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AN ANALYTIC APPROACH TO THE EVOLUTION OF DEGENERATE CARBON CORES OF STARS

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ABSTRACT

An analytic procedure for calculation of the evolution of degenerate massive stellar cores is described and compared with detailed numerical results. This procedure provides a simple computational method for investigating such objects as well as a simple physical picture for understanding them.

I. INTRODUCTION

The work of Rose (1969), Paczyński (1970), and Arnett (1969) has shown interesting behavior in the evolution of stars whose core masses are well below the Chandrasekhar limit at the conclusion of the core helium-burning. As was shown by Weigert (1966), the weak interactions predicted by a direct e-ν coupling can cool such a core sufficiently to inhibit the ignition of carbon burning by the $^{12}C + {}^{12}C$ reaction. The subsequent evolution cannot be followed in a straightforward way because of the appearance of a thin-shell thermal instability associated with the helium-burning shell source. Rose (1969), Paczyński (1970), and Arnett (1969) avoided this difficulty by the assumption that the average behavior was similar to that predicted by calculations in which the thermal instabilities were artificially suppressed. These authors found that the $^{12}C + {}^{12}C$ reaction eventually did ignite at high density ($\rho \sim 2 \times 10^9$ g cm^{-3}). Under such conditions a detonation wave develops, and if electron capture is unimportant the core explodes and leaves no remnant (Arnett 1969; Wheeler, Barkat, and Buchler 1970). However, if the density at ignition is sufficiently high, electron capture can occur rapidly in the detonated material, and a decrease in pressure will result from the removal of electrons. The precise nature of subsequent behavior is unclear at present, but it seems likely that, in some cases at least, a neutron star might be formed. This possibility is an intriguing one for the theoretician, as it may identify the stellar progenitors of pulsars (Gunn and Ostriker 1970). If stars in the mass range of, say, 4–8 $M_\odot$ underwent supernova explosions and formed pulsars, then the predicted supernova rate and the predicted number density of pulsars would be reasonably consistent with observations. It is therefore an important theoretical problem to reinvestigate the approach to $^{12}C + {}^{12}C$ ignition, with special emphasis on the effect of input physics upon the ignition density. Some research of this type has been hampered by a lack of numerical detail in the published models (Barkat 1971). The purpose of this paper is to illustrate that such details may be obtained from an analytic technique with adequate accuracy. Sophisticated physics may be readily incorporated into this procedure, and the necessity of performing complex numerical integrations of stellar structure is eliminated.

In § I the approximations made possible by the thin-shell nuclear burning and the highly degenerate core are discussed. In § II the method is demonstrated for the evolution of a pure ^{12}C-^{16}O core, and compared with previously published investigations. In § III the question of $^{12}C + {}^{12}C$ ignition is considered.

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II. THIN-SHELL AND DEGENERATE-CORE APPROXIMATIONS

The luminosity of a thin shell is (see Eggleton 1967, and references therein):

$$L \simeq 4\pi c G M_c (1 - \beta)/\kappa \,, \tag{1}$$

where the symbols have their usual meaning and M_c is the mass interior to the shell source. If we assume that the hydrogen- and the helium-burning shells process mass at the same rate, we have

$$dM_{\mathrm{He}}/dt \approx dM_{\mathrm{H}}/dt = L_{\mathrm{H}}/(Q_{\mathrm{H}} X_{\mathrm{H}}) \,, \tag{2}$$

where L_{H} is the luminosity of the hydrogen shell, Q_{H} the energy released per gram of hydrogen consumed, and X_{H} the initial fraction by mass of hydrogen.

In the hydrogen-burning shell the opacity is mostly due to Thomson scattering. Therefore, the rate of increase of the mass of the core is given by

$$dM_c/dt = 2.2 \times 10^{-14} \left(\frac{M_c}{M_\odot}\right) \left(\frac{0.75}{X_{\mathrm{H}}}\right) (1 - \beta) M_\odot \,\mathrm{s}^{-1} \,. \tag{3}$$

The first law of thermodynamics may be written as

$$dT/dt = [dq/dt - (P + E_V) dV/dt]/E_T \,, \tag{4}$$

where $V \approx 1/\rho$, $E_V \equiv (\partial E/\partial V)_T$, and $E_T \approx (\partial E/\partial T)_V$. Now the stellar core is very degenerate, and can be approximated well by a white-dwarf model of Chandrasekhar (1939). For such models the central density ρ_c (and in fact the density at any given Lagrangian mass coordinate) is a function of the mass of the core. Therefore $\rho_c = \rho_c(M_c)$, and

$$d\rho_c/dt = \frac{\partial \rho_c}{\partial M_c} dM_c/dt \,. \tag{5}$$

If we denote Chandrasekhar's (1939) mass variable M_c/μ_e^2 by M^*, and his density ρ/μ_e by ρ^*, then a useful approximation is

$$\frac{d \ln \rho_c}{d \ln M_c} = N \left(\frac{M^*}{5.75 - M^*}\right) , \tag{6}$$

where the quantity

$$N \equiv -d \ln (2\rho^*)/d \ln (5.75 - M^*) \tag{7}$$

varies relatively slowly with M^*. This approximation was obtained by graphical examination of Chandrasekhar's (1939) numerical results. At $\rho_c \simeq 2 \times 10^9$ g cm^{-3}, we have $N \approx 1.7$. Now, take

$$\frac{dT}{dt} = \frac{\alpha T dV/dt}{V} , \tag{8}$$

where α is a parameter which may be determined by iteration. The energy loss by neutrino radiation dominates the thermal evolution of the core until after the ignition of $^{12}C + {}^{12}C$. Because of this, and because of the sensitive dependence of these neutrino energy-loss rates upon temperature, we may take $\alpha = 0$ as an excellent first approximation. Then, having determined a path of evolution (by the methods to be discussed below), we may make a better guess for α by using $\alpha_1 = (\partial \ln T/\partial \ln V)$ evaluated along this evolutionary path, and recalculate the evolution with $\alpha \approx \alpha_1$. The two paths so computed for the evolution described in § II are virtually identical, so that convergence is rapid.

If we assume for the moment that α has been specified by this or some other procedure, equation (4) becomes

$$\frac{dq}{dt} = \left[(P + E_V) + \frac{\alpha T E_T}{V}\right]\frac{dV}{dt} \approx -\left(\frac{P + E_V}{\rho} + \alpha T E_T\right)\frac{1}{\rho}\frac{d\rho}{dt}. \tag{9}$$

Concentrating upon the center (at which ignition of $^{12}C + ^{12}C$ will eventually occur for the imput physics used here), we combine equations (3), (5), (6), and (7) to give

$$\frac{1}{\rho_c}\frac{d\rho_c}{dt} = \frac{N}{1.4375 - M_c}(2.2 \times 10^{-14} M_c)\ \mathrm{s}^{-1}, \tag{10}$$

where M_c is measured in solar units and $0.75\,(1 - \beta)/X_{\mathrm{H}}$ is taken to be unity.

For the degenerate electron gas, we find the expression

$$\left(\frac{p + E_V}{\rho}\right) = \frac{\pi^2 m_e c^2 \mathcal{A}}{\mu_e x}\frac{kT}{m_e c^2} \tag{11}$$

where $\mathcal{A}$ is Avogadro's number, μ_e the number of atomic mass units per electron, $x^3 = 1.0268 \times 10^{-6}\ \rho/\mu_e$, and the other symbols have their usual meaning. Note that for a *completely* degenerate gas ($T = 0$), $P + E_V = 0$. For the ions (to the extent they may be assumed to be an ideal gas), we find

$$\left(\frac{P + E_V}{\rho}\right)_{\mathrm{ions}} = \Sigma\left(\frac{X_i}{A_i}\right)\mathcal{R}T = 6.0 \times 10^{15}\,T_9\,, \tag{12}$$

where X_i is the abundance mass fraction and A_i the atomic number of species i, $\mathcal{R}$ is the gas constant, and $T_9 \equiv T/10^9\ ^\circ\mathrm{K}$. The numerical factor is for an equal mixture by mass of ^{12}C and ^{16}O. In the region of interest here, i.e., extreme degeneracy, the electron contribution is small (about one-fifth of the ion contribution). If (for simplicity in exposition) we ignore the contribution from electrons (eq. [11]), and take $\alpha = 0$ in equation (8), then

$$dq_c/dt = -231 T_9\left(\frac{M_c}{1.4375 - M_c}\right)\ \mathrm{ergs\ g^{-1}\ s^{-1}}, \tag{13}$$

where M_c is the core mass in solar units. This equation relates the rate of energy loss at the center of the core (by neutrinos in this case) to the core mass M_c and the central temperature. We could equally well have picked some other point in the core to consider. If we pick a particular process which will dominate the energy loss, i.e., define a relation:

$$dq_c/dt = f(T_9, \rho_c)\,, \tag{14}$$

then for a given M_c we can solve for the temperature. Clearly a similar procedure will work even if electron contributions to $(P + E_V)/\rho$, and $\alpha \neq 0$ corrections from equation (8), are included. Thus we can determine the trajectory defined by the evolution in time of a core of mass $M_c(t)$ in the temperature-density plane, and hence determine its complete structure.

III. EVOLUTION OF A PURE ^{12}C-^{16}O CORE

In order to illustrate the power of the technique just described we examine the evolution of a pure ^{12}C-^{16}O core. The composition was chosen to avoid computational complexities associated with the nuclear Urca process upon nuclei of low abundance; this neglect is to be remedied in a future paper. We note that both ^{12}C and ^{16}O are resistant to electron capture; for these nuclei this process may be neglected below $\rho \approx 2 \times 10^{10}$ g cm^{-3}.

First we consider the neutrino-pair bremsstrahlung process of Festa and Ruderman (1969). The rate of energy loss by neutrino emission is

$$dq/dt \simeq 1.01 T_8{}^6 \text{ ergs g}^{-1}\text{ s}^{-1}\,, \tag{15}$$

where $T_8 \equiv T/10^8\,^\circ$ K. Equations (13) and (15) may be solved to give temperature as a function of core mass:

$$T_8 \simeq \{23.1 M_c/(1.4375 - M_c)\}^{1/5}\,. \tag{16}$$

The path in the (ρ, T)-plane implied by this relation is shown in Figure 1.

In the region of interest the analytic fits to the neutrino energy-loss rates of Beaudet, Petrosian, and Salpeter (1967) reduce to the plasma-neutrino rate:

$$\frac{dq}{dt} \approx 8.94 \times 10^9 \left(\frac{1}{\mu_e}\right)\left(\frac{\rho_9}{\mu_e}\right)^2 \frac{1}{\xi} \exp\,(-0.56457\xi)\,, \tag{17}$$

where $\rho_9 = \rho/(10^9 \text{ g cm}^{-3})$ and $\xi = (\rho_9/\mu_e)^{1/3}\, 60/T_8$. Solving equations (17) and (13), we find central temperature as a function of core mass. The corresponding trajectory in the (ρ, T)-plane is shown in Figure 1. Note that for all $\rho < 10^{10}$ g cm^{-3} the plasma-neutrino trajectory lies at a lower temperature than the neutrino-pair bremsstrahlung trajectory; this means that the plasma-neutrino loss rate is dominant. For reference the core trajectory of Arnett (1969) is also plotted in Figure 1. In that paper the choice of an approximate numerical procedure at the core boundary resulted in an overestimate of dM_c/dt and hence of dq/dt and T_c. The kink at $\rho_c \approx 1.8 \times 10^9$ g cm^{-3} is due to $^{12}C + ^{12}C$ ignition, a process to be discussed in the next section.

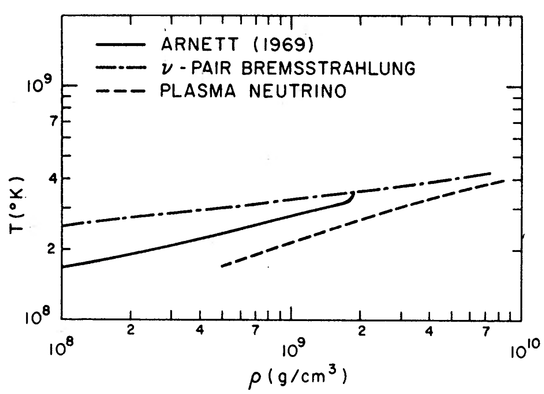

FIG. 1

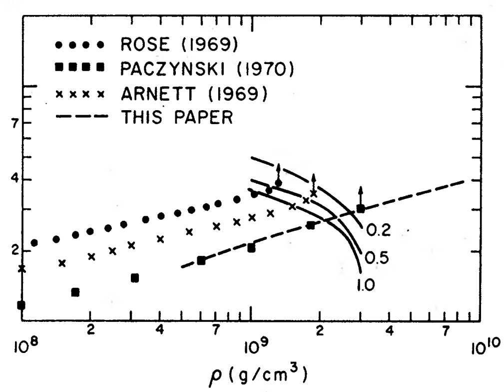

FIG. 2

FIG. 1.—Evolutionary tracks of the stellar centers in the (ρ, T)-plane. Tracks calculated by considering only the plasma-neutrino process, and only the neutrino-pair bremsstrahlung process, are shown. The figure shows that the plasma-neutrino process is the important one under these conditions. The evolutionary path previously calculated by Arnett (1969) is also shown for reference.

FIG. 2.—Comparison of evolutionary tracks by various authors of the stellar centers in the (ρ, T)-plane. *Solid squares*, track of Paczyński (1971); *solid circles*, that of Rose (1969); *crosses*, that of Arnett (1969); *dashed lines*, this paper. Vertical arrows mark the point of ignition of $^{12}C + ^{12}C$ for the computer calculations (the corresponding points for this paper are given in Table 1). Solid curves labeled 1.0, 0.5, and 0.2 show the ignition contours (see eq. [21]) for those abundance fractions by mass of ^{12}C. The differences between the evolutionary path predicted in this paper and previous results seem to be due to different input physics (Rose 1969) or to mathematical errors (Arnett 1969; see text for details). The excellent agreement between the dashed curve and the points of Paczyński (1971) indicates the accuracy of simple technique explained in this paper.

In Figure 2 the trajectory obtained by using the plasma-neutrino rate is compared with the published results of Rose (1969), Paczyński (1970), and Arnett (1969). Since Rose dealt with helium stars, the shell luminosity (eq. [1]) had to be produced by helium-burning. Since $Q_{He} \approx 0.1Q_H$, the rate of growth of the core was estimated to be higher by a factor of 10, resulting in higher temperatures for the same density. The trajectory of Paczyński (1971) agrees with the plasma-neutrino trajectory to excellent accuracy.

IV. IGNITION OF $^{12}C + ^{12}C$

Arnett (1969) used the rate of energy generation derived by Arnett and Truran (1969) and based upon the experimental results of $^{12}C + ^{12}C$ obtained by Patterson, Winkler, and Zaidins (1969). Arnett (1969) used the strong-screening factor for $^{12}C + ^{12}C$ quoted by Reeves (1965, p. 171):

$$f = \exp (U_{SO}) , \tag{18}$$

where

$$U_{SO} = 35\rho_9^{1/3}/T_8 . \tag{19}$$

Salpeter and Van Horn (1969) give expressions[1] from which we find

$$U_{SO} = 37.9\rho_9^{1/3}/T_8 \tag{20}$$

and also give a correction term U_{SI} which *decreases* f. In the region of interest ($\rho \approx 2.5 \times 10^9$ g cm^{-3}, and $T \simeq 3 \times 10^8$ ° K), f(Reeves), and f(Salpeter–Van Horn) = $\exp (U_{SO} + U_{SI})$ are the same to within about a factor or 2. This is quite a small difference since $f \simeq 10^7$. The Salpeter–Van Horn formulation (which was used here) should give essentially the same results as that of Reeves.

We define an ignition temperature by equating nuclear energy generation to neutrino energy loss:

$$dq_{^{12}C}(\rho, T, X_{12})/dt = dq_\nu(\rho, T)/dt , \tag{21}$$

where X_{12} is the fraction by mass of ^{12}C. For a choice of X_{12} and ρ, an "ignition temperature" is determined. The correct choice of X_{12} is uncertain because of the uncertainty in the rate of $^{12}C(\alpha,\gamma)^{16}O$. Contours obtained from equation (21) are plotted in Figure 2 for $X_{12} = 1.0$, 0.5, and 0.2. Note that the ignition of $^{12}C + ^{12}C$ for the Arnett (1969) model and the Rose (1969) model is accurately predicted by the intersection of the evolutionary trajectory and the $X_{12} = 0.5$ contour, as it should be. Ignition for the Paczyński (1970) model is less well predicted by this contour. This could be due to numerical inconsistencies, but is probably due to a different ^{12}C abundance in the core. Paczyński actually calculated the ^{12}C abundance left by core helium-burning (with $\theta_\alpha^2 = 0.08$ for $^{12}C(\alpha,\gamma)^{16}O$) rather than assume $X_{12} = 0.5$. Unfortunately, I do not now know what Paczyński's X_{12} value was. Nevertheless, the basic procedure used here must be correct because Paczyński's models satisfy equation (21) at $^{12}C + ^{12}C$ ignition as shown in his Figure 2. The difference in the two values of ρ_{ign} is quite small ($\rho_{ign} \approx 2.5 \times 10^9$ versus 3.0×10^9 g cm^{-3}), and the actual *error* introduced by our analytic procedure certainly is much less than that difference.

Table 1 gives the ignition densities ρ_{ign} for $^{12}C + ^{12}C$ obtained from equation (21). If we correct for the finite thermal relaxation time, we must increase them by a factor 1.08 to get ρ_{ign}(corrected). This increase in density was obtained by calculating the growth of the core in the e-folding time for temperature increase. For such densities detailed calculations indicate that electron capture has relatively minor effects (Arnett, Truran, and Woosley 1971), and is incapable of reversing the explosion. We must emphasize, however, that the *neglect of Urca processes upon nuclei other than* ^{12}C *and* ^{16}O *is*

[1] Beware of typographical errors!

TABLE 1

IGNITION DENSITIES FOR $^{12}C+^{12}C$
(in units of 10^9 g cm^{-3})

X_{12}	ρ_{ign}	ρ_{ign}(corrected)
1.0	2.0	2.16
0.5	2.3	2.48
0.2	2.6	2.81

invalid for all but extreme Population II objects. Consequently, the ignition densities in Table 1 may need revision when calculations including Urca processes for Population I stars and other refinements of the input physics are completed.

V. SUMMARY

An analytic procedure for calculating the evolution of degenerate stellar cores has been presented. Even in its simplest form the procedure reproduces the results of sophisticated computer codes with excellent accuracy, and should aid in further investigation of this problem.

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