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COHERENT EMISSION FROM EXPANDING SUPERNOVA SHELLS

STIRLING A. COLGATE AND PETER D. NOERDLINGER
New Mexico Institute of Mining and Technology, Socorro, New Mexico 87801
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ABSTRACT

When a stellar core of magnetic moment μ undergoes supernova explosion, at some radius r_0 the expanding shell of photons and hot gas may attain a relativistic velocity V, with associated Lorentz factor $\gamma = (1 - V^2/c^2)^{-1/2}$ on the order of 3×10^3 or more. This expanding shell will comb the initial dipole field into a radial configuration, and the associated current sheet will produce coherent radio emission of power as a function of frequency ν

$$P(\nu)d\nu = \tfrac{4}{3}\mu^2 c^{-1} r_0^{-2} \psi(q) d\nu \text{ ergs} ,$$

where $q = \pi\nu r_0 c^{-1}\gamma^{-2}$ and the function $\psi(q)$ is unity for $q \ll 1$ and falls to roughly $\frac{1}{3}$ at $q = 1$. The resulting emission in the range of 300 MHz corresponds to an antenna temperature of about 1.5×10^7 ° K for a supernova event per day per steradian at 100 Mpc, dispersed by the galactic and intergalactic electron density.

I. INTRODUCTION

If presupernova stars have an external magnetic field, then the ejection of the outermost layers of matter at relativistic velocity will generate a powerful pulse of electromagnetic radiation by pushing on this magnetic field. We show herein that radiofrequency pulses detectable at distances of many megaparsecs are expected.

Colgate and Johnson (1960) predicted that the outer layers of a supernova should be accelerated to relativistic energies. The numerical hydrodynamic calculations of Colgate and White (1966) quantitatively evaluated the *nonrelativistic* shock acceleration in spherically symmetric distributions of matter, and showed that the velocity of the material behind the shock increases as $F^{-\alpha}$, where $\alpha \approx \frac{1}{4}$ and F is the fraction of mass exterior to the point in question. This behavior was independent of the exact density distribution (polytrope or red-giant structure) and was predictable from elementary laws of hydrodynamics (Ono, Sakashita, and Ohyama 1961; Grover and Hardy 1966). Hence, regardless of the mechanism of the supernova, one would expect the shock wave from the explosion to increase markedly in speed as it approaches the stellar surface. If the average velocity of the bulk of the ejected matter is 2×10^9 cm sec^{-1} as derived for Tycho's supernova (Minkowski 1968) in its early stages, only a thirteenfold increase in velocity leads to the relativistic regime. Extrapolation of the $F^{-1/4}$ law in this direction indicates that relativistic energies are reached at an external mass fraction $F_r = (13)^{-4} \approx 3 \times 10^{-5}$. Although this mass fraction may seem small, it is still deep (10^{11} g cm^{-2}) within the star, and the total mass of matter (6×10^{28} g) thus accelerated to greater than 1 GeV per nucleon is large. As discussed by Colgate and Johnson, the shock wave is expected to propagate relativistically until it "breaks out" of the star, but at the present state of the development of the theory, the exact hydrodynamic behavior is not known; instead, limits can be set which will be narrowed by further theoretical and numerical work. For our purposes it will be sufficient to determine whether the nature of the breakout is such as to liberate the shock energy mainly in incoherent radiation (photons or particles), or whether the energy of the driving gas can be expended in doing work on the exterior magnetic field; the latter process leads to our powerful electromagnetic pulse.

II. SHOCKS

A strong nonrelativistic shock divides the energy equally between kinetic and internal energy in the fluid behind the shock. The compression ratio of the fluid is fixed and is relatively small (7:1) across the discontinuity. In the case of a shock in a density gradient, the aforementioned numerical calculations show that the subsequent expansion of the internal energy behind the shock further increases the velocity of the matter by the fixed ratio $\times 2$.

Relativistic shocks are somewhat different, as described by the equivalent junction relations (May and White 1967; Johnson and McKee 1970). The feature that most nearly resembles nonrelativistic shocks is that the energy is divided equally between translational energy (relative to the laboratory frame) and internal energy behind the shock (measured in the proper frame of the moving fluid). The velocities of the shock itself and the fluid behind it nearly match, because the compression ratio is no longer bounded, but increases for strong shocks ($\rightarrow 4\gamma_s$) in proportion to the Lorentz factor $\gamma_s = (1 - \beta^2)^{-1/2}$ of the post-shock fluid, where βc is the laboratory velocity of that fluid and the compression ratio is defined as the ratio of pre- to post-shock fluid densities, measured in each case in the local rest frame of the fluid. If the specific internal energy density ϵ is equal to $c^2\gamma_s$, then the total *laboratory* value for the energy per gram of rest mass in the post-shock fluid becomes $\gamma_s(c^2 + \epsilon) \approx c^2\gamma_s^2$. Hence, in the relativistic case, the total energy per unit rest mass behind the shock is very much larger than its translational kinetic energy. As a result, in a subsequent expansion into vacuum, a very large change in translational energy takes place ($\cong \gamma_s$) as compared with a relatively modest increase ($\times 4$) in the nonrelativistic case. This is in accord with the fact that the equivalent mass density of the energy in the moving frame, ϵ/c^2, is much greater than the rest mass. The theory predicts

$$\gamma_s \propto F^{-\alpha} , \tag{1}$$

where $0.234 \leq \alpha \leq 0.33$, and a final kinetic (translational) energy factor after expansion, $\gamma_F \propto F^{-K}$, where

$$0.64 \leq K \leq 0.67 .$$

The numerical work of May and White resulted in $K = 0.42$ for partial expansion, although this value may perhaps contain a significant error.

For the analysis of the electromagnetic pulse, we choose $\alpha = \frac{1}{3}$, $K = \frac{2}{3}$, which in previous work (Colgate and Johnson 1960) is a reasonable estimate and which, in addition, is a self-consistent assumption that produces the integral cosmic-ray spectrum $N(>E) \propto F \propto \gamma_F^{-3/2}$. The shock propagates within matter, and at some boundary the expanding matter will perturb the "vacuum" magnetic field. We then ask, "At what γ_s will the fluid behind the shock 'break out' and 'displace' the magnetic field?" We can define "breakout" on a number of bases:

1. Breakout occurs when the matter density is sufficiently low such that the hydromagnetic approximation breaks down due to lack of current carriers. External to this boundary the volume currents in the field can give rise to only small perturbations of the field.

2. Breakout occurs when the residual thickness of matter is less than that required to contain the Planck radiation behind the shock. The energy density in the shocked fluid resides entirely in the gas of photons and e^+e^- pairs, so that if the photons escape the fluid, the shock no longer propagates.

3. Breakout occurs when the mass fraction of external matter is equal to the equivalent mass of the tangential component of the dipole magnetic field. The time average of the shock velocity increases if the total mass ahead (external mass fraction) is smaller

than the expanding mass behind. Conversely, if the mass ahead is larger (including all the equivalent rest mass of the magnetic field), the expansion will slow down.

Density condition 1 can be shown to be much less restrictive than conditions 2 and 3, and so we start with the latter two conditions, calculate the pertinent mass fraction, and confirm that condition 1 is not the limiting condition for breakout.

If we demand that the Planck radiation reaches equilibrium by radiation processes in the shock front, then the residual thickness of the star must be approximately 10 g cm^{-2}, which corresponds to several Compton mean free paths. In order to calculate the corresponding mass fraction, we must assume a stellar radius.

Current presupernova models (Finzi and Wolf 1967; Chiu 1966; Rakavy and Shaviv 1967; Arnett 1969) all depend upon dense (10^9–10^{10} g cm^{-3}) cores of degenerate matter where the degenerate electrons are relativistic and, hence, behave with an adiabatic index $\gamma = \frac{4}{3}$. The resulting structure, although not exactly a polytrope of index 3, is nevertheless sufficiently close to give a reasonable approximation to the radius as $1 \times 10^8 \leq r \leq 1.5 \times 10^8$ cm when $\rho_c = 2 \times 10^9$ g cm^{-3} for a total mass $1.5\,\mathfrak{M}_\odot \leq \mathfrak{M} \leq 2\,\mathfrak{M}_\odot$. The central density of $\geq 2 \times 10^9$ g cm^{-3} is presumably the point at which neutrino processes become rapid enough to initiate either a thermonuclear detonation or collapse to a neutron star—either of which should result in a similar explosion shock wave.

Using $r = 1.25 \times 10^8$ cm gives a mass fraction corresponding to a surface layer of 10 g cm^{-2} at radiation breakout of

$$F_{\mathrm{surface}} = \frac{4\pi 10 r^2}{\mathfrak{M}_\odot} = 10^{-15} . \tag{2}$$

The shock becomes relativistic at $F_R = 3 \times 10^{-5}$, so that at breakout

$$\gamma_s = \left(\frac{F_{\mathrm{surface}}}{F_R}\right)^{-1/3} = 3 \times 10^3 . \tag{3}$$

The condition for breakout, based upon the equivalent magnetic-field mass, assumes that if a mass fraction is to accelerate relativistically a finite volume of magnetic field, then the total mass energy of the field should not be larger than the rest mass of the moving piston ($\mathfrak{M}_\odot F_{\mathrm{surface}}$).

The equivalent mass fraction of the tangential component of the field is

$$F_B = \frac{1}{c^2 \mathfrak{M}_\odot} 4\pi \int_r^\infty \frac{B_\theta^2}{8\pi} r^2 dr . \tag{4}$$

For a dipole field, the maximum $B_\theta = \mu r^{-3}$, where μ is the magnetic moment of the star.

If we choose μ such that the maximum field at the surface of the possibly resulting neutron star is 2×10^{12} gauss at $r = 1.2 \times 10^6$ cm, as required for pulsars, and scale the field proportional to r^{-2} (constant flux), then B_θ max $= 10^8$ gauss. (The corresponding field at a solar radius would be 200 gauss, and recently circularly polarized light has been observed from a white dwarf, which implies a magnetic field $\geq 10^7$ gauss [Kemp *et al.* 1970].) Then at the magnetic equator, the equivalent mass fraction becomes

$$F_B = \frac{4\pi}{c^2 \mathfrak{M}_\odot} \frac{r^3}{3} \frac{B_\theta^2}{8\pi} = 10^{-15} ,$$

which is the same as the mass fraction of matter corresponding to the radiation breakout condition. Therefore, we consider the coherent radiation from a conducting surface expanding within a vacuum dipole magnetic field at a constant velocity corresponding to $\gamma_B = \gamma_s = 3 \times 10^3$.

Actually, as a refinement, we expect the velocity of expansion to be a function of both radius and angle. At the poles, where we have free expansion into vacuum, $\gamma_B \simeq \gamma_s^2$. Between the poles and equator of the dipole, conservation of total energy of the combined surface layer (mass fraction) and B_θ field implies (averaged over the radial expansion)

$$B_\theta^2 \gamma_B^2 = \text{constant or } \gamma_B = \frac{\gamma_s}{\sin\theta}\,. \tag{5}$$

Finally, the condition of "conducting" is equivalent to breakout condition 1 and requires that there exist sufficient current carriers moving at a limiting velocity c within the thickness of 10 g cm^{-2} to bound the compressed field $\gamma_B B_\theta$ in the moving frame. Therefore, the current sheet is $k = c\gamma_B B_\theta/(4\pi) = Nec$, or $N = 5 \times 10^{19}$ electrons cm^{-2} or 9×10^{-5} g cm^{-2}. This is well satisfied by the "surface layer" condition 2.

Therefore, we calculate the ensuing radiation on the assumptions that (*a*) the initial exterior field is of dipole form; (*b*) the conducting surface starts at some radius r_0 and expands at constant speed V; and (*c*) what happens in the interior of the conducting surface is irrelevant, the essential boundary condition being that in the rest frame of the expanding shell the tangential electric field should vanish at the shell surface. Behind the contact surface, we expect radial field lines, as in the solar wind of a nonrotating Sun. The expanding surface essentially "combs" the dipolar lines into radial lines. Surface currents arrange themselves on the expanding shell so as to destroy the tangential component of $\boldsymbol{B}$ behind the surface (see Fig. 1). In the sequel we explicitly write down the "combing" condition, which amounts to conservation of radial magnetic flux within any cone whose apex is at the center of the star, and we show that it is *redundant* with the boundary condition given in (*c*) above. This redundancy gives us confidence in the consistency of our model. Before the explosion, the magnetic field is

$$B_r = 2\mu r^{-3}\cos\theta\,, \qquad B_\theta = \mu r^{-3}\sin\theta\,, \tag{6}$$

where μ is the magnetic moment of the star. We now claim that the correct field outside the expanding shell must have the form of equation (6) plus a pure magnetic-dipole radiation and induction field, whose time dependence must, of course, be determined. This form was suggested by an earlier approximate analysis (Colgate and Noerdlinger 1970) and is justified by the fact that we are able to satisfy Maxwell's equations and all boundary conditions this way. Thus, we write $\boldsymbol{B} = \boldsymbol{B}_0(r) + \boldsymbol{B}_1(r, t)$, $\boldsymbol{E} = \boldsymbol{E}_1(r, t)$, where $\boldsymbol{B}_0$ is given by equation (6) and $(\boldsymbol{B}_1, \boldsymbol{E}_1)$ have the angular form of magnetic-dipole type

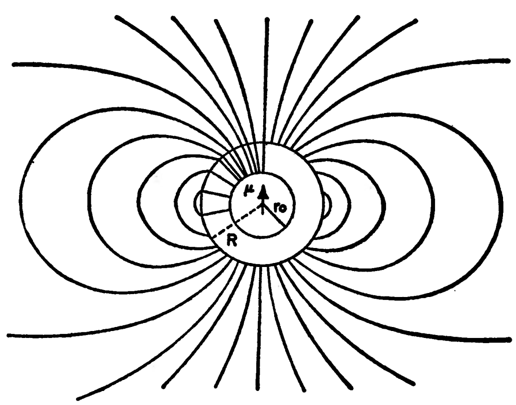

FIG. 1.—A dipole magnetic field of initial magnetic moment as at r_0 "combed" by an expanding current sheet to the radius R. The current sheet corresponds to the conducting surface of the exploding star.

containing only outgoing waves (Jackson 1962). The general dipole term is readily available only in Fourier-transform form, which we take directly from Jackson's equation (16.44). A single Fourier component at angular frequency ω is of the form

$$\boldsymbol{B}_1(\omega) = \mu f(\omega) e^{i\omega r/c}\left[2\hat{\boldsymbol{e}}_r \frac{\cos\theta}{r^2}\left(1 + \frac{ic}{\omega r}\right) - \hat{\boldsymbol{e}}_\theta \frac{i\omega \sin\theta}{cr}\left(1 + \frac{ic}{\omega r} - \frac{c^2}{\omega^2 r^2}\right)\right], \quad (7)$$

$$\boldsymbol{E}_1(\omega) = \mu f(\omega) e^{i\omega r/c} \hat{\boldsymbol{e}}_\phi \frac{i\omega \sin\theta}{cr}\left(1 + \frac{ic}{\omega r}\right). \quad (8)$$

Here we have introduced an undetermined Fourier strength $f(\omega)$ from which the expected μ-dependence has been factored.

Equations (7) and (8) are to be used only exterior to the shell

$$r(t) = r_0 \qquad (t \leq 0), \quad (9a)$$

$$r(t) = r_0 + Vt \qquad (t > 0). \quad (9b)$$

Since the boundary conditions must be expressed in terms of the time-dependent forms of $\boldsymbol{E}$ and $\boldsymbol{B}$, we introduce the Fourier and inverse Fourier operators

$$\mathfrak{F} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\omega t} dt\,; \qquad \mathfrak{F}^{-1} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega\,. \quad (10)$$

The time-dependent forms of $\boldsymbol{E}$ and $\boldsymbol{B}$ will be designated by the same symbols, with time as an argument, but to avoid confusion we denote $\mathfrak{F}^{-1} f(\omega) = F(t)$. The boundary conditions are that for $t < 0$ *and* $r > r_0$, $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ vanish, and that for $t > 0$, on the expanding shell we have

$$E_\phi + Vc^{-1} B_\theta = 0\,, \qquad r = r_0 + Vt\,. \quad (11)$$

Because the formal expressions (7) and (8) do not incorporate any means for excluding the region interior to the star before the explosion, it is necessary to ignore the spurious fields which accidentally appear in this region ($r < r_0$, $t < 0$), much as in simple image-charge problems the fields in the image region have no significance. Alternatively, one can physically interpret the fields that seem to be present in the excluded region as those that would have been produced by a hypothetical time-dependent point dipole at $r = 0$, which varied in such a way as to satisfy condition (11) for $t > 0$. In computing the emitted energy spectrum it is proper and most convenient to treat these fields as if they are real; at sufficiently large times they are all realized as actual fields outside the shell, and so they do contribute to the energy. In the sequel, then, we allow these fields to exist.

Before going into the analysis based on boundary condition (11), we present an alternative boundary condition based on the idea of "combing out" the lines of force. Not only does this concept give a cross-check, but the actual equation combines with that arising from condition (11) to yield a much simpler equation for $f(\omega)$. If the shell is a perfect conductor, the magnetic-field lines cannot slide tangentially along it, so in any fixed region of (θ, ϕ)-space (i.e., in any fixed cone centered at the origin), there must always emerge from the expanding shell as many flux lines as emerged at $t = 0$. This gives the condition

$$B_r(\xi) = 2\mu r_0^{-1} \xi^{-2} \cos\theta\,, \qquad t > 0\,, \quad (12)$$

where

$$\xi \equiv r_0 + Vt\,. \quad (13)$$

Now when equations (7) and (8) are put into the inverse transform (10), a certain operator M results which merits separate definition:

$$M = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\,[-i\omega t + i\omega c^{-1}(r_0 + Vt)]d\omega\,. \tag{14}$$

It will also be necessary to have the time derivative of M:

$$(d/dt)Mf(\omega) = -iM[\omega(1-\beta)f(\omega)]\,, \qquad \beta \equiv V/c\,. \tag{15}$$

Armed with these formulae, we substitute equations (6)–(8) into equations (10) and (11), remembering that in equation (11) B_θ includes the initial *and* perturbed fields. The result is

$$M\{[(1-\beta)(i\omega c^{-1}\xi^{-1} - \xi^{-2}) + i\beta c\omega^{-1}\xi^{-3}]f(\omega)\} + \beta\xi^{-3} = 0\,. \tag{16}$$

In principle, this may be solved for $f(\omega)$. It should be noted that ξ and M freely commute. Although it is possible to choose suitable contours for integrations such as those in equation (16), to encircle the poles properly at $\omega = 0$, it will turn out a little later that the expressions finally used have no ambiguities of this type, so the problem is ignored. When equation (12) is put in similar form, there results

$$M[(\beta\xi^{-2} + i\beta c\omega^{-1}\xi^{-3})f(\omega)] - \beta Vtr_0^{-1}\xi^{-3} = 0\,. \tag{17}$$

Fortunately, equations (16) and (17) are redundant; they must be lest f be overdetermined. The redundancy is exhibited by comparing the *difference* of equations (16) and (17) with the *time derivative* of equation (17); the two results are identical. We omit the algebra, which is simply subtraction of equations or the application of equation (15), respectively. The common result is

$$Mf(\omega) - iA\xi c^{-1}M\omega f(\omega) - \beta r_0^{-1} = 0\,, \tag{18}$$

where $A \equiv 1 - \beta$. Now, although any of equations (16)–(18) seems difficult to solve, the situation is quite different when looked at in terms of the *time*-dependent form, $F(t)$. In fact, one has

$$Mf(\omega) = F(\tau)\,, \tag{19}$$

where

$$\tau \equiv At - r_0c^{-1}\,. \tag{20}$$

Similarly, $M\omega f(\omega)$ may be reduced to $F'(\tau)$ via equation (15), viz.:

$$M\omega f(\omega) = iF'(\tau)\,. \tag{21}$$

In the foregoing, the identity $(r_0 + Vt)A = r_0 + V\tau$ has been used. Using equations (19) and (20), we reduce equation (18) to the form

$$(r_0 + V\tau)F'(\tau) + cF(\tau) - Vr_0^{-1} = 0\,. \tag{22}$$

This is a first-order linear differential equation for $F(\tau)$, or equivalently for $F(t)$, since the variable is dummy. The boundary condition to be used is

$$F(t) = 0 \quad \text{at} \quad t = -r_0/c\,. \tag{23}$$

To understand this condition, recall the remarks previously made about the meaning of the fields when $r < r_0$ and $t < 0$. A hypothetical image source at the origin must start up at such time that its disturbance reaches r_0 at $t = 0$, but not before. To see that $F(t)$ has the significance of the source at the origin, one may compare with Jackson's (1962) equation (16.47). Alternatively, one may verify by inspection that our final solution has the form of a pulse that originates at $r = r_0$ when $t = 0$, so far as the region exterior to r_0 is concerned, and that any different boundary condition would not fit this requirement. The solution to equation (22) subject to condition (23) is

$$F(t) = \frac{V}{cr_0}\left[1 - \left(\frac{r_0 A}{r_0 + Vt}\right)^{+1/\beta}\right], \qquad t > -\frac{r_0}{c}, \quad (24)$$

and, of course, F is zero at smaller times. Now, to recover the fields, we must in principle substitute expression (24) into equation (10) to get $f(\omega)$, and thence into equations (7) and (8), returning to (10) if we want the time dependence. This onerous procedure may be skirted by using the convolution theorem; clearly the effect of the chain of operations just described is to convolve $F(t)$ with the inverse Fourier transform of the functions in equations (7) and (8). Since we are interested mainly in the radiation field, let us find only the radiation part of $B_1(t)$, which we denote as

$$B_{\text{rad}} = \sin\theta\, \hat{e}_\theta B_2(r, t)\,. \quad (25)$$

Clearly, B_2 is the convolution of $F(t)$ with

$$S(t) = -\mathfrak{F}^{-1}[\mu e^{i\omega r/c}(i\omega/cr)] \quad (26)$$

$$= (2\pi)^{1/2}\mu(cr)^{-1}\delta'(t - rc^{-1})\,, \quad (27)$$

where δ' means the derivative of the Dirac δ-function. Thus,

$$B_2(r, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} F(t')S(t - t')dt'\,. \quad (28)$$

Equation (28) will introduce the derivative of $F(t)$, which is largest at $t = -r_0/c$. In order to be able to study the behavior there, it is convenient to introduce a displaced time coordinate defined by

$$\bar{t} = t + r_0/c\,. \quad (29)$$

In addition, the time interval $T = r_0/c$ for light to travel one stellar radius enters frequently. In terms of these variables, F has the form

$$F(\bar{t}) = \frac{\beta}{cT}\left[1 - \left(1 + \frac{\beta\bar{t}}{AT}\right)^{-1/\beta}\right], \quad \bar{t} > 0\,; \quad F(\bar{t}) = 0\,, \quad \bar{t} < 0\,. \quad (30)$$

Equation (30) nicely exhibits the rapid fluctuations that F undergoes near $t = 0$ when $V \approx c$, that is, when $A \approx 0$. Specifically, in this limit of large $\gamma = (1 - \beta^2)^{-1/2}$, we have $A \approx \frac{1}{2}\gamma^{-2}$, so that most of the variation of F occurs in the brief interval $0 < \bar{t} < \frac{1}{2}T\gamma^{-2}$. Thus, we expect the Fourier spectrum of the pulse to extend not just to frequencies of order T^{-1}, but to a factor γ^2 higher. If equation (30) is put into equation (28), there results

$$B_2(r, \bar{t}) = \frac{\mu\beta}{crr_0TA}\left(1 + \frac{\beta t^*}{AT}\right)^{-(\beta+1)/\beta}, \quad (31)$$

where

$$t^* = \bar{t} - r/c > 0\,, \qquad r > r_0\,. \quad (32)$$

The first inequality in expression (32) is essential, as it represents the correct form of the pulse, but the second inequality is just an arbitrary restriction to the physical region; eventually, the entire pulse is seen in the observable region, and we may consider its properties then, ignoring the restriction $r > r_0$. Because of dispersion, we shall probably not see the pulse shape, but will be interested in the total energy emitted per Hz bandwidth, $P(\omega)$. This is obtained from

$$P(\omega) = 4(8\pi)^{-1}cr^2\int d\Omega \left| \int_0^\infty B_2 e^{i\omega t^*}dt^* \right|^2 \sin^2\theta \text{ ergs Hz}^{-1}\,. \tag{33}$$

The factor 4 takes into account the energy present in the electric field and the fact that we define P only for positive frequency, although formally there is energy present at negative frequency. A factor 2π has been used to convert to ergs Hz^{-1} from units based on angular frequency ω; it makes no difference that ω itself is left as the argument. In general, the Fourier transform of expression (31) will be an incomplete Γ-function, but it is possible to achieve simple, accurate results when γ is large. Then, we may replace β by 1, except in A, where we use $A \approx \frac{1}{2}\gamma^{-2}$. It may then be shown that

$$\int_0 B_2(r, t^*) \exp(i\omega t^*)dt^* = \mu(rcr_0)^{-1}\Phi(q)\,, \tag{34}$$

where

$$\Phi(q) = \int_1^\infty p^{-2}\exp(ipq)dp = \exp(iq) - iq[\text{ci}(q) + i\,\text{si}(q)] \tag{35}$$

and

$$q = \omega T/(2\gamma^2)\,. \tag{36}$$

Here, si and ci are the sine and cosine integrals (Gradshteyn and Ryzhik 1962). The fact that γ has disappeared from all the results except in the definition of q means that the spectrum of radiation emitted will always be of identical form and maximum strength, but will simply extend to higher frequencies when γ is large (Colgate and Noerdlinger 1970). Upon substituting into equation (32), we obtain the emitted power (using $\int \sin^2\theta d\Omega = 8\pi/3$)

$$P(\nu)d\nu = \tfrac{4}{3}\mu^2 c^{-1} r_0^{-2}\psi(q)d\nu \text{ ergs}\,, \tag{37}$$

where ν is the frequency, $q = \pi\nu T/\gamma^2$, and $\psi = |\Phi|^2$. A graph of ψ versus q is shown in Figure 2. The fact that it starts out nonzero even at zero frequency is associated with the fact that the pulse of radiation is "unipolar," i.e., that at any point of observation, B_θ and E_ϕ do not reverse sign. The behavior at large q is $\sim q^{-2}$. The function ψ was integrated numerically, and to the 1 percent accuracy of the method used, the integral

$$\int_0^\infty \psi(q)\,dq$$

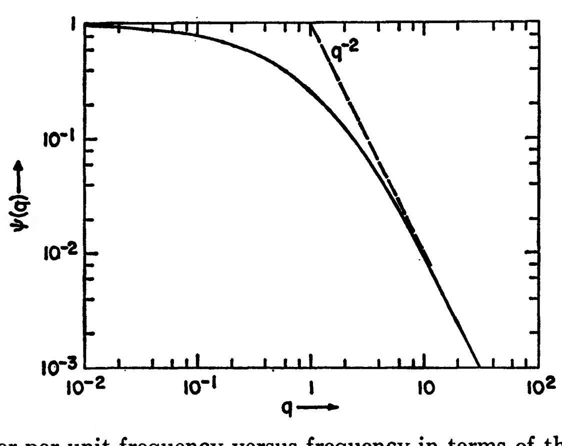

FIG 2 —Emitted power per unit frequency versus frequency in terms of the reduced variables $\psi(q)$ (*ordinate*) and q (*abscissa*). The value $q = 1$ corresponds to that frequency such that the vacuum wavelength equals approximately the thickness of the relativistically compressed dipole magnetic field is just unity. This implies that, upon converting to an integral over frequency, the total energy U emitted in the pulse is

$$U = \frac{4\gamma^2\mu^2}{3\pi r_0^3} = 0.425 \frac{\gamma^2\mu^2}{r_0^3} \text{ ergs} . \quad (38)$$

III. HEURISTIC ANALYSIS

A heuristic argument can be given for the emission based upon the shock jump conditions. The tangential magnetic field is compressed by $(4\gamma_B)$ as in a strong shock, giving a thickness of $r_0/(4\gamma_B)$ for the magnetic field in the moving frame, moving at a velocity corresponding to γ_B. The apparent thickness as viewed by a stationary observer is then $r_0/(4\gamma_B^2)$. This pulse of magnetic field will have constant Fourier coefficients up to a frequency $\nu_c \simeq 4c\gamma_B^2 r_0^{-1}\pi^{-1}$. The total energy radiated should then be

$$4\gamma_B^2 \int (B_\theta^2/8\pi) d \text{ Vol} = \tfrac{4}{9}\gamma_B^2\mu^2 r_0^{-3}$$

which is close to equation (38).

We now consider the emission above ν_c. Following equation (37), $P(\nu) \propto \nu^{-2}$, $\nu > \nu_c$. The emission at high frequency ($\nu > \nu_c$) is due to the approximation that breakout of the shock is a sudden phenomenon. Actually, breakout involves the gradual acceleration of the matter by the radiation stress of the photon flux from the shocked fluid. The corresponding scale height h of the breakout layer of 10 g cm^{-2} is roughly $h \simeq 2 \times 10^5$ cm. Therefore, the spectrum behavior $P(\nu) \propto \nu^{-2}$ should extend up to a frequency $c\gamma_B^2 h^{-1}$.

Finally, we consider the effect of the variation of $\gamma_B \propto (\sin\theta)^{-1}$. If we assume from the previous arguments that the radiation per unit solid angle

$$\frac{d}{d\Omega} P(\nu) \propto B_\theta^2 \propto (B_\theta \sin\theta)^2 \quad (39)$$

and that the upper frequency limit of the plateau $\nu_c \propto \gamma_B^2 \propto \gamma_s^2 (\sin\theta)^{-2}$, then the total radiated energy per solid angle remains constant, and the power at $\nu < \nu_c$ becomes proportional to $1/\nu_c$. For frequencies larger than ν_c, the radiation flux will be larger than the equatorial value by $(\sin\theta)^{-1}$ with a probability $\frac{1}{2}\sin\theta d\theta$ of viewing that orientation of the dipole.

From equation (31), it is possible to show that the maximum laboratory value for B_1, at the leading edge of the pulse, is $\sim\gamma_B^2 B_0$. However, in the shock frame, this is *not* further increased, because B_1 has an associated electric field, which causes it to move radially outward at speed c in any frame, and thus it is found that in the *moving* frame, B_1 and B_0 are of comparable magnitude. This yields a remarkably simple picture in the shock frame. Think of the expanding shock as a moving mirror. From the standpoint of the mirror, an incoming Lorentz-contracted magnetic field of strength $\gamma_B B_0$ and scale height r_0/γ_B is reflected into a similar outgoing wave. When this is transformed back to the laboratory, its strength is increased another factor γ_B while its scale height is decreased a factor γ_B, and we get out a strong, brief pulse. It does not seem to be essential that the "incident" magnetic field has an associated electric field slightly too weak to make it an electromagnetic wave.

The thicknesses and energies per unit surface area present in the original field B_0 and the perturbed field B_1 as seen in the two reference frames, together with the relevant scale heights, are summarized in Table 1.

IV. DETECTION

The total energy in the radiation pulse for $\gamma_B = 3 \times 10^3$ is

$$U = \frac{\mathfrak{M}_\odot}{2} F_B \gamma_B^2 c^2 = 10^{46} \text{ ergs} . \quad (40)$$

TABLE 1

THICKNESSES AND ENERGIES OF THE FIELDS IN THE TWO REFERENCE FRAMES

Field	Laboratory Frame		Shock Frame	
	Scale	Energy	Scale	Energy
B_0	r_0	$B_0^2 r_0$	r_0/γ_B	$\gamma_B B_0^2 r_0$
B_1	r_0/γ_B^2	$B_0^2 r_0 \gamma_B^2$	r_0/γ_B	$\gamma_B B_0^2 r_0$

NOTE.—Factors of order unity are ignored.

(This is also the energy of cosmic rays of $E > \gamma_B^2 c^2 \mathfrak{M}_p = 10^{16}$ eV per supernova required to fill the Galaxy if one assumes a cosmic-ray lifetime of 3×10^6 years and 1 supernova per 100 years.) The electromagnetic radiation will be dispersed both within galaxies and in intergalactic space such that the time dispersion is

$$D = \frac{10^8}{\nu^2 2c} \int n_e dl \text{ sec} . \tag{41}$$

The dispersion due to the galactic electrons, $n \simeq 0.02$ cm^{-3} (Bridle and Venugopal 1969), has a minimum value toward the galactic pole of $N = \int n_e dl \simeq 2 \times 10^{19}$ cm^{-2} or $D \geq 3 \times 10^{16} \nu^{-2}$ sec. Therefore, all frequencies $\leq 5 \times 10^{12}$ will be dispersed more by propagation than by the time structure of the source.

As a consequence, the received energy S in a band $\Delta\nu$ Hz arrives in a time $\Delta D = \Delta\nu(dD/d\nu)$ seconds. The received energy for equatorial emission ($\sin\theta = 1$) is

$$S \simeq \frac{U}{4\pi R^2} \frac{\Delta\nu}{\nu_c} \text{ ergs cm}^{-2} \qquad (\nu \leq \nu_c) \tag{42}$$

and

$$S \simeq \frac{U}{4\pi R^2} \left(\frac{\Delta\nu}{\nu_c}\right) \left(\frac{\nu_c}{\nu}\right)^2 \text{ ergs cm}^{-2} \qquad (\nu \geq \nu_c) . \tag{43}$$

A receiver has a bandwidth $\Delta\nu$ and integrates for a time Δt. The bandwidth can never be less than one period in the time Δt, or $\Delta t \geq \Delta\nu^{-1}$. In addition, during the time Δt, the signal will change in frequency due to dispersion. The time that the signal remains within the receiver band $\Delta\nu$ is ΔD, where $\Delta D = \Delta\nu(dD/d\nu)$. The optimum detection occurs when the receiver integrates for just the time that the signal remains within the band. A longer integration includes noise with no signal, and a shorter time neglects signal. Therefore, for optimum detection $\Delta D = \Delta t$. Finally, in order to minimize the noise power received within the bandwidth $\Delta\nu$ in the time Δt (power $\propto \Delta\nu \Delta t kT$), we wish to minimize the product $\Delta\nu\Delta t$, or $\Delta t \leq \Delta\nu^{-1}$. This, combined with our first condition, implies $\Delta t = \Delta\nu^{-1}$, so that the optimum detection is

$$\Delta\nu = \frac{1}{\Delta t} = \left(\frac{dD}{d\nu}\right)^{-1/2} = \left(\frac{2D}{\nu}\right)^{-1/2} . \tag{44}$$

Therefore, detection occurs when the received signal energy in an antenna of area πa^2 and in a time Δt equals the system noise energy in a bandwidth $\Delta\nu$ and time Δt. A combined radio antenna and receiver is characterized by a system noise kT ergs Hz^{-1} sec^{-1}, so that for detection

$$\pi a^2 S \geq (kT)\Delta\nu\Delta t \text{ ergs} ,$$

or

$$S \geq \frac{kT}{\pi a^2} \text{ ergs cm}^{-2} . \tag{45}$$

If we further impose the condition that for detection we require one observed event per day, then for a supernova rate of 10^{-2} per year and a galaxy density of 5×10^{-2} $(\text{Mpc})^{-3}$, a beamwidth of $\lambda^2/\pi a^2$ steradians will include one supernova per day at a distance R_d

$$R_d = 100(a/\lambda)^{2/3}\pi^{1/3} \text{ Mpc} . \tag{46}$$

For $a/\lambda = 1$, $n_e = 10^{-6}$ in the Metagalaxy, $\nu = 3 \times 10^8$ Hz, $D = 9$ sec, and $\Delta\nu = 4 \times 10^3$ Hz.

This optimum detection criterion for $\Delta\nu$ presupposes that the broadbanding due to multiple paths in both galactic and metagalactic space is small. The thin-screen model of scintillation (Salpeter 1967), which is compared with the current measurements by Lang (1971), predicts a decorrelation-frequency bandwidth which reflects different path lengths for various scattered paths. The inverse decorrelation frequency is a measure of the time difference $\Delta\tau$ of various paths. The bandwidth $\Delta\nu$ of a dispersed, $d\nu/dt$, initially coherent signal becomes

$$\Delta\nu = \Delta\tau \frac{d\nu}{dt} = \frac{1.3 \times 10^4 \langle \Delta n_e^2 \rangle R^2}{a\nu^4} \left(\frac{10^8}{\nu^3 c} \int n_e dl \right)^{-1} = 4 \times 10^6 \frac{R}{a} \frac{\langle \Delta n_e^2 \rangle}{\langle n_e \rangle} \frac{1}{\nu} , \tag{47}$$

where $\langle \Delta n_e^2 \rangle$ is the mean square fluctuating electron density and a is the turbule dimension. If we assume that $R/a \simeq 10^{10}$, that $\langle \Delta n_e^2 \rangle / \langle n_e^2 \rangle = 10^{-4}$ is the same in the Metagalaxy as within the Galaxy, and that $n_e = 2 \times 10^{-2}$ cm^{-3} within the Galaxy and 10^{-6} between galaxies, then at $\nu = 3 \times 10^8$ Hz, $\Delta\nu \approx 3 \times 10^3$ Hz within the Galaxy and 10^{-1} Hz in metagalactic space. Therefore, one should keep in mind the possible increase in bandwidth due to propagation when considering the signal-to-noise ratio.

Detection results in the condition

$$U \geq \frac{kTR^{5/2} n e^{1/2} \nu_c}{2^{1/2} \pi a^2 \nu^{3/2}} \text{ ergs}$$

for $\nu \leq \nu_c$, and

$$U \geq \frac{kTR^{5/2} n_e^{1/2} \nu^{1/2}}{2^{1/2} \pi a^2 \nu_c} \text{ ergs} \tag{48}$$

for $\nu \geq \nu_c$. In each case, the maximum signal occurs at $\nu = \nu_c$. From Figure 2, we estimate that the half-power point occurs at $\nu_c = \frac{1}{3} c \gamma_B^2 r_0^{-1} \pi^{-1}$. The most likely value of $\gamma_B^2 = 10^7$ and $r_0 = 1.2 \times 10^8$ cm, $\nu_c = 3 \times 10^8$ Hz. Substituting the condition (eq. [46]) of one event per day at $a/\lambda = 1$ and $R_d = 145$ Mpc into the above detection condition and letting $T = 50°$ K represent the combined sky and system noise temperature, we obtain the condition

$$U \geq 7 \times 10^{30} n_e^{1/2} \nu_c \nu^{1/2} \text{ ergs} \qquad (\nu \leq \nu_c)$$

and

$$U \geq 7 \times 10^{30} n_e^{1/2} \frac{\nu^{5/2}}{\nu_c} \text{ ergs} \qquad (\nu \geq \nu_c) . \tag{49}$$

If one is fortunate enough to be observing at $\nu = \nu_c$, and if $n_e \approx 10^{-6}$ cm^{-3}, then we require $U \geq 3.5 \times 10^{40}$ ergs, which is roughly 3×10^5 smaller than the most optimistic emitted pulse.

This very large expected signal-to-noise ratio is predicated upon the assumption of a highly condensed presupernova star surrounded by a relatively large vacuum-dipole magnetic field. In discussing breakout condition 1, we have shown that there is expected to be sufficient plasma in the magnetic field to provide a good hydromagnetic piston surface. The question also arises as to whether there is so much plasma that the wave is attenuated by reflection (plasma cutoff) or absorption (energization of plasma particles). One of us has shown (Noerdlinger 1970) that if radiation reaction is neglected, a plane electromagnetic pulse of thickness X is assured of penetrating a column density $N_e \sim 1/(r_e X)$ of electrons and ions without reflection, where $r_e = e^2/mc^2$. Thus our pulse 3 cm thick will certainly penetrate 10^{12} electrons or ions per cm^2. This is a rather tenuous atmosphere; however, a thicker atmosphere might very well only disperse the wave, not reflect it, since the foregoing criterion is sufficient for good transmission but may not be necessary. In fact, the usual condition $\omega > \omega_p$ is probably adequate (Noerdlinger 1971) for transmission, which allows a plasma density in excess of $n_e = 10^8\ \text{cm}^{-3}$.

The acceleration of ions by the wave (Ostriker and Gunn 1969) results in ions of Lorentz factor

$$\gamma_i = \left(\frac{r_0}{r_\text{L}}\right)^{2/3} \leq 1.4 \times 10^6\,, \tag{50}$$

where r_L is the Larmor radius of an ion of kinetic energy $\mathfrak{M}c^2$ in the field B_0. Thus, losses to the ions will be small provided $N_i < 10^{25}\ \text{cm}^{-2}$. If radiation reaction is neglected, the electrons, like the ions, stay in phase well with the wave (Ostriker and Gunn 1969), and pick up an energy per particle that is smaller by the factor $(1836)^{-1/3}$. The force of radiation reaction can only throw the particle out of phase, or retard its progress along the local electric field vector in the laboratory reference frame. Thus, the total energy $\int E dr$ taken out of the wave by an electron (which must be shared by kinetic energy and radiated photons) cannot be larger than the estimated kinetic energy in the absence of radiation reaction, and we may neglect losses to the electrons as compared with those to the ions. A brief discussion of the effect of radiation reaction in spherical geometry, when the field is due to a rotating dipole, is also given by Gunn and Ostriker (1970).

With a possible signal-to-noise ratio as large as 3×10^5 at 145 Mpc for an antenna of $\lambda/a = 1$, it is natural to inquire what the maximum possible redshift z is for detection with a large radio telescope. The dispersion must be integrated along the path at the Doppler-shifted frequency $\nu' = \nu(1+z)$ in an electron density $n_e = n_{e0}(1+z)^3$ and path length

$$ds = cH_0^{-1}(1+z)^{-2}(1+2q_0 z)^{-1/2}dz\,.$$

Hence, the dispersion becomes

$$D = D_0(1+z)^2 \int_0^z (1+x)^{-1}(1+2q_0 x)^{-1/2}dx\,, \tag{51}$$

where $D_0 = cH_0^{-1}n_e 1.6 \times 10^{-3}/\nu^2$. For z large and $q_0 \simeq \frac{1}{2}$, $D \simeq 1.6 \times 10^{-3}(cH_0^{-1}) n_e \nu_c^{-2}(1+z)^2$ seconds at an observed frequency $\nu = \nu_c(1+z)^{-1}$ if ν_c is assumed to be the optimum source frequency.

The receiver bandwidth $\Delta\nu$ corresponds to a source bandwidth $\Delta\nu' = \Delta\nu(1+z)$. The energy emitted is $U\Delta\nu'\nu_c^{-1}$. The received energy S becomes

$$S = \frac{U\Delta\nu(1+z)}{4\pi R_L^2 \nu_c}\ \text{ergs cm}^{-2}\,, \tag{52}$$

where R_L is the luminosity distance $= cH_0^{-1}q_0^{-2}\{q_0 z + (q_0 - 1)[(1+2q_0 z)^{1/2} - 1]\}$. Using equation (44) for $\Delta\nu$ and taking $q_0 = \frac{1}{2}$, we obtain

$$S = \frac{U\nu_c^{1/2}2^{1/2}}{4(cH_0^{-1})^{5/2}n_e^{1/2}[1+z-(1+z)^{1/2}]^2(1+z)^{1/2}}\ \text{ergs cm}^{-2}\,.$$

The Arecibo antenna has an approximate area $\pi a^2 \simeq 3 \times 10^8$ cm^2, and the sky noise temperature increases with decreasing frequency below $\nu_c = 3 \times 10^8$ Hz as $\nu^{-2.5}$, so that, for an emitted frequency ν_c, $T \simeq 30(1+z)^{2.5}$ ° K and $n_e = 10^{-6}$; and substituting these values into the detection condition $kT = \pi a^2 S$ results in

$$(1+z)^3[1+z-(1+z)^{1/2}]^2 = \frac{U\nu_c^{1/2}2^{1/2}\pi a^2}{4(cH_0^{-1})^{5/2}n_e^{1/2}kT_0} = 1.5 \times 10^5 , \tag{53}$$

or $1+z=12$ at $\nu = 2.4 \times 10^7$ Hz. Ionospheric reflection and absorption limits ν to values not less than 3×10^7 Hz, setting a practical limit of $1+z=10$. If, instead, observation is restricted to $\nu = \nu_c$ which demands emission at $\nu_c(1+z)$, then the signal-to-noise ratio increases by $(1+z)^{3/2}$ from dispersion and $(1+z)^{2.5}$ from noise, but it decreases as $(1+z)^{-2}$ due to the roll-off at the source for emission above ν_c (Fig. 2). The result is that the left-hand side of the equation (53) is reduced by $(1+z)^{-2}$, resulting in a maximum $(1+z) = 100$. The addition to the measured noise (30° K) in the direction of the North Galactic Pole by all supernovae in the Universe at 3×10^8 Hz would be 10 percent if U were the maximum value of 10^{46} ergs per supernova.

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REFERENCES

Arnett, W. D. 1969, *Ap. and Space Sci.*, **5**, 180.
Bridle, A., and Venugopal, V. 1969, *Nature*, **224**, 545.
Chiu, H. Y. 1966, in *Stellar Evolution*, ed. R. Stein and A. G. W. Cameron (New York: Plenum Press), p. 279.
Colgate, S A., and Johnson, M. H. 1960, *Phys. Rev. Letters*, **5**, 235.
Colgate, S. A., and Noerdlinger, P. D. 1970, post-deadline paper presented at Amer Phys. Soc , Washington, D.C.
Colgate, S. A., and White, R. H 1966, *Ap. J.*, **143**, 626.
Finzi, A., and Wolf, R. 1967, *Ap. J* , **150**, 115.
Gradshteyn, I., and Ryzhik, I. 1965, *Tables of Integrals, Series, and Products* (New York: Academic Press).
Grover, R., and Hardy, J. W. 1966, *Ap. J.*, **143**, 48.
Gunn, J. E., and Ostriker, J. P. 1970, preprint No. OAP 221.
Jackson, J. D. 1962, *Classical Electrodynamics* (New York: John Wiley & Sons), p 566.
Johnson, M. H., and McKee, C. F. 1971, *Phys. Rev.* (in press).
Kemp, J. C., Swedlund, J. B., Landstreet, J. D., and Angel, J. R. P. 1970, *Ap J. (Letters)*, **161**, L77.
Lang, K. 1971, *Ap. J.* (in press).
May, M. M., and White, R. H. 1967, *Methods in Computational Physics*, Vol. **7**, ed. B. Alder (New York: Academic Press), p. 219.
Minkowski, R. L. 1968, in *Stars and Stellar Systems*, Vol. **7**, ed. B. M. Middlehurst and L. H. Aller (Chicago: University of Chicago Press), chap. 11.
Noerdlinger, P. 1971, *Phys. Fluids* (in press)
Ono, Y., Sakashita, S., and Ohyama, N. 1961, *Progr. Theoret. Phys. Suppl.*, No. 20, p. 85.
Ostriker, J. P., and Gunn, J. E. 1969, *Ap. J.*, **157**, 1395.
Rakavy, G., and Shaviv, G. 1967, *Ap. J.*, **148**, 803.
Salpeter, E. 1967, *Ap. J.*, **147**, 433.