are listed in table 3. The central temperature (T_c) and density (ρ_c) are given in units of 10^8 ° K and 10^3 g cm^{-3}, respectively. The evolution after the establishment of the convective core may be approximated by

$$\rho_c \propto T_c^{\,n} , \tag{31}$$

TABLE 3

PROPERTIES OF HELIUM STARS NEAR HELIUM IGNITION

$M_\alpha/M_\odot$	$T_c/10^8$ ° K	$\rho_c/10^3$ g cm^{-3}	n	L/M_α (ergs g^{-1} s^{-1})
2......	1.52	2.37	2.38	2.5 (3)
4......	1.69	1.14	2.46	9.2 (3)
8......	1.83	0.653	2.59	2.3 (4)
16......	2.02	0.454	2.66	4.2 (4)
32......	2.08	0.296	2.74	6.2 (4)
64......	2.20	0.220	2.81	8.0 (4)
100......	2.25	0.179	2.89	8.9 (4)

where the value of n in table 3 was chosen by fitting the numerical models at core helium abundances of $X_\alpha \simeq 1$ and $X_\alpha \simeq 0.1$. Growth of the convective core could modify n somewhat. The luminosity L of such stars is only a slowly varying function of time, and is given in table 3 as the ratio L/M_α in cgs units. In these units the solar light-to-mass ratio, $L_\odot/M_\odot$ is about 2 ergs g^{-1} s^{-1}. Neutrino losses were always less than 1 percent of photon energy losses. Using the method of Fowler and Hoyle (1964), and table 3, it is possible to conduct a reasonably accurate investigation of helium-burning nucleosynthesis without having to reconstruct detailed models of the stellar structure. This is particularly useful when the effect of a newly measured cross-section is to be found.

Table 4 contains several evolutionary characteristics of helium burning in these stars. The "evolutionary" time $t_{\rm evol}$, defined as that interval between helium ignition and depletion to the $X_\alpha \simeq 0.01$ level, is given in units of millions of years. Since the luminosity is a slowly varying function for these stars, equality of energy liberated and energy radiated implies

$$t_{\rm evol} \simeq \left(\frac{M_c}{M_\alpha}\right) q \left(\frac{L}{M_\alpha}\right)^{-1} , \tag{32}$$

where q is the average energy release per gram of ^{4}He consumed, L/M_α is given in table 3, and the fractional size of the convective core M_c/M_α is given in table 4. The "error estimates" quoted for M_c/M_α refer only to those obvious errors due to finite mass zoning. The changes in $t_{\rm evol}$ due to growth of the convective core, changes in the rate of $^{12}C(\alpha, \gamma)^{16}O$, or variation in luminosity can easily be estimated from equation (32).

The gravitational binding energy, per unit mass, of a hydrostatic star of mass M_α is

$$\mathcal{B} = \frac{1}{M_\alpha} \int (3P/\rho - E) dm , \tag{33}$$

integrated from $m = 0$ to M_α. This quantity is only a slowly varying function of time during core helium burning; the value of $\mathcal{B}$ at helium ignition, in units of 10^{16} ergs g^{-1}, is quoted in table 4. Notice that the gravitational binding energy is small compared to the thermonuclear fuel reserve of these stars.

Because of the uncertainty in the value of the reduced width θ_α^2 of the 7.12-MeV level in ^{16}O [and therefore the rate of $^{12}C(\alpha, \gamma)^{16}O$], it is of interest to investigate the variation of the characteristics of these models with θ_α^2. The features of this variation can be illustrated by a simple argument. For a thermally static star, the rate of nuclear energy generation at the center of the star obeys

$$\epsilon_c \propto L/M , \tag{34}$$

to reasonable accuracy (see Fowler and Hoyle 1964, for a detailed discussion). Also,

$$\epsilon_c \propto X_\alpha^3 \rho^u T^s , \tag{35}$$

TABLE 4

Some Evolutionary Characteristics of Helium-burning Stars

$M_\alpha/M_\odot$	$t_{\rm evol}/10^6$ years	M_c/M_α	$\mathcal{B}$ (10^{16} ergs g^{-1})
2.........	2.07	0.31 ± 0.02	0.86
4.........	0.84	0.43 ± 0.02	0.96
8.........	0.51	0.61 ± 0.02	1.08
16.........	0.36	0.76 ± 0.02	1.21
32.........	0.28	0.86 ± 0.03	1.26
64.........	0.23	0.90 ± 0.03	1.30
100.........	0.22	0.93 ± 0.02	1.34

and using equation (31), we have

$$\epsilon_c \propto X_\alpha{}^3 T^{s+un} .$$

Since $L/M_\alpha \simeq$ const. for a given mass,

$$\log X_\alpha \approx -\tfrac{1}{3}(s + un) \log T + \text{const.} \tag{36}$$

This behavior is shown in figure 3. The solid heavy line represents the evolution of the $M_\alpha/M_\odot = 8$ star, with $\theta_\alpha{}^2 = 0.0425$. After the helium abundance drops, equation (36) gives an oversimplified description of the evolution because equation (35) must be modified to include $^{12}C(\alpha, \gamma)^{16}O$, and primarily because equation (34) must also include compressional work. The evolution predicted by equation (36) is represented by the heavy dashed line.

The details of the evolution are changed, for example, if (1) M_α is changed, and (2) $\theta_\alpha{}^2$ is changed. The latter effect is the smaller. The evolution for $M_\alpha/M_\odot = 8$ and $\theta_\alpha{}^2 = 0.340$ is denoted in figure 3 by crosses, and differs little from the $\theta_\alpha{}^2 = 0.0425$ case. The evolution for $M_\alpha/M_\odot = 4$ is denoted by solid circles, and for $M_\alpha/M_\odot = 16$ by solid triangles. In both these cases $\theta_\alpha{}^2 = 0.085$. As far as temperature evolution is concerned, an error of only 10 percent in the size of the convective core is as important as an enormous change (a factor of 8) in $\theta_\alpha{}^2$. From comparisons of this type it is concluded that the ρ-T history of core helium burning is insensitive to changes in $\theta_\alpha{}^2$.

d) Nucleosynthesis

While the precise value of $\theta_\alpha{}^2$ for the 7.12-MeV level in ^{16}O does not much affect the ρ-T history of helium burning, it is of crucial importance for nucleosynthesis. The reduced width $\theta_\alpha{}^2$ determines which of the two major constituents, ^{12}C and ^{16}O, is the dominant product of helium burning. Figure 4 illustrates this point for the $M_\alpha/M_\odot = 8$ star; the fraction by mass of ^{12}C is shown, as a function of ^4He remaining, for several values of $\theta_\alpha{}^2$. For $\theta_\alpha{}^2 \lesssim 0.04$; the fraction by mass of ^{12}C is $X_C \gtrsim 0.5$, while for $\theta_\alpha{}^2 \simeq 0.3$ there is almost no ^{12}C left when ^4He is depleted. Notice that X_C decreases very rapidly for small X_α. This can be understood from equation (8): the production of ^{12}C is proportional to $X_\alpha{}^3$ while its destruction goes as X_α. For small X_α, destruction becomes relatively more effective. Also note that figure 4 resembles figure 2 of Deinzer and

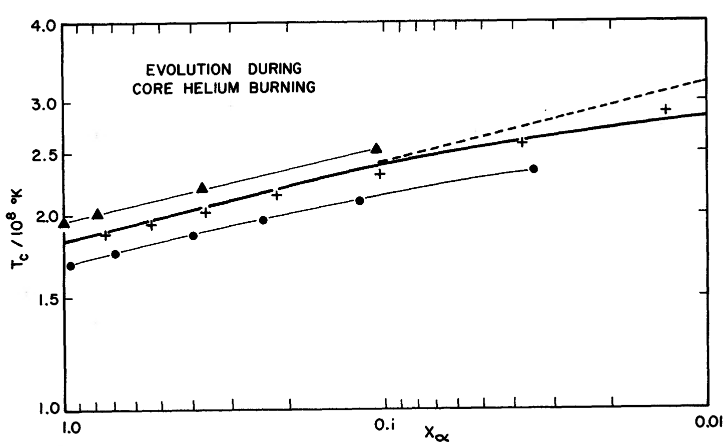

FIG. 3.—Variation of central temperature with ^4He abundance. A heavy solid line represents the evolution of T_c (in units of 10^8 ° K) for $M_\alpha = 8\ M_\odot$ and $\theta_\alpha{}^2 = 0.0425$; the heavy dashed line represents eq. (36). The evolution for $M_\alpha = 8\ M_\odot$ but $\theta_\alpha{}^2 = 0.34$ is represented by plus signs. Changing $\theta_\alpha{}^2$ causes little change in T_c for a given X_α, but changing M_α has a large effect. The evolution for $\theta_\alpha{}^2 = 0.085$ but $M_\alpha/M_\odot = 4$ and 16 is shown by filled circles and filled triangles, respectively.

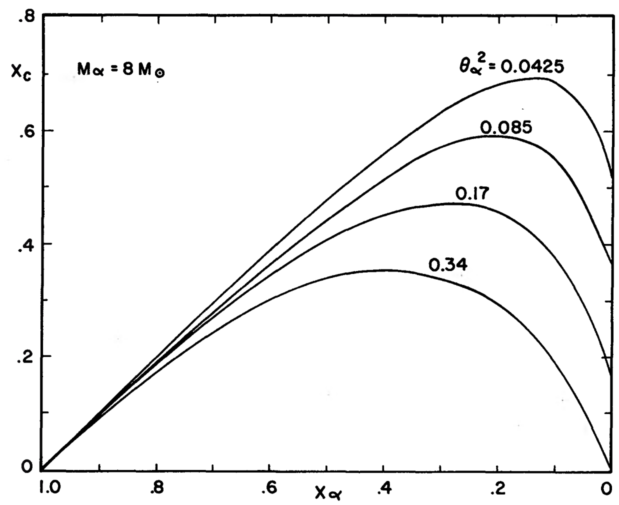

FIG. 4.—Variation of ^{12}C abundance with ^4He abundance, with various values of θ_α^2, for $M_\alpha = 8\ M_\odot$. Note the rapid decrease in ^{12}C abundance as ^4He becomes rare.

Salpeter (1964), where their parameter is not θ_α^2 but M_α. Increasing the mass M_α changes T^3/ρ in such a way as to increase the rate of $^{12}C(\alpha, \gamma)^{16}O$ relative to $3\alpha \rightarrow {}^{12}C$, and therefore has an effect similar to increasing θ_α^2.

Figure 5 summarizes the results of nucleosynthesis in core helium burning for various values of θ_α^2. These results are very similar to those of Deinzer and Salpeter (1964). Since even for $M_\alpha/M_\odot \simeq 100$, little ^{20}Ne is produced, a plot of X_C also gives an approximate value of X_O, the mass fraction of ^{16}O, since $X_O \simeq 1 - X_C$. For a given θ_α^2, X_C is almost linear in log M_α (at least for $M_\alpha/M_\odot \gtrsim 2$, see however Deinzer and Salpeter's figure 3 for $M_\alpha < 2\ M_\odot$). Also, for a given mass, X_C is nearly linear in log θ_α^2. Therefore, if we define $m = M_\alpha/M_\odot$ and $x = \theta_\alpha^2/0.085$, we find

$$X_C \simeq 0.60 - 0.667 \log x - 0.267 \log m \,. \tag{37}$$

When equation (37) gives a negative X_C, it should be taken to mean $X_C = 0$. A discussion of some implications of this result has been published (Arnett 1971*b*).

In Population I stars it is expected that previous CNO cycling will have produced some ^{14}N (of order of 1 percent or so). During core helium burning these nuclei can undergo some interesting nuclear processing. Such thermonuclear evolution can be calculated separately because it affects the energy-generation rate for these stars even less than $^{12}C(\alpha, \gamma)^{16}O$ does; consequently it will not be discussed here. As mentioned

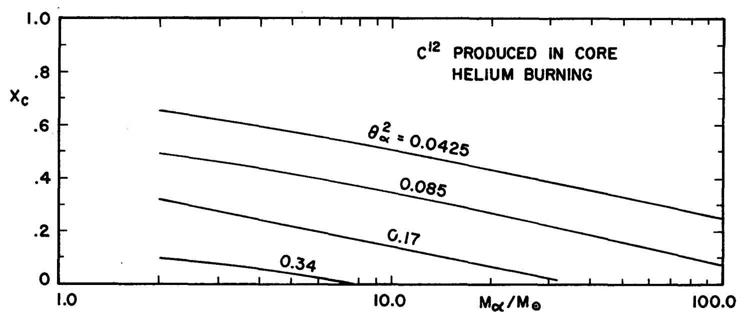

FIG. 5.—Final fractions by mass of ^{12}C as a function of $M_\alpha/M_\odot$ for various values of θ_α^2. The solid lines represent smooth interpolation of calculations for $M_\alpha/M_\odot$ = 2, 4, 8, 16, 32, 64, and 100.

above, new stellar-structure models need not be recalculated for such an investigation. As an aid to applying the technique of Fowler and Hoyle (1964) as well as for use in their own right, details of the history of ρ, T, X_α, and $X_{\rm C}$ are tabulated for a few interesting values of M_α in the Appendix.

VI. SUMMARY

This is the first in a series of papers which will describe a survey of the later stages of evolution of massive stars. In this paper, we have (1) developed and discussed the motivation and astrophysical background for the survey, (2) described the methods and approximations to be used, (3) developed initial models for the survey, (4) checked the techniques to be used by comparison with other investigations of the evolution of helium stars, (5) examined the "single-star" approximation (of replacing helium cores by helium stars of the same mass) by explicit comparison with published evolutionary sequences, (6) calibrated the helium-core mass M_α as a function of total stellar mass M, and (7) explored helium-burning nucleosynthesis in stellar cores with the revised triple-alpha rate and various values for the reduced width θ_α^2 for the 7.12-MeV level in ^{16}O. These latter results were expressed as a simple analytic function of θ_α^2 and M_α.

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APPENDIX

In the course of this investigation it was necessary to make quantitative comparison of the evolutionary results obtained here with those of others. In several cases this was extremely difficult because properties of the evolutionary sequences were presented only in graphical form, and often with a choice of scale that rendered quantitative comparison untrustworthy. For this reason, and to aid in the investigation of secondary nucleosynthesis during helium burning, some numerical data concerning the history of core helium burning for $M_\alpha/M_\odot = 4$, 8, and 32 are presented in Table A1. In this table, MODEL refers to the number of time steps elapsed since the beginning of the sequence, t refers to an elapsed time (in seconds), T_c is the central temperature and ρ_c the central density, and X_α and $X_{\rm C}$ are the fractions by mass of ^{4}He and ^{12}C, respectively.

TABLE A1

HISTORIES OF CORE HELIUM BURNING

MODEL	t/sec.	$T_c/10^8$ °K	ρ/g cm^{-3}	X_α	X_c
$M = 4\ M_\odot$					
83	3.11(11)	1.27	6.86(2)	1.0	1.74(-6)
93	3.78(11)	1.40	9.11(2)	1.0	7.21(-5)
103	4.46(11)	1.55	1.14(3)	1.0	1.03(-3)
113	5.26(11)	1.67	1.24(3)	9.97(-1)	2.73(-3)
123	6.60(11)	1.69	1.17(3)	9.93(-1)	6.77(-3)
133	1.66(12)	1.69	1.14(3)	9.60(-1)	3.91(-2)
143	9.91(12)	1.76	1.23(3)	6.80(-1)	3.07(-1)
153	1.76(13)	1.87	1.42(3)	3.97(-1)	5.31(-1)
163	2.18(13)	1.97	1.63(3)	2.38(-1)	6.14(-1)
173	2.50(13)	2.11	1.97(3)	1.20(-1)	6.15(-1)
183	2.78(13)	2.34	2.65(3)	3.47(-2)	5.24(-1)
193	2.90(13)	2.62	3.76(3)	4.80(-3)	4.48(-1)
203	2.92(13)	2.88	5.29(3)	1.17(-4)	4.34(-1)
$M = 8\ M_\odot$					
1	4.59(11)	1.38	3.26(2)	1.0	8.50(-4)
10	4.90(11)	1.51	4.34(2)	1.0	9.20(-4)
20	5.24(11)	1.67	5.68(2)	9.98(-1)	1.35(-3)
30	5.67(11)	1.80	6.71(2)	9.97(-1)	2.36(-3)
40	6.38(11)	1.83	6.67(2)	9.94(-1)	5.93(-3)
50	8.54(11)	1.83	6.52(2)	9.80(-1)	1.80(-2)
60	4.71(12)	1.89	6.98(2)	7.48(-1)	2.40(-1)
70	8.11(12)	1.97	7.67(2)	5.32(-1)	4.22(-1)
80	1.08(13)	2.06	8.59(2)	3.60(-1)	5.39(-1)
90	1.30(13)	2.18	9.95(2)	1.96(-1)	5.87(-1)
100	1.49(13)	2.35	1.22(3)	1.05(-1)	5.54(-1)
110	1.64(13)	2.61	1.67(3)	2.81(-2)	4.34(-1)
120	1.69(13)	2.93	2.37(3)	3.90(-3)	3.68(-1)
130	1.71(13)	3.28	3.35(3)	1.69(-4)	3.58(-1)
$M = 32\ M_\odot$					
73	1.83(11)	1.54	1.23(2)	1.0	3.68(-6)
83	1.98(11)	1.70	1.65(2)	1.0	8.73(-5)
93	2.16(11)	1.87	2.19(2)	1.0	2.62(-4)
103	2.38(11)	2.04	2.83(2)	9.99(-1)	1.04(-3)
113	3.22(11)	2.08	2.96(2)	9.87(-1)	1.11(-2)
123	1.98(12)	2.14	3.18(2)	7.88(-1)	2.02(-1)
133	3.81(12)	2.22	3.53(2)	5.68(-1)	3.83(-1)
143	5.30(12)	2.33	3.98(2)	3.88(-1)	4.85(-1)
153	6.58(12)	2.45	4.62(2)	2.42(-1)	5.09(-1)
163	7.72(12)	2.63	5.64(2)	1.24(-1)	4.45(-1)
173	8.68(12)	2.92	7.65(2)	3.09(-2)	3.04(-1)
183	9.08(12)	3.27	1.07(3)	7.82(-3)	2.20(-1)
193	9.19(12)	3.66	1.51(3)	7.92(-4)	1.98(-1)
203	9.22(12)	4.07	2.09(3)	2.36(-6)	1.96(-1)

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