THE ASTROPHYSICAL JOURNAL, 175:243–252, 1972 July 1

TIME EVOLUTION OF A ROTATING BLACK HOLE IMMERSED IN A STATIC SCALAR FIELD*

WILLIAM H. PRESS†
California Institute of Technology, Pasadena, California
Received 1971 November 5; revised 1972 January 21

ABSTRACT

We compute the time evolution of a Kerr (rotating) black hole which is immersed in a perturbing scalar field, uniform at large distances from the hole. The perturbing field produces a torque on the hole which (i) is perpendicular to the field lines; (ii) causes the perpendicular component $J_\perp$ of the hole's angular momentum to decrease exponentially with time $J_\perp = (J_\perp)_{\text{initial}} \exp(-t/\tau)$, $\tau = (3c^5/8\pi G) \times$ (mass of hole)$^{-1} \times$ (energy density of field)$^{-1}$, bringing the hole's total angular momentum J into eventual alignment with the field; and (iii) accomplishes this alignment by converting rotational energy of the black hole into irreducible mass. We conjecture extensions of these results to black holes perturbed by external electromagnetic or gravitational fields. According to these conjectures "spin-orbit coupling" in a binary star system should not remove a significant fraction of a black hole's intrinsic angular momentum during the system's lifetime against gravitational-radiation damping.

I. INTRODUCTION

Hawking (1972) has recently proved by global methods an important theorem whose physical content can be stated very concisely: "A stationary black hole must be either static or axisymmetric." In particular, a rotating, Kerr black hole immersed in a nonaxisymmetric perturbing field must become nonstationary; it must evolve in time, until either (i) it has lost its angular momentum and become a static (Schwarzschild) hole or (ii) it has achieved an axisymmetric orientation with respect to the perturbing field, if one exists. The perturbing field can be of any sort: gravitational, electromagnetic, scalar, or whatever.

A number of interesting questions come to mind: How does the black hole choose between options (i) and (ii) above? Or does it choose a combination, both losing angular momentum and changing its orientation? What is the timescale of Hawking's process? (Will an astrophysical black hole align itself with the galactic magnetic field in 1 msec? or not even in 10^{10} years?) These are questions which cannot be answered by using Hawking's global methods of investigation.

Luckily, Ipser (1971) has recently described a process for Kerr black holes which is precisely the "microscopic" physical description of Hawking's global prediction. Imagine a Kerr geometry perturbed nonaxisymmetrically by a static field (for example, the field generated by distant static sources). Inside the ergosphere there are no static, timelike world lines. Hence any observer will see the perturbing field as dynamic in this region (cf. Bardeen 1970). In fact, Ipser points out, local observers will see a flux of energy through the event horizon ("down the hole"). This energy cannot have come from infinity where things are static; rather, it must come from the rotational energy of the black hole. The hole is evolving to a new configuration with a different "irreducible" mass (cf. Christodoulou 1970), and this evolution must continue until the conditions of Hawking's theorem are satisfied. In changing its angular momentum, the hole exerts a torque back on the perturbing field and (in principle) on the sources of that field.

In this paper we examine the quantitative details of Ipser's mechanism for the case of a perturbing scalar field. We calculate the time evolution of a Kerr black hole im-

* Supported in part by the National Science Foundation [GP-27304 and GP-28027].
† Fannie and John Hertz Foundation Fellow.

mersed in a scalar field. For convenience we take the field to be uniform and constant at great distances from the black hole, although this restriction could be relaxed easily. The results of our calculation justify in detail the qualitative description presented above. Moreover, we find that the intrinsic angular momentum J of the hole evolves with time according to a very simple law:

$$J_{\|} = \text{constant} , \quad J_{\perp} = J_{\perp}(t = 0) \exp(-t/\tau) . \tag{1}$$

Here $\|$ and $\perp$ denote components of angular momentum respectively parallel and perpendicular to the direction of the gradient of the scalar field (direction of the field lines), and τ is the characteristic time

$$\tau = (3/8\pi)(c^5/G)(\text{mass of hole})^{-1}(\text{energy density of field})^{-1} . \tag{2}$$

In particular, this evolution means that only an initial orientation precisely perpendicular to the field will cause a Kerr hole to evolve into a Schwarzschild hole; other initial orientations will give a partial loss of angular momentum and a reorientation along the field lines.

Why do we limit ourselves to a scalar field (which is unknown in nature) rather than studying a vector (electromagnetic) or tensor (gravitational) field? Only for practical reasons: the vacuum field equation (wave equation) for a perturbing scalar field is separable and can be solved analytically in the stationary case; by contrast, the vacuum Maxwell equations in a Kerr geometry are probably inseparable (Fackerell and Ipser 1972) and have not (to date) been solved, while the equations governing gravitational perturbations of the Kerr metric have not fully been written down in manageable from, and will almost certainly be inseparable when they are finally derived (cf. Teukolsky 1972). Also, since Ipser's description of the inflow of rotational energy through the horizon does not depend on the details of the field, we should expect that the main results of the scalar case are extendable (at least qualitatively) to the case of other fields. Below we conjecture the extension to electromagnetic and gravitational perturbations, and we discuss implications for a black hole in a binary system. (In what follows we take units with $c = G = 1$.)

II. STATIONARY SCALAR FIELDS IN A KERR METRIC

We begin with the unperturbed Kerr metric for a rotating black hole in the form (Boyer and Lindquist 1967)

$$ds^2 = (r^2 + a^2 \cos^2\theta)[dr^2/(r^2 - 2Mr + a^2) + d\theta^2] + (r^2 + a^2)\sin^2\theta d\varphi^2$$
$$- dt^2 + [2Mr/(r^2 + a^2\cos^2\theta)](a \sin^2\theta d\varphi - dt)^2 , \tag{3}$$

where M is the hole's mass and aM is its angular momentum ($0 \leq a \leq M$), oriented in the direction $\theta = 0$. The event horizon is at $r = M + (M^2 - a^2)^{1/2}$, and the outer boundary of the ergosphere is at $r = M + (M^2 - a^2\cos^2\theta)^{1/2}$.

The perturbing scalar field Ω and its source (a scalar charge density ρ) satisfy

$$\Box\Omega = 4\pi\rho , \tag{4}$$

where $\Box$ denotes the covariant d'Alembertian in the Kerr geometry. (Note the assumption that the field is a weak perturbation: we ignore the second-order effect of the field's stress energy in distorting the geometry of the Kerr solution.) If the charged sources are static in the background geometry, the force of the field acting back on its sources is

$$dF_\mu = \Omega_{,\mu}\rho d(\text{proper volume}) . \tag{5}$$

(See Chase 1970 for details and references about the theory of scalar fields in curved spacetime.)

The wave equation (4) is known to be separable in the vacuum Kerr geometry (Carter 1968; Brill *et al.* 1972). In the case of stationary fields, the separation is particularly simple:

$$\Omega = R_{lm}(r) Y_{lm}(\theta, \varphi) , \qquad (6)$$

where $Y_{lm}(\theta, \varphi)$ is a spherical harmonic and $R_{lm}(r)$ satisfies, in source-free regions,

$$\frac{d}{dr}\left[(r^2 - 2Mr + a^2)\frac{d}{dr}R_{lm}\right] + \left[\frac{m^2a^2}{r^2 - 2Mr + a^2} - l(l+1)\right]R_{lm} = 0 . \qquad (7)$$

One fundamental solution of equation (7) is

$$R_{lm} = (r - r_-)^{-ima/\delta}(r - r_+)^{+ima/\delta}{}_2F_1[l + 1, -l; 1 + 2ima/\delta; (r - r_-)/\delta] , \qquad (8)$$

where $\delta \equiv 2(M^2 - a^2)^{1/2}$, $r_\pm \equiv M \pm (M^2 - a^2)^{1/2}$. The other fundamental solution is obtained by complex conjugation. ${}_2F_1$ is a hypergeometric function, actually a polynomial in r of degree l since l is an integer. Appropriate linear combinations of these solutions (the details need not concern us here) yield two "physical" solutions, conveniently normalized and satisfying appropriate boundary conditions at the event horizon $r = r_+$ and at $r = \infty$ respectively:

$$R^+{}_{lm} \to \begin{cases} \text{const. } (r - r_+)^{+ima/\delta} & \text{as } r \to r_+ , \\ r^l + \text{const. } r^{l-1} + \ldots & \text{as } r \to \infty , \end{cases} \qquad (9a)$$

$$R^\infty{}_{lm} \to r^{-l-1} \qquad \text{as } r \to \infty . \qquad (9b)$$

The boundary condition on $R^+{}_{lm}$ corresponds to a requirement that all timelike observers see inward-going waves, and an inward energy flux, at the event horizon; details are given in an appendix. The condition on $R^\infty{}_{lm}$ merely requires a well-behaved solution at infinity.

Even without exhibiting the construction of solutions (9a) and (9b) explicitly (which is straightforward, but laborious), we can prove their existence and derive some important properties. In the asymptotic region $r \to \infty$ equation (7) has solutions whose leading terms are r^l and r^{-l-1}. $R^\infty{}_{lm}$ is the unique analytic continuation of the latter; since the differential equation is real, all coefficients in the asymptotic series solution are real, and analytic continuation gives the result $\text{Im}(R^\infty{}_{lm}) = 0$ everywhere. The existence of $R^+{}_{lm}$ is immediate: it is just R_{lm} (eq. [8]) normalized to unity in its leading asymptotic term (that term must be r^l since we have just shown that expression [9b] is real everywhere—while eq. [8] is evidently complex). Again because the differential equation (7) is real, the coefficients in the asymptotic expansion of $R^+{}_{lm}$ are real, at least up to the term in r^{-l-1}—the point at which the other asymptotic solution may enter (and *must* enter to satisfy the complex boundary condition at the horizon). Hence we have

$$\text{Im}(R^+{}_{lm}) \to C_{lm}(a) r^{-l-1} + O(r^{-l-2}) \qquad \text{for large } r , \qquad (10)$$

where $C_{lm}(a)$ is a function which is obtained by (tediously!) expanding equation (8) in powers of r and picking out the first nonvanishing imaginary coefficient.

The general inhomogeneous solution to (4) can now be constructed. Because the field is linear in its sources, it suffices to give the solution for a shell of scalar charge at $r = b$ with surface density (charge per proper area)

$$\sigma = \sigma_0 Y_{lm}(\theta, \varphi)/(g_{rr})^{1/2} . \qquad (11)$$

Here g_{rr} is a component of the metric tensor (3). Straightforward calculation gives the result

$$\Omega = [4\pi/(2l + 1)]\sigma_0 Y_{lm}(\theta, \varphi)(b^2 - 2Mb + a^2)\begin{cases} R^+{}_{lm}(r)R^\infty{}_{lm}(b) , & r \le b \\ R^\infty{}_{lm}(r)R^+{}_{lm}(b) , & r \ge b . \end{cases} \qquad (12)$$

Appropriate sums over l and m, and an integral over shells of various radii b, allow one to construct an explicit solution for a general static charge distribution $\rho(r, \theta, \varphi)$. The construction is identical to that for flat space, and we omit it.

III. THE TORQUE ON A KERR BLACK HOLE

On first glance it would seem that in taking the Kerr metric as a fixed geometrical background we have lost the possibility of following its evolution in time. This is not the case. Our perturbation calculation will give the small rate of change of the black hole's total angular momentum J. It is the nature of the perturbation approach that we cannot integrate this small rate directly over very long times—the total angular-momentum change would become large and no longer a perturbation, while the background geometry would remain "unlawfully" fixed. Given the rate of change, then, how do we follow the evolution over long times? As the field deposits (or extracts) a finite amount of angular momentum, the Kerr solution must evolve into *something*, but what? The answer is given by Carter's theorem (1971) and its extensions (Ipser 1971; Wald 1971; Hawking 1972): the class of Kerr metrics is "analytically complete," is completely specified by M and J, and (see Chase 1970) admits no stationary scalar field except one generated by external sources; in short, we can be confident that a Kerr metric will evolve into another Kerr (or Schwarzschild) metric. Thus, once we know the rate of change of J—i.e., the torque on the hole—from a perturbation calculation, we can apply this rate to the family of Kerr solutions and obtain the hole's complete time evolution.

In principle, one might obtain the torque $N \equiv dJ/dt$ by examining the flow of angular momentum in the perturbing field very near the event horizon. In practice, there is an easier way: We suppose that the scalar field is generated by a shell of scalar charge located at some large radius b. This field, influenced by the Kerr black hole, acts back on the shell to produce a net torque. By the global conservation of angular momentum (which holds in asymptotically flat spacetime [see, e.g., Misner, Thorne, and Wheeler 1972]) this torque must be exactly the negative of the rate of change of the hole's angular momentum. Finally, we take the limit of this torque as the sources are made infinitely distant, $b \rightarrow \infty$.

Let the shell of scalar charge at $r = b$ have the surface charge density

$$\sigma = [E/(g_{rr})^{1/2}]\{(\tfrac{3}{4}\pi)^{1/2} \cos\gamma\ Y_{10}(\theta, \varphi) - (\tfrac{3}{8}\pi)^{1/2} \sin\gamma\,[Y_{11}(\theta, \varphi) - Y_{1-1}(\theta, \varphi)]\}\ . \quad (13)$$

For $b \geq r \gg M$ this source distribution generates a field with uniform gradient in the (x, z)-plane, at angle γ to the z-axis. The magnitude of the gradient is $|\nabla\Omega| = E$.

We define gradients in the direction of z, x, and y rotations by

$$L_z \equiv (\partial/\partial\varphi)\ , \quad L_x \equiv -[\sin\varphi(\partial/\partial\theta) + \cot\theta\cos\varphi(\partial/\partial\varphi)]\ ,$$
$$L_y \equiv [\cos\varphi(\partial/\partial\theta) - \cot\theta\sin\varphi(\partial/\partial\varphi)]\ . \quad (14)$$

Since the background metric has symmetry around the z-axis, L_z is uniquely defined. L_x and L_y are unique only up to an additional term of order $(M/b)^2$. However as we take limits as $b \rightarrow \infty$, this lack of uniqueness gives vanishing contribution to our result.

The calculation is detailed in an appendix and proceeds as follows: (i) Use equations (11), (12), and (13) to obtain Ω; (ii) with definition (14), calculate $\sigma L\Omega$, the torque per unit proper area of the shell; (iii) integrate over proper area $d\Sigma$ using the axisymmetry of the metric to eliminate terms with orthogonal φ-dependence, and using the relation $R^+{}_{lm} = R^+{}_{l-m}{}^*$ (* denotes complex conjugation). The result (the net torque on the charge distribution) is equal and opposite to the torque N on the black hole:

$$-N_z = \int \sigma L_z \Omega d\Sigma = \int E^2 C_{11}(a) \sin^2\gamma\,|Y_{11}|^2 d\Sigma/b^2$$
$$+ \text{ fractional corrections of } O(M/b)\ , \quad (15a)$$

$$-N_x = \int \sigma L_x \Omega d\Sigma = -\int E^2 C_{11}(a) \sin\gamma \cos\gamma \,|Y_{10}|^2 d\Sigma/b^2$$
$$+ \text{ fractional corrections of } O(M/b)\,, \tag{15b}$$

$$-N_y = \int \sigma L_y \Omega d\Sigma = \int [E^2 \cos\gamma \sin\gamma/(g_{rr})^{1/2}](b^2 - 2Mb + a^2)$$
$$\times [R^{\infty}{}_{11} \operatorname{Re}(R^{+}{}_{11})|Y_{10}|^2 - R^{\infty}{}_{10} \operatorname{Re}(R^{+}{}_{10})|Y_{11}|^2] d\Sigma\,. \tag{15c}$$

We now take the limit $b \to \infty$, using $C_{11}(a) = \frac{1}{3}aM^2$ which is obtainable by comparing equation (10) with the asymptotic expansion of equation (8). The resulting limit is

$$N_z \to -\tfrac{1}{3}E^2 aM^2 \sin^2\gamma\,, \tag{16a}$$

$$N_x \to \tfrac{1}{3}E^2 aM^2 \sin\gamma \cos\gamma\,. \tag{16b}$$

The third component of the torque, N_y, would give a precession of the hole's $\boldsymbol{J}$ around the scalar field lines. Its limit is harder to compute than N_x and N_z, since it depends on a delicate balance of coordinate-orthogonal spherical harmonics integrated over proper area. However, one can prove easily that N_y must vanish exactly: (i) By equation (15c) N_y is unchanged when the direction of the field is reversed ($E \to -E$). (ii) If we imagine reversing the black hole's spin ($a \to -a$), then boundary condition (9a) is affected, but the differential equation (7) and boundary condition (9b) are left unchanged; consequently $R^{+}{}_{1m}$ goes to its complex conjugate; but $\operatorname{Re}(R^{+}{}_{1m})$ and $R^{\infty}{}_{1m}$ are unaffected. This means that the reversal of spin direction leaves all terms in the integrand (15c) for N_y unchanged. Since N_y (and the precession it produces) is invariant under inversions of both field direction and the direction of black-hole spin, it must vanish exactly, since (in fig. 1) the directions out of or into the paper are symmetrical. We conclude

$$N_y = 0\,. \tag{16c}$$

IV. TIME EVOLUTION OF THE BLACK HOLE

Describe the black hole's angular momentum vector $\boldsymbol{J}$ by its magnitude J, and by the angle γ between its direction and the (rigidly fixed) direction of the scalar field lines at infinity (direction of $\nabla\Omega$). The torque (16) on the hole produces changes in J and γ

$$dJ/dt = -(1/\tau)J \sin^2\gamma\,, \quad J d\gamma/dt = -(1/\tau)J \sin\gamma\cos\gamma\,, \tag{17}$$

where

$$\tau = 3/E^2 M\,. \tag{18}$$

The solution of equations (17) is

$$J_{\parallel} \equiv J\cos\gamma = \text{constant}\,, \quad J_{\perp} \equiv J \sin\gamma = \text{const.} \times \exp(-t/\tau)\,. \tag{19}$$

This is the result quoted in equation (1), since the energy density of the scalar field at infinity is equal to $E^2/8\pi$.

V. CONCLUSIONS, DISCUSSION, AND CONJECTURES

A Kerr black hole, then, immersed in a uniform scalar field, responds to Hawking's theorem in a very simple manner: exponentially in time, it loses the component of its angular momentum perpendicular to the field.

What about a nonuniform scalar field? If the scalar field itself is not axisymmetric far from the hole, then there is *no* possible final axisymmetric configuration, so by Hawking's theorem the hole must eventually lose all of its angular momentum. Is this consistent with our conclusions about the uniform case? Yes. Let $\mathfrak{R}$ be a characteristic distance over which the field's gradient changes magnitude or direction, i.e. $\mathfrak{R} \sim |\nabla\Omega|/|\nabla(\nabla\Omega)|$. Assume that the sources are astronomically distant from the black hole,

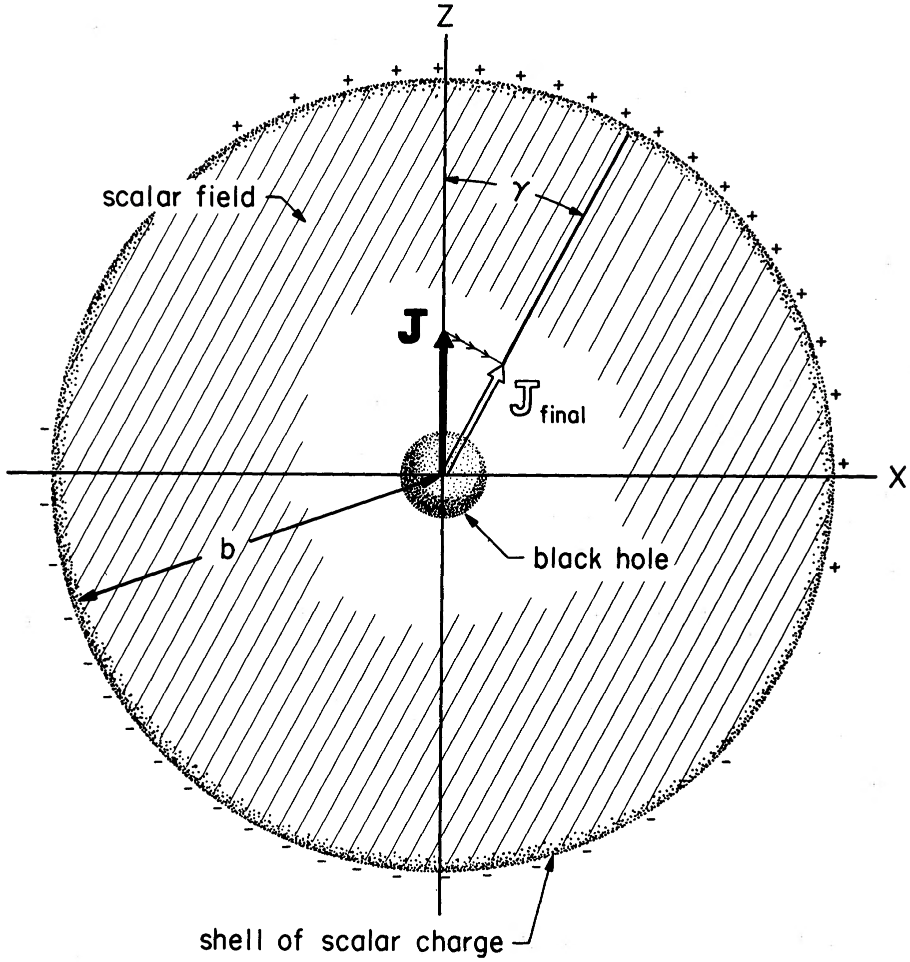

FIG. 1.—The rotating black hole is immersed in a scalar field which becomes uniform far from the hole. For convenience we take the field to be generated by a shell of scalar charge at radius b, and consider the limit as $b \to \infty$. The x-, y-, and z-components of torque on the charged shell correspond respectively to alignment, precession, and loss of angular momentum of the black hole. J is the initial angular momentum, which evolves asymptotically to J_{final} (see text for details).

so that $M/\Re \ll 1$. A generalization of § III above to include multipoles $l > 1$ yields the order-of-magnitude results

$$dJ_{\perp}/dt \sim -(J_{\perp}/\tau)[1 + O(M/\Re)] , \quad dJ_{\parallel}/dt \sim -(J_{\parallel}/\tau)O(M/\Re) . \tag{20}$$

In other words, nonuniformity of the field gives rise to a higher-order effect which saps parallel angular momentum as well as perpendicular, but acts only $\sim M/\Re \sim$ (kilometers)/(parsecs) times as fast.

In a more speculative vein, we consider the generalization of these scalar-field results to the physically interesting cases of electromagnetic and gravitational perturbations—which are far more difficult to analyze quantitatively.

For stationary, uniform electric or magnetic fields, it is obviously reasonable that equations (1) and (2) should continue to hold, at least in terms of qualitative behavior, with the simple replacement (energy density of scalar field) $\to$ (energy density of electromagnetic field). If we imagine a Kerr black hole immersed in an interstellar magnetic field, this yields a time constant for its angular momentum loss of

$$\tau \sim 10^{37} \text{ years } (M_\odot/M)(B/10^{-5} \text{ gauss})^{-2}\ ! \tag{21}$$

For electromagnetic fields the effect of Hawking's theorem would seem to be quite negligible.

For gravitational perturbations, we might suppose that the qualitative behavior of equation (1) (gradual loss of $J_\perp$) still holds. In equation (2), however, there is no unique analog of scalar-field energy density. The Einstein field equations dimensionally equate an energy density to a spacetime curvature, so we might be tempted to try the substitution

$$(\text{scalar energy density}) \rightarrow (\text{Riemann curvature}) \sim m/r^3\ , \tag{22}$$

where m is a perturbing mass at distance r. However, for this conjecture the torque on a small perturbing mass affects its orbit even in the limit $m \rightarrow 0$—an inadmissible violation of geodesic motion for test particles.

As the next alternative, we might try the substitution

$$(\text{scalar energy density}) \sim q^2/r^4 \rightarrow m^2/r^4\ ; \tag{23}$$

i.e., we replace a perturbing scalar charge q by a perturbing mass m, even though the resulting quantity (23) can no longer be interpreted as an energy density. However, preliminary results from work by Hartle and Hawking (1972) indicate that this substitution neglects an important effect: since the black hole cannot avoid "falling freely" in the external field, it cannot experience a dipole gravitational perturbation. Thus the gravitational perturbation which acts on it must be at least quadrupole; the spin-down time will be increased from the dipole estimate by *at least* one power of (distance to perturbing mass)/(size of hole) $\sim r/M$. Thus we are led to a spin-down time (eqs. [2] and [23] with extra r/M)

$$\tau \sim \left(\frac{1}{M}\right)\left(\frac{r^4}{m^2}\right)\left(\frac{r}{M}\right)^n , \qquad n \geq 1\ . \tag{24}$$

This information is sufficient to derive an important limit on the astrophysical importance of the effect. Consider a black hole in a binary star system; let the mass m of the "perturbing" companion be equal to the mass M of the hole. The system has a Keplerian period (disregarding constant factors)

$$\tau_{\text{Kepler}} \sim r(r/M)^{1/2}\ . \tag{25}$$

There is also a second timescale $\tau_{\text{radiation}}$ for the decay of its orbit by gravitational radiation damping,

$$\tau_{\text{radiation}} \sim r(r/M)^3\ . \tag{26}$$

Equation (24) gives a third timescale, that for the loss of the Kerr hole's angular momentum:

$$\tau_{\text{ang mom loss}} \sim r(r/M)^{3+n}\ , \qquad n \geq 1\ . \tag{27}$$

Thus, for astrophysical black holes, it appears that there will never be sufficient time for the spin-down effect to operate significantly. Hawking's theorem places no restrictions on the possibility of finding rotating black holes in nature.

I thank James Ipser for making unpublished work available to me and for helpful discussion; and I thank S. Teukolsky, K. S. Thorne, J. B. Hartle, and R. P. Feynman for their helpful suggestions. Partial stimulus for this work came from a private letter to Thorne from B. Carter which described discussions with S. W. Hawking and Hartle.

APPENDIX A

DERIVATION OF BOUNDARY CONDITION (9a)

Consider an observer at some radius r very near the event horizon, so that the relation

$$r^2 + a^2 = 2Mr(1 + \epsilon) \tag{A1}$$

defines a small positive parameter ϵ. The requirement that the observer follow a time-like world line, and the form of the metric (3) yield a restriction on the observer's angular velocity

$$\frac{a}{2Mr}(1 - d) < \frac{d\varphi(t)}{dt} < \frac{a}{2Mr}(1 + d) , \tag{A2}$$

where

$$d = \frac{r^2 + a^2 \cos^2 \theta}{2Mr}(\epsilon)^{1/2} + O(\epsilon) , \tag{A3}$$

is also small. Choose a particular space-filling congruence of observers near the horizon with the world lines

$$r(t) = r_+ + \left(\frac{Mr_+}{r_+ - M}\right)\epsilon + O(\epsilon^2) , \quad \theta(t) = \theta_0 , \quad \varphi(t) = \varphi_0 + \frac{a}{2Mr}t . \tag{A4}$$

(To order ϵ these are Bardeen's "locally nonrotating observers.") Any timelike observer momentarily differs from one member of this congruence by at most a Lorentz transformation.

The stress-energy tensor for the scalar field is given by

$$T_{\mu\nu} = \frac{1}{4\pi}(\Omega_{,\mu}\Omega_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\Omega^{,\alpha}\Omega_{,\alpha}) , \tag{A5}$$

so that

$$-T_{\hat{t}\hat{r}} = -\frac{1}{4\pi}\Omega_{,\hat{t}}\Omega_{,\hat{r}} \tag{A6}$$

is the radial energy flux seen by an observer in the congruence. Here $\Omega = \Omega[t, r(t), \theta(t), \varphi(t)]$ is the field seen by the observer and "hats" denote his orthonormal frame.

The metric (3) and world line (A4) combine to give

$$\frac{dr}{d\hat{r}} = \left(\frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right)^{1/2}(\epsilon)^{1/2}[1 + O(\epsilon)] \tag{A7a}$$

$$\frac{dt}{d\hat{t}} = \left(\frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right)^{1/2}(\epsilon)^{-1/2}[1 + O(\epsilon)] , \tag{A7b}$$

while for the solution (9a) (with the angular dependence of eq. [6]),

$$\Omega_{,r} = \frac{ima\Omega}{2Mr\epsilon}[1 + O(\epsilon)] \tag{A8a}$$

$$\Omega_{,t} = \frac{ima\Omega}{2Mr} . \tag{A8b}$$

Combining (A7) and (A8), one obtains

$$\Omega_{,\hat{r}} = \Omega_{,\hat{t}} = \frac{ima\Omega}{[(2Mr)(r^2 + a^2 \cos^2 \theta)\epsilon]^{1/2}}[1 + O(\epsilon)] , \tag{A9}$$

which shows that the wave is a function of $\hat{r} + \hat{t}$ only, i.e., inward propagating. The radial energy flux (A6), with the real part of equation (A9) taken as the physical field, is

$$-T_{\hat{t}\hat{r}} = \frac{-m^2a^2\Omega^2}{8\pi Mr(r^2 + a^2\cos^2\theta)\epsilon}[1 + O(\epsilon)] \tag{A10}$$

which is negative, i.e., an inward energy flux. The other independent solution, the complex conjugate to boundary condition (9a), would give the opposite sign in equation (A8a) and an outward flux. It is therefore unacceptable. Any linear *combination* of expression (9a) and its complex conjugate has a "standing-wave" component and is likewise not acceptable. Since the sign of equation (A8) is preserved under all Lorentz transformations, all timelike observers agree on the correct boundary condition (9a). In fact, it is not difficult to show that a radial Lorentz transformation to the frame of an inward-falling observer removes the $(1/\epsilon)$ singularity in equation (A10), while for an outward-falling observer, the singularity is increased to $(1/\epsilon^2)$. This shows that the correct solution (9a) is regular on the future (ingoing) event horizon, and singular on the past (outgoing) horizon. For an astrophysical black hole, of course, there is no past horizon, there are no outgoing observers, and the past singularity is fictitious.

APPENDIX B

DERIVATION OF EQUATIONS (15)

Equation (11) is used to write the charge distribution (13) as a sum of three terms

$$(g_{rr})^{1/2}\sigma = \sigma_0{}^1 Y_{11} + \sigma_0{}^0 Y_{10} + \sigma_0{}^{-1} Y_{1-1}\,, \tag{B1}$$

where

$$\sigma_0{}^0 = E(3/4\pi)^{1/2}\cos\gamma\,, \tag{B2a}$$

$$\sigma_0{}^1 = -\sigma_0{}^{-1} = -E(3/8\pi)^{1/2}\sin\gamma\,. \tag{B2b}$$

Equation (12) then gives

$$\Omega = (4\pi/3)(b^2 - 2Mb + a^2)\sum_{m=-1}^{+1}\sigma_0{}^m R^+{}_{1m}(r_<)R^\infty{}_{1m}(r_>)\,Y_{1m} \tag{B3}$$

with $r_<$, $r_>$ denoting the lesser or greater of r and b, respectively. It is convenient to define

$$f_m(r) = (4\pi/3)(b^2 - 2Mb + a^2)R^+{}_{1m}(r_<)R^\infty{}_{1m}(r_>)\,, \tag{B4}$$

so that

$$\Omega = \sigma_0{}^0 f_0 Y_{10} + \sigma_0{}^1(f_1 Y_{11} - f_1{}^* Y_{1-1})\,, \tag{B5}$$

where we have used the fact $f_1 = f_{-1}{}^*$. Using definitions (14) and elementary properties of the spherical harmonics Y_{1m}, one obtains

$$L_z\Omega = i\sigma_0{}^1(f_1 Y_{11} + f_1{}^* Y_{1-1})\,, \tag{B6a}$$

$$2^{1/2}L_x\Omega = i\sigma_0{}^0 f_0(Y_{11} + Y_{1-1}) + i\sigma_0{}^1(f_1 Y_{10} - f_1{}^* Y_{10})\,, \tag{B6b}$$

$$2^{1/2}L_y\Omega = \sigma_0{}^0 f_0(Y_{11} - Y_{1-1}) - \sigma_0{}^1(f_1 Y_{10} + f_1{}^* Y_{10})\,. \tag{B6c}$$

Now multiply by σ to get $\sigma L\Omega$, evaluated on the shell $r = b$. Note that one is entitled to suppress terms with products of φ-orthogonal spherical harmonics, since everything is φ-symmetric, even near the black hole:

$$\begin{aligned}(g_{rr})^{1/2}\sigma L_z\Omega &= i(\sigma_0{}^1)^2(Y_{11}Y_{1-1}f_1{}^* - Y_{11}Y_{1-1}f_1) + (\varphi\text{-orthogonal term})\\ &= -2\,\mathrm{Im}\,f_1(\sigma_0{}^1)^2|Y_{11}|^2 + (\varphi\text{-orthogonal terms})\,.\end{aligned} \tag{B7a}$$

Likewise,

$$(g_{rr})^{1/2}\sigma L_x\Omega = -2^{1/2}\sigma_0{}^1\sigma_0{}^0 |Y_{10}|^2 \operatorname{Im} f_1 + (\varphi\text{-orthogonal terms}) , \quad \text{(B7b)}$$

$$(g_{rr})^{1/2}\sigma L_y\Omega = +2^{1/2}\sigma_0{}^1\sigma_0{}^0 (f_0 |Y_{11}|^2 - \operatorname{Re} f_1 |Y_{10}|^2) + (\varphi\text{-orthogonal terms}) . \quad \text{(B7c)}$$

Definitions (9), (10), and (B4) then give

$$\operatorname{Im} f_1(b) \rightarrow (4\pi/3)C_{11}(a)/b^2 \, [1 + O(m/b)] \quad \text{for large } b . \quad \text{(B8)}$$

Substitution of expressions (B8) and (B2) into equations (B7a) and (B7b) yields equations (15a) and (15b). Equation (15c) is just equation (B7c) with the fact that f_0 is real emphasized. In equations (15) the formal integration over proper area $d\Sigma$ entitles one to omit the φ-orthogonal terms.

REFERENCES

Bardeen, J. M. 1970, *Ap. J.*, **162**, 71.
Boyer, R. H., and Lindquist, R. W. 1967, *J. Math. Phys.*, **8**, 265.
Brill, D. R., Chrzanowski, P. L., Pereira, C. M., Fackerell, E. D., and Ipser, J. R. 1972, paper to be published.
Carter, B. 1968, *Comm. Math. Phys.*, **10**, 280.
———. 1971, *Phys. Rev. Letters*, **26**, 331.
Chase, J. E. 1970, *Comm. Math. Phys.*, **19**, 276.
Christodoulou, D. 1970, *Phys. Rev. Letters*, **25**, 1596.
Fackerell, E. D., and Ipser, J. R. 1972, *Phys. Rev., D* (in press).
Hartle, J. B., and Hawking, S. W. 1972, to be published.
Hawking, S. W. 1972, *Comm. Math. Phys.*, **25**, 152.
Ipser, J. R. 1971, *Phys. Rev. Letters*, **27**, 529.
Misner, C., Thorne, K., and Wheeler, J. 1972, *Gravitation* (San Francisco: W. H. Freeman & Co.) (in press).
Teukolsky, S. A. 1972, paper to be published.
Wald, R. M. 1971, *Phys. Rev. Letters*, **26**, 1653.